General asymptotic formulas for fixed-length quantum entanglement concentration

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Abstract — **General formulas of entanglement concentration are derived by using an information-spectrum approach not only for i.i.d. sequences, but also for the general sequences of partially entangled pure states. From these formulas, not only can the achievable rates be calculated, but also the error exponents, resulting in a general relation between the performance of the entanglement concentration and the eigenvalues of the partially traced state.**

For a rigorous treatment of fixed-length entanglement concentrations (FLECs), we give mathematical definitions for the two kinds of fixed-length entanglement concentrations. We call a triple $\mathcal{F}_1 = (C, \psi, L)$ a first kind fixed-length entanglement concentration (1-FLEC) when it consists of a natural number L, a trace-preserving completely positive (TP-CP) map C of local operations and classical communications(LOCCs), and a maximally entangled state ψ of a subspace $\mathcal{H}'_A \otimes \mathcal{H}'_B$, where $\mathcal{H}'_A$ ($\mathcal{H}'_B$) is an L-dimensional subspace of $\mathcal{H}_A$ ($\mathcal{H}_B$), respectively. When it is transformed to a pure state $\phi \in \mathcal{H}_A \otimes \mathcal{H}_B$, its performance is characterized by $\epsilon(\mathcal{F}_1) = 1 - \langle \psi | C(|\phi\rangle\langle\phi|) | \psi \rangle$ and the size $L(\mathcal{F}_1) = L$.

A rigid analysis of the second setting requires using an instrument that describes a quantum measurement with the final state as well as the probability distribution obeyed by data. The instrument $I = \{I_i\}_i$ is a CP map valued measure whose sum is a TP-CP map; i.e., every I_i is a CP map, and $\sum_i I_i$ is a TP-CP map. When we perform a quantum measurement corresponding to the instrument $I = \{I_i\}_i$, to a state ρ, we obtain the data i and the final state $\frac{1}{\operatorname{Tr} I_i(\rho)} I_i(\rho)$ with the probability $\operatorname{Tr} I_i(\rho)$. We call a pair $\mathcal{F}_2 = (I = \{I_0, I_1\}, L)$ a second kind fixed-length entanglement concentration (2-FLEC) of a pure state $\phi \in \mathcal{H}_A \otimes \mathcal{H}_B$ when it consists of a natural number L and a pair of CP maps, i.e., 2-valued instruments $I = \{I_0, I_1\}$ of LOCCs, such that $I_1(|\phi\rangle\langle\phi|)/\operatorname{Tr} I_1(|\phi\rangle\langle\phi|)$ is a maximally entangled state of the subspace $\mathcal{H}'_A \otimes \mathcal{H}'_B$ with $\dim \mathcal{H}'_A = \dim \mathcal{H}'_B = L$. Its performance is characterized by the failure probability $\epsilon(\mathcal{F}_2) = \operatorname{Tr} I_0(|\phi\rangle\langle\phi|)$ and the size $L(\mathcal{F}_2) = L$. For any 2-FLEC $(I = \{I_0, I_1\}, L)$ of ϕ, the triple $(I_1 + I_0, I_1(|\phi\rangle\langle\phi|)/\operatorname{Tr} I_1(|\phi\rangle\langle\phi|), L)$ becomes a 1-FLEC and its fidelity of ϕ is greater than the success probability of the 2-FLEC $(I = \{I_0, I_1\}, L)$:
$\operatorname{Tr}\left[(I_1 + I_0)(|\phi\rangle\langle\phi|)\frac{I_1(|\phi\rangle\langle\phi|)}{\operatorname{Tr} I_1(|\phi\rangle\langle\phi|)}\right] \geq \operatorname{Tr} I_1(|\phi\rangle\langle\phi|)$.

In the following, we discuss asymptotic performances of FLECs based on a general sequence of sources $\phi_n \in \mathcal{H}_{A,n} \otimes \mathcal{H}_{B,n}$. In this notation, $\mathcal{H}_{A,n}$ and $\mathcal{H}_{B,n}$ are regarded as generalizations of $\mathcal{H}_A^{\otimes n}$ and $\mathcal{H}_B^{\otimes n}$, and ϕ_n is regarded as a generalization of the n-tensor product vector $\phi^{\otimes n}$. For the 1-FLECs class and the 2-FLEC class, we focus on the following values:

$$B_1(\epsilon) := \sup\left\{ \underline{\lim} \frac{1}{n} \log L(\mathcal{F}_{1,n}) \middle| \ \exists\{\mathcal{F}_{1,n}\} : \underline{\lim}\, \epsilon(\mathcal{F}_{1,n}) \leq \epsilon \right\}$$

