The Mean Values and the Universality of Rankin-Selberg L-Functions

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1 Statement of results

Let $\phi(z)$ be a holomorphic normalized Hecke-eigen cusp form of weight κ with respect to the full modular group $SL(2,\mathbf{Z})$, and denote by $a(n)$ the nth Fourier coefficient of $\phi(z)$. The Rankin-Selberg L-function attached to $\phi(z)$ is defined by

$$\begin{aligned} Z(s) &= \zeta(2s)\sum_{n=1}^{\infty} a(n)^2 n^{1-\kappa-s} \\ &= \sum_{n=1}^{\infty} c_n n^{-s}, \end{aligned} \tag{1.1}$$

where $s=\sigma+it$ is a complex variable, $\zeta(s)$ is the Riemann zeta-function, and

$$c_n = n^{1-\kappa}\sum_{m^2|n} m^{2(\kappa-1)} a(n/m^2)^2. \tag{1.2}$$

The above Dirichlet series is absolutely convergent in the half plane $\sigma>1$, and can be continued meromorphically to the whole complex plane. The aim of the present paper is to study the analytic behaviour of $Z(s)$ in the strip

$1/2 \leq \sigma \leq 1$. By $m(A)$ we mean the Lebesgue measure of the set A. In what follows ϵ denotes an arbitrarily small positive number, not necessarily the same at each occurrence. One of the major result in this paper is the following universality theorem.

Theorem 1 *Let $D = \{s \,|\, 3/4 < \sigma < 1\}$, and K be any compact subset of D with connected complement. Let $f(s)$ be a continuous function on K such that $f(s) \neq 0$ for any $s \in K$ and holomorphic in the interior of K. Then, for any $\epsilon > 0$ we have*

$$\liminf_{T\to\infty} \frac{1}{T} m\{\tau \in [0,T] | \sup_{s\in K} |Z(s+i\tau) - f(s)| < \epsilon\} > 0. \tag{1.3}$$

This theorem gives an analogy of Voronin's universality theorem [21] for $\zeta(s)$, which asserts that the analogous conclusion for $\zeta(s)$ holds for any compact subset with connected complement of $\{s \,|\, 1/2 < \sigma < 1\}$. The reason why there is the restriction $\sigma > 3/4$ in our Theorem 1 is that, in the case of $Z(s)$, we can prove the necessary mean square estimate

$$\int_0^T |Z(\sigma+it)|^2 dt = O(T) \tag{1.4}$$

only for $\sigma > 3/4$.

It is not difficult to prove (1.4) for $\sigma > 3/4$; in fact, it is sufficient to apply Potter's classical result (Theorem 3 of [15]). In this paper we will study the mean square of $Z(s)$ more closely and will obtain the following

Theorem 2 *(i) In the case of $1/2 \leq \sigma \leq 3/4$, we have*

$$\int_0^T |Z(\sigma+it)|^2 dt = O(T^{4-4\sigma}(\log T)^{1+\epsilon}) \tag{1.5}$$

for any $\epsilon > 0$.

(ii) In the case of $3/4 < \sigma \leq 1$, we have

$$\int_0^T |Z(\sigma+it)|^2 dt = T\sum_{n=1}^{\infty} c_n^2 n^{-2\sigma} + O(T^{\theta(\sigma)+\epsilon}), \tag{1.6}$$

where

$$\theta(\sigma) = \begin{cases} \frac{5}{2} - 2\sigma & \text{if } \frac{3}{4} < \sigma < \frac{12+\sqrt{19}}{20} = 0.8166\ldots, \\ \frac{60(1-\sigma)}{29-20\sigma} & \text{if } \frac{12+\sqrt{19}}{20} \leq \sigma \leq 1. \end{cases}$$

Note that the estimate (1.5) especially implies

$$\int_0^T |Z(\frac{1}{2}+it)|^2 dt = O(T^2(\log T)^{1+\epsilon}), \tag{1.7}$$

which is a slight improvement of (7.31) of Ivić [4].

From Section 2 to Section 6, we will present several approaches to the mean square problem. As a by-product, an improvement on a result of Ivić, Matsumoto and Tanigawa [5] will be shown in Section 2. The universality theorem will be proved in the final section.

The author expresses his gratitude to Professor Yoshio Tanigawa and Mr. Yuichi Kamiya for useful discussions. In particular, Mr. Kamiya kindly pointed out an inaccurate point in the original version of the manuscript.

2 Auxiliary estimates

From Deligne's estimate $|a(n)| \le n^{(\kappa-1)/2} d(n)$, where $d(n)$ is the number of positive divisors of n, it follows immediately that

$$c_n = O(n^\epsilon). \tag{2.1}$$

The asymptotic formula

$$\sum_{n\le x} c_n = A_0 x + \Delta(x;\phi) \tag{2.2}$$

with

$$\Delta(x;\phi) = O(x^{3/5}) \tag{2.3}$$

for $x>0$, where A_0 is the residue of $Z(s)$ at $s=1$, is due to Rankin [16] and Selberg [18]. Also we can show that for any $\epsilon>0$, there exists an $x_0 = x_0(\epsilon)$, for which the estimate

$$\sum_{n\le x} c_n^2 = O(x(\log x)^{1+\epsilon}) \tag{2.4}$$

holds for any $x \geq x_0$. Note that the result of Moreno-Shahidi [12] and Theorem 1 of Rankin [17] suggest that the true order of the left-hand side of (2.4) would probably be $x \log x$.

