A Relationship between the Continuous Wavelet Transformation and the Algebra su(1,1)

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Abstract—It is well known that the Cauchy wavelets are the eigenfunctions of the linear operator $T+kD^{-1}$ (T: multipication by time, D^{-1}: integral operator, k: positive real number). A rational function of this linear operator is an annihilation operator of the orthonormal wavepacket system related to the associated Laguerre polynoimal. In this paper, this relation is discussed in terms of the algebra su(1,1), where the Cauchy wavelets are regarded as the generalized coherent states associated with the algebra su(1,1).

Keywords—Cauchy Wavelet, Eigenvector, Generalized coherent state, Algebra su(1,1), Associated laguerre polynomial, Annihilation/creation retation

1 Introduction: Cauchy wavelets and related operators

As is well known, wavelet systems used in the continuous wavelet transformation [1] [2] [3] [4] are non-orthogonal over-complete wavapacket systems with continuous-valued parameters a and b, where the function systems are made by the scale change (dilation) and the time shift. It is also well known that the generator of the time shift is the differential operator D while the generator of the scale change is the operator $(DT+TD)/2$ (where T denotes the multiplication operator by the time t), in the sense that the wavelet $h^{(a,b)}(t)) := |a|^{-1/2}h_k(\frac{t-b}{a})$ from the basic wavelet function $h(t)$ by

$$h^{(a,b)}(t) = \Big(\exp(bD)\exp\big(-(\ln a)(DT+TD)/2\big)\ h\Big)(t).$$

(The action of these generators is corresponding to a representation of the $ax+b$ -group)

One of the over-complete wavelet systems of this kind is made of Cauchy wavelets where the function $h_k(t) := \frac{G_k}{(t-i)^{k+1}}$ (k: positive real number, G_k: normalization constant) is used for the basic wavelet $h(t)$. (NB: The basic Cauchy wavelet $h_k(t)$ satisfies the 'admissible condition' for the continuous wavelet transformation and has k vanishing moments) This Cauchy wavelet system is equivalent to the eigenfunction system of the linear operator $Y_k := T+kD^{-1}$ (D^{-1} denotes the integral operator which is the inverse of the differential operator), as

$$\Big(Y_k h_k^{(a,b)}\Big)(t) = (b+ai)\ h_k^{(a,b)}(t) \tag{1}$$

[5] [6] [7]. This relation is very analogous to the relation between the short-time Fourier transform(STFT) or Gabor transform with the Gaussian window function and the eigenfunction system of the operator $T+D$. In this case, it is well known that this operator is the annihilation operator (down-ladder) of the eigenfunction of the 'number operator' $(T^2-D^2-I)/2$, where the 'number states' are the orthogonal system of Gaussian-weighted Hermite polynomials. (This relation is known as the boson annihilation relation in quantum mechanics.)

A similar annihilation relation for Cauchy wavelet has been known [10]. In this relation, the annihilation operator is

$$W_k := (Y_k - iI)^{-1}\ (Y_k + iI) \tag{2}$$

which is the Cayley transform of the operator $-Y_k$ (NB: therefore the Cauchy wavelets are the eigenfunctions of W_k), and the annihilation and creation relations are

$$\big(W_k\psi_{k,n}\big)(t) = \sqrt{\frac{n}{n+2k}}\ \psi_{k,n-1}(t) \tag{3}$$

$$\big(W_k^\dagger\psi_{k,n}\big)(t) = \sqrt{\frac{n+1}{n+2k+1}}\ \psi_{k,n+1}(t) \tag{4}$$

(† denotes the adjoint) where the 'number state' $\psi_{k,n}$ is the eigenfunction of the self-adjoint operator

$$M_k := -\frac{i}{2}(Y_k+iI)^\dagger\ D\ (Y_k+iI) \tag{5}$$

associated with the positive-integer eigenvalue n. [10] This 'number state' is closely related to the associated Laguerre (or Sonine) polynomial L_n^m in the sense that the Fourier transform of $\psi_{k,n}(t)$ is

$$\Psi_{k,n}(\omega) = \begin{cases} C_{k,n}\ e^{-\omega}\omega^k L_n^{2k}(2\omega) & (\text{if } \omega \geq 0) \\ 0 & (\text{otherwise}). \end{cases}$$