$$B_{e,1}(r) := \sup\left\{ \underline{\lim} \frac{1}{n} \log L(\mathcal{F}_{1,n}) \middle| \begin{array}{l} \exists\{\mathcal{F}_{1,n}\} : \\ \underline{\lim} \frac{-\log \epsilon(\mathcal{F}_{1,n})}{n} \geq r \end{array} \right\}$$

$$B^*_{e,1}(r) := \sup\left\{ \underline{\lim} \frac{1}{n} \log L(\mathcal{F}_{1,n}) \middle| \begin{array}{l} \exists\{\mathcal{F}_{1,n}\} : \\ \overline{\lim} \frac{-\log(1-\epsilon(\mathcal{F}_{1,n}))}{n} \leq r \end{array} \right\}$$

$$B_2(\epsilon) := \sup\left\{ \underline{\lim} \frac{1}{n} \log L(\mathcal{F}_{2,n}) \middle| \ \exists\{\mathcal{F}_{2,n}\} : \overline{\lim}\, \epsilon(\mathcal{F}_{2,n}) \leq \epsilon \right\}$$

$$B_{e,2}(r) := \sup\left\{ \underline{\lim} \frac{1}{n} \log L(\mathcal{F}_{2,n}) \middle| \begin{array}{l} \exists\{\mathcal{F}_{2,n}\} : \\ \underline{\lim} \frac{-1}{n} \log \epsilon(\mathcal{F}_{2,n}) \geq r \end{array} \right\}$$

$$B^*_{e,2}(r) := \sup\left\{ \underline{\lim} \frac{1}{n} \log L(\mathcal{F}_{2,n}) \middle| \begin{array}{l} \exists\{\mathcal{F}_{2,n}\} : \\ \overline{\lim} \frac{-\log(1-\epsilon(\mathcal{F}_{2,n}))}{n} \leq r \end{array} \right\},$$

where $\{\mathcal{F}_{1,n}\}$ is a sequence of 1-FLECs and $\{\mathcal{F}_{1,n}\}$ is a sequence of 2-FLECs. It is trivial from the definition that $B_1(\epsilon) \geq B_2(\epsilon), B_{e,1}(r) \geq B_{e,2}(r), B^*_{e,1}(r) \geq B^*_{e,2}(r)$.

In this paper, we treat a quantum analogue of information spectrums to analyze the above values. To do this, we need the following definitions. For a self-adjoint operator X, we can denote the projection $\sum_{x_i \geq c} E_i$ by $\{X \geq c\}$, where the spectral decomposition is given by $X = \sum_i x_i E_i$. We can define the projections $\{X > c\}, \{X < C\}, \{X \leq c\}$, etc. in a similar manner. Let ρ_n be the state $\operatorname{Tr}_{\mathcal{H}_{B,n}} |\phi_n\rangle\langle\phi_n|$ and define

$$K(R) := \overline{\lim} \operatorname{Tr} \rho_n \{\rho_n \geq e^{-na}\}$$

$$\underline{\zeta}^c(a) := \underline{\lim} \frac{-1}{n} \log \operatorname{Tr} \rho_n \{\rho_n - e^{-na} > 0\}.$$

When the limit $\lim \frac{-1}{n} \log \operatorname{Tr} \rho_n \{\rho_n - e^{-na} < 0\}$ exists, we denote it by $\zeta(a)$.

Note that the function $\underline{\zeta}^c(a)$ decreases monotonically, while the function $\zeta(a)$ increases monotonically.

Theorem 1 *Without any assumption, for every $\epsilon \in [0,1]$ we have*

$$B_1(\epsilon) = B_2(\epsilon) = \sup_R \{R | K(R) \leq \epsilon\}$$

$$B_{e,1}(r) = B_{e,2}(r) = \sup_R \{R | \underline{\zeta}^c(R) \geq r\}.$$

When $\zeta(a)$ exists and there exists a real number a such that $\zeta(a) \leq \underline{\zeta}^c(a)$, we have

$$B^*_{e,1}(r) = \sup_a \left\{ a - r \middle| \inf_{a'} \left\{ \zeta(a') - \frac{a'}{2} \middle| a' \leq a \right\} + \frac{a}{2} \leq r \right\}$$

$$B^*_{e,2}(r) = \sup_a \{a - \zeta(a) | \zeta(a) \leq r\}.$$

For the proof and the references, please see Hayashi[1].

References

[1] M. Hayashi, "General formulas for fixed-length quantum entanglement concentration," LANL eprint quant-ph/0206187, 2002.