Fiber bundle over manifolds of quantum channels and its application to quantum statistics

10th January 2008

It is well known that a quantum channel Γ is represented by using a finite number of operators $\{A_i\}_{i=1}^q$ (called a generator of Γ) satisfying $\sum_{i=1}^q A_i^* A_i = I$ as $\Gamma(\cdot) = \sum_{i=1}^q A_i(\cdot)A_i^*$. We introduce a fiber bundle over a manifold of quantum channels from the freedom of $\mathrm{U}(q)$ action of generators as

$$U : [A_1, \ldots, A_q] \mapsto \left[\sum_{j=1}^q A_j u_{j1}, \ldots, \sum_{j=1}^q A_j u_{jq} \right].$$

There is a natural fiber bundle structure over a manifold of quantum states which is derived from the freedom of $\mathrm{U}(q)$ action of extreme point decompositions $\rho = \sum_{i=1}^q |\psi_i\rangle \langle\psi_i|$ as $U : [\psi_1, \ldots, \psi_q] \mapsto [\sum_{j=1}^r \psi_j u_{j1}, \ldots, \sum_{j=1}^r \psi_j u_{jq}]$. We make use of the canonical correspondence $[A_1, \ldots, A_q] \mapsto [A_1\psi, \ldots, A_q\psi]$ between these fiber bundles, given a $\psi \in \mathcal{H}$. This enables us to analyze quantum channel estimation problems in a unified manner from a differential geometrical point of view.