($C_{k,n}$: normalization constant) Moreover, the 'vacuum state' $\psi_{k,0}(t)$ is identical to the Cauchy wavelet $h_k^{(-1,0)}(t)$ $(= h_k(-t))$, (NB: This relation is quite analogous to the STFT case where the 'vacuum state' is

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identical to the Gabor function with the parameters $0, 0$)

In this paper, it will be shown that these relations can be interpreted from the viewpoint of the algebra su(1,1), by adding another extra generator more than the above-mentioned generators D and $(DT+TD)/2$ of the $ax+b$-group. Recently, the authors found a mathematical framework on the annihilation and creation relations and the generalized coherent states associated with the algebra su(1,1) [11] [12]. By using this framework, we can characrerize the Cauchy wavelets in more comprehensive mathematical contexts, and the annihilation and creation relations in (3) and (4) will be shown to be spacial cases of these mathematical framework. From this algebraic viewpoint, we will regard the continuous wavelet transform of a signal by the Cauchy wavelets as a kind of 'coherent-state representation' related to the algebra su(1,1).

In Section 2, we will discuss general mathematical framework of su(1,1)-annihilation and creation relations and su(1,1)-coherent states. In Section 3, we will interpret the Cauchy wavelets and related annihilation and creation relations as special cases of the results given in Section 2. In Section 4, we will discuss a possibility of application of these results for the operator method in signal processing which is similar to the operator method used in the Wiener-Hermite expansion of random signals.

2 su(1,1)-annihilation/creation relation

First, we will start with more general algebraic formalism. If the triplet of the skew-adjoint operators E_0, E_+ and E_- on a Hilbert space $\mathcal{H}$ satisfies the commutation relations $[E_0, E_\pm] = \pm 2E_\pm$ and $[E_+, E_-] = E_0$, it is called the unitary representation of the Lie algebra su(1,1). Define another triplet of the operators $L_0 := i(E_- - E_+)$ and $L_\pm := \big(E_0 \pm i(E_+ + E_-)\big)/2$. ($E_0 = L_+ + L_-$ and $E_\pm = \pm(L_0 \mp L_+ \pm L_-)/2$). Then the same type of commutation relations $[L_0, L_\pm] = \pm 2L_\pm$, $[L_+, L_-] = L_0$ hold, and these are another basis of the same Lie algebra. However, in this basis system, L_0 is self-adjoint while $L_\pm$ are neither self-adjoint nor skew-adjoint, and $(L_\pm)^* = -L_\mp$. In this paper, we investigate only the cases where the representation is irreducible and is not trivial. Then the corresponding Casimir operator should be a scalar by the Schur's lemma,

$$\begin{aligned} &C := L_0^2 + 2(L_+L_- + L_-L_+) = \beta \\ &(= L_0^2 \pm 2L_0 + 4L_\mp L_\pm = E_0^2 \pm 2E_0 + 4E_\mp E_\pm). \end{aligned} \tag{6}$$

where the scalar parameter β depends on the representation of the Lie algebra. If v is the eigenvector of L_0 associated with the eigenvalue value κ, then $L_0(L_\pm v) = L_+(L_0 \pm 2)v = (\kappa \pm 2)(L_+ v)$, from the commutation relations. From this relation and the self-adjoint property of L_0, the eigenvector system of L_0 is an orthogonal system, and the eigenvalues of L_0 are the real numbers spaced uniformly. The irreducibility and the unitarity imply that the dimension of the kernel of $L_\pm$ can not be more than one and the dimension of the kernel either of L_- or of L_+ is zero. From now, we are investigating the case where dim Ker $L_+ = 0$ and dim Ker $L_- = 1$. Let v_0 be the unit vector in Ker L_-, from the above relation, v_0 should be the eigenvector of L_0 associated with the minimum eigenvalue λ (otherwise the existence of a smaller eigenvalue were contradictory to $L_- v_0 = 0$). Then the characteristic equation

$$L_0\big((L_+)^n v_0\big) = (\lambda + 2n)\big((L_+)^n v_0\big) \tag{7}$$

holds. It is well known that this constant λ determine the representation of the algebra su(1,1) uniquely though the Casimir operator ($= \beta$) does not. In this case, the unitarity means that $\lambda > 0$ [9]. From the irreducibility and the self-adjointness of L_0, the set $\{(L_+)^n v_0\}_{n=0}^\infty$ is a CONS of $\mathcal{H}$. From the relation(6) and $L_- v_0 = 0$ and $L_0 v_0 = \lambda v_0$, we have $\beta = \lambda(\lambda - 2)$. Now, we define the su(1,1)-number operator N_λ and su(1,1)-number states vector $|n\rangle_{N_\lambda}$

$$N_\lambda := \tfrac{1}{2}(L_0 - \lambda), \quad |n\rangle_{N_\lambda} := \sqrt{\tfrac{\Gamma(\lambda)}{n!\ \Gamma(\lambda+n)}} L_+^n v_0. \tag{8}$$

Then, the equation (7) and the definition (8) imply that $N_\lambda |n\rangle_{N_\lambda} = n|n\rangle_{N_\lambda}$. The relations (6) and $L_+^* = -L_-$ show that the vector $|n\rangle_{N_\lambda}$ defined in (8) is an unit vector [11].

Next we will define the su(1,1)-annihilation operator a_λ. The relation (7) implies that $(L_0 - \lambda)|n\rangle_{N_\lambda}$ belongs to the range of L_+ for any n. Because $\dim \mathrm{Ker}(L_+) = 0$, we can define the annihilnation operator a_λ as the bounded operator

$$a_\lambda |n\rangle_{N_\lambda} := \tfrac{1}{2} L_+^{-1}(L_0 - \lambda)|n\rangle_{N_\lambda} = \sqrt{\tfrac{n}{n+\lambda-1}}\ |n-1\rangle_{N_\lambda}, \tag{9}$$

(where we mean $a_\lambda |0\rangle_{N_\lambda} = 0$ by the second equality of (9) in the case of $\lambda = 1$ and $n = 0$.) The second equality of (9) is derived from (7) and (8). From the relations (8) and (9), we have $[a_\lambda, N_\lambda] = a_\lambda$. Thus the

creation operator can be defined as the ajoint operator a_λ^*, and

$$a_\lambda^*|n\rangle_{N_\lambda} = \sqrt{\tfrac{n+1}{n+\lambda}}\ |n+1\rangle_{N_\lambda}. \tag{10}$$

Therefore, from the completeness and orthogonality of the eigenvectors of L_0, we have

$$a_\lambda^* a_\lambda = (N_\lambda+\lambda-1)^{-1}N_\lambda, \quad a_\lambda a_\lambda^* = (N_\lambda+\lambda)^{-1}(N_\lambda+1) \tag{11}$$

in the case of $\lambda \neq 1$. By eliminating N_λ from these, we have the re-ordering relation between the annihilation operator a_λ and the creation operator a_λ^*

$$\begin{array}{l} a_\lambda a_\lambda^* = -(a_\lambda^* a_\lambda + \lambda - 2)^{-1}(\lambda a_\lambda^* a_\lambda - 1), \\ a_\lambda^* a_\lambda = (a_\lambda a_\lambda^* - \lambda)^{-1}\big((2-\lambda)a_\lambda a_\lambda^* - 1\big) \end{array} \tag{12}$$

in the case of $\lambda \neq 1$. In the case of $\lambda = 1$, we have $a_\lambda a_\lambda^* = 1$, $\quad a_\lambda^* a_\lambda = 1 - |0\rangle_{N_\lambda\ N_\lambda}\langle 0|$. These relations are important for the calculating the quantum characteristic function.

With the unitary displacement operator $D(\xi) := \exp\big(\xi L_+ - \bar{\xi}L_+^*\big)$, define the su(1,1)-coherent state

$$\begin{array}{l} v(\zeta) := D\left(\frac{1}{2}e^{i\arg\zeta}\ln\frac{1+|\zeta|}{1-|\zeta|}\right)\ |0\rangle_{N_\lambda} \\ = \exp(\zeta L_+)\exp\left(\frac{1}{2}\ln(1-|\zeta|^2)\ L_0\right)\exp(\bar{\zeta}L_-)\ |0\rangle_{N_\lambda} \end{array} \tag{13}$$

(for $|\zeta| < 1$). The latter "normal-order" form of the right hand side is obtained from the relations given in pp.73-74 of [8]. Because we can show that $\exp\left(\frac{1}{2}\ln(1-|\zeta|^2)\ L_0\right)\exp(\bar{\zeta}L_-)\ |0\rangle_{N_\lambda} = (1-|\zeta|^2)^{\lambda/2}\ |0\rangle_N$ and $[a_\lambda, \exp(\zeta L_+)] = \zeta\exp(\zeta L_+)$ [11], we have

$$a_\lambda v(\zeta) = \exp(\zeta L_+)\ a_\lambda\ |0\rangle_{N_\lambda} + \zeta\exp(\zeta L_+)|0\rangle_{N_\lambda} = \zeta v(\zeta). \tag{14}$$

These relations show that the vector $v(\zeta)$ is the eigenvector of a_λ associated with the eigenvalue ζ. Thus we can denote $v(\zeta)$ by $|\zeta\rangle_{a_\lambda}$, and the set of eigenvalues of a_λ is the unit disk.

From the relations (6),(9) and the relation $\big[L_0, L_+^{-1}\big] = L_+^{-1}[L_+, L_0]L_+^{-1} = -2L_+^{-1}$, we can show the relation $(E_0-\lambda)(a_\lambda - 1) = -2iE_+(a_\lambda+1)$ [11]. Since we can show that $(E_0-\lambda)|n\rangle_{N_\lambda}$ belongs to the range of E_+ [11], we can define the operator

$$A_\lambda := \tfrac{1}{2}E_+^{-1}(E_0-\lambda) = -i(a_\lambda+1)(a_\lambda-1)^{-1}, \tag{15}$$

whose domain is dense in $\mathcal{H}$. Hence $A_\lambda|\zeta\rangle_{a_\lambda} = -i\frac{\zeta+1}{\zeta-1}|\zeta\rangle_{a_\lambda}$, which shows that the vector $|\zeta\rangle_{a_\lambda}$ can be denoted by $\left|-i\frac{\zeta+1}{\zeta-1}\right\rangle_{A_\lambda}$. Since $-i$ doesn't belong to the spectrum of A_λ, the latter relation in (15) indicates that

$$a_\lambda = (A_\lambda + i)^{-1}(A_\lambda - i). \tag{16}$$

Since a_λ is given by the Möbius transform of A, the set of eigenvalues of A is the upper half plane, and they are mapped to the unit disk by the Möbius transform.

Define the 'normal-ordered affine coherent vector' as $v'_n(r,s) := \exp(rE_+)\exp(sE_0)|0\rangle_{N_\lambda}$. Then, we can show[11] the relation similar to (14)

$$\begin{array}{l} A_\lambda v'_n(r,s) \\ = e^{2s}\exp(rE_+)\exp(sE_0)\ A_\lambda\ |0\rangle_{N_\lambda} \\ \qquad + r\exp(sE_+)\exp(sE_0)|0\rangle_{N_\lambda} \\ = (e^{2s}i + r)v'_n(r,s). \end{array} \tag{17}$$

Note that the operators $\exp(rE_+)$ and $\exp(sE_0)$ are unitary because of E_0 and E_+ are skew-adjoint, and this is equivalent to the unitary representation of ax+b-group[11].

3 Interpretation of Cauchy wavelets in terms of su(1,1)

In this section, we will show the relations (3) and (4) in Section 1 can be interpreted as a special case of the results given in Section 2, and will discuss the Cauchy wavelets in terms of the algebra su(1,1). If we choose

$$E_0 = -(DT+TD), \quad E_+ = D, \quad E_- = (-TDT+k^2D^{-1}),$$

by using the time-shift generator D and the scale-change generator $(DT+TD)/2$ in Section 1, and by adding an extra generator E_-, then the triplet E_0, E_+, E_- satisfies the commutation relation for the algebra su(1,1) in Section 2. (In this choice, the displacement operator $\exp(rE_+)\exp(sE_0)$ of the eigenvector of A used in (17) is the product of the time-shift operator $\exp(rD)$ and the scale-change (dilation) operator $\exp\big(-s(DT+TD)\big)$. These are corresponding to the $ax+b$-group of the wavelets.) Then, in terms of the mathematical framework in Section 2,

$$\begin{array}{l} L_0 = (1/2)\big(-iTDT + ik^2D^{-1} - iD\big), \\ L_\pm = \pm(i/2)\big(-iTDT \mp i(DT+TD) + ik^2D^{-1} - iD\big), \end{array}$$

and $\beta = 4k^2 - 1$. When we choose the function space $L^2(\mathrm{R}^+)$ in the frequency domain, then $\lambda = 2k+1$, and from (15), (16) (2) and (9)

$$A_{2k+1} = -\frac{i}{2}D^{-1}\big(-i(DT+TD) - (2k+1)iI\big) = -Y_k \tag{18}$$

$$a_{2k+1} = W_k$$

$$N_{2k+1} = \big(-iTDT + ik^2D^{-1} - iD - (2k-1)I\big)/2 = M_k.$$

(The second equality of the last relation is obtaned by substituting (1) into (2).) The wavefunction of the 'vacuum vector' $|0\rangle_N$ in frequency domain is (const.) $\cdot$

$\omega^k e^{-\omega}$. In the time domain, the corresponding function space is $\{f(t) \in L^2 \mid \big(\mathcal{H}f\big)(t) = if(t)\}$ ($\big(\mathcal{H}f\big)$: Hilbert tansform of f) or the space of normalizable analytic signals in signal analysis, and the eigenfunction of the vacuum vector is the Cauchy wavelet $h_k^{(-1,0)}(t) = -\frac{G_k}{(t+i)^{k+1}}$ $(= h_k(-t))$. The annihilation and creation relations (9) and (10) in this case are

$$a_{2k+1}|n\rangle_{N_{2k+1}} = \sqrt{\frac{n}{n+2k}}\,|n-1\rangle_{N_{2k+1}} \qquad (19)$$

$$a^*_{2k+1}|n\rangle_{N_{2k+1}} = \sqrt{\frac{n+1}{n+2k+1}}\,|n+1\rangle_{N_{2k+1}} \qquad (20)$$

which are equivalent to (3) and (4) in Section 1, and then the nonlinear-type re-ordering relations (12) between the annihilation and creation operators are equivalent to the relations in Ref[10]

$$W_k W_k^\dagger = \big(W_k^\dagger W_k - (2k+1)I\big)^{-1} \cdot \big((2k-1)W_k^\dagger W_k + I\big)$$

$$W_k^\dagger W_k = \big(W_k W_k^\dagger + (2k-1)I\big)^{-1} \cdot \big((2k+1)W_k W_k^\dagger - I\big).$$

From the relation (18) between A in Section 2 and Y_k in Section 1, the Cauchy wavelet with the scale parameter a and the shift parameter b is the wavefunction of $|-b-ai\rangle_{A_{2k+1}} = |\frac{b+ai+i}{b+ai-i}\rangle_{a_{2k+1}}$.

From these relations, we can regard the continuous wavelet transform of a signal $f(t)$ by the Cauchy wavelets

$$\begin{aligned}\big(\mathcal{T}_{(h_k)}\ f\big)(a,b) &:= \int_{-\infty}^{\infty} h_k^{(a,b)*}(t) f(t) dt \\ &= G_k\ |a|^{-1/2} \int_{-\infty}^{\infty} \frac{f(t)}{(\frac{t-b}{a}+i)^{k+1}} dt\end{aligned}$$

as a kind of 'coherent-state representation of $f(t)$' associated with the algebra su(1,1), which maps a function defined on the real time axis into an analytic function on the upper half plane in the complex plane.

4 Possibility of application in signel processing

As is well known, the annihilation and creation relations related to the short-time Fourier (Gabor) tansform (or boson annihilation/creation in quantum mechanics) has been applied to the operator method in Wiener-Hermite expansion of random signals. There are some possibilies of discussing similar operator methods for the Wiener-Laguerre expansion of random signals in terms of the results given in the previous sections.

5 Conclusions

The Cauchy wavelets and the eigenvector problem and the annihilation/creation relation closely related to them has been discussed from the viewpoint of the algebra su(1,1).

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