The proof of (2.4) is as follows. The function $f(n) = c_n^2$ is non-negative multiplicative, and by using Deligne's estimate it can be easily shown that $f(p^l) = O(l^6)$ for any prime power p^l. Hence we can apply Shiu's theorem [19] to $f(n) = c_n^2$, and obtain

$$\begin{aligned} \sum_{n \leq x} c_n^2 &\ll \frac{x}{\log x} \exp\left(\sum_{p \leq x} c_p^2 p^{-1} \right) \\ &= \frac{x}{\log x} \exp\left(\sum_{p \leq x} |a(p)|^4 p^{1-2\kappa} \right), \end{aligned} \tag{2.5}$$

where p denotes prime numbers. A special case of Theorem 2 of Rankin [17] asserts that

$$\sum_{p \leq x} |a(p)|^4 p^{2(1-\kappa)} = \frac{2x}{\log x}(1 + o(1)). \tag{2.6}$$

Hence by partial summation we have

$$\sum_{p \leq x} |a(p)|^4 p^{1-2\kappa} \leq (2 + \epsilon) \log \log x$$

for $x \geq x_0(\epsilon)$. Therefore (2.5) implies (2.4).

We will use (2.4) essentially in the proof of Theorem 2. But before discussing it, here we mention another application of (2.4).

The Riesz means of the coefficients of $Z(s)$ were studied by Ivić, Matsumoto and Tanigawa [5]. In particular, they proved

$$\int_1^X \Delta_1(x; \phi)^2 dx = \frac{2}{13}(2\pi)^{-4} \left(\sum_{n=1}^{\infty} c_n^2 n^{-7/4} \right) X^{13/4} + O(X^{3+\epsilon}) \tag{2.7}$$

for any $X > 1$, where $\Delta_1(x; \phi)$ is the error term of the asymptotic formula of the Riesz mean defined by

$$\sum_{n \leq x} (x - n) c_n = \frac{1}{2} A_0 x^2 + Z(0) x + \Delta_1(x; \phi).$$

Now we show the following

Theorem 3 *The error term $O(X^{3+\epsilon})$ on the right-hand side of (2.7) can be replaced by the better estimate $O(X^3(\log X)^{3+\epsilon})$.*

This improvement is small, but is of interest in view of Theorem 3 of [5]. Let M be a parameter satisfying $M \gg X^{2+\epsilon}$, and let

$$\delta_1(x; M) = (2\pi)^{-2} x^{9/8} \sum_{n \le M} c_n n^{-7/8} \sin(8\pi (nx)^{1/4} + \pi/4).$$

Then, in the proof of Theorem 2 of [5], it is shown that

$$\begin{aligned}\int_X^{2X} \delta_1(x; M)^2 dx &= \frac{2}{13}(2\pi)^{-4} \left(\sum_{n=1}^{\infty} c_n^2 n^{-7/4} \right) \left((2X)^{13/4} - X^{13/4} \right) \\ &\quad + O(X^{13/4} M^{-3/4+\epsilon}) + O(X^3 M^{\epsilon}). \qquad (2.8)\end{aligned}$$

To prove the above theorem, it is sufficient to see that the second error term on the right-hand side of (2.8) can be replaced by $O(X^3(\log M)^{3+\epsilon})$. In [5] it is just indicated that the proof of the above (2.8) is similar to that of Theorem 13.5 of Ivić [3]. Following this indication, we encounter the sum

$$S_M = \sum_{m \le M} \frac{c_m}{m^{7/8}} \sum_{m/2 \le n < m} \frac{c_n}{n^{7/8}(m^{1/4} - n^{1/4})},$$

which corresponds to the sum S_1'' in the proof of Theorem 13.5 of [3]. We have easily

$$S_M \ll \sum_{m \le M} \frac{c_m}{m} \sum_{m/2 \le n < m} \frac{c_n}{m - n}, \qquad (2.9)$$

and in [5], we simply estimate the inner sum by using (2.1) to obtain $S_M = O(M^{\epsilon})$, which gives the ϵ-factor in the second error term of (2.8). Here we put $r = m - n$ in (2.9), change the order of summation, and use the Cauchy-Schwarz inequality to get

$$S_M \ll \sum_{1 \le r \le M/2} \frac{1}{r} \left(\sum_{2r \le m \le M} \frac{c_m^2}{m} \right)^{\frac{1}{2}} \left(\sum_{2r \le m \le M} \frac{c_{m-r}^2}{m} \right)^{\frac{1}{2}}.$$

Evaluating these sums by using (2.4) and partial summation, we obtain $S_M = O((\log M)^{3+\epsilon})$ as desired. Hence the assertion of Theorem 3 follows.

3 A general mean value theorem of Perelli

Perelli [13] considered analytic properties of a general class of L-functions. Let

$$\Delta(s) = \prod_{j=1}^{N} \Gamma(\alpha_j s + \beta_j),$$

where $\Gamma(s)$ denotes the gamma function, α_j's are real and β_j's are complex. Perelli's "general L-function" is defined by the Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \qquad a_n \ll n^{\epsilon},$$

which is assumed to be continued meromorphically to the whole complex plane with at most a simple pole at $s = 1$, and satisfy a certain growth condition, an Euler product expansion, and the functional equation of the form $\Phi(s) = W\Phi^*(1-s)$. Here W is a complex number with $|W| = 1$, $\Phi(s) = Q^s\Delta(s)L(s)$ with a certain real Q, and $\Phi^*(s)$ is defined similarly with replacing $L(s)$ by some other L-function

$$L^*(s) = \sum_{n=1}^{\infty} a_n^* n^{-s}.$$

For the rigorous definition of "general L-function", see his paper [13]. Let

$$A = \sum_{j=1}^{N} \alpha_j, \qquad B = \sum_{j=1}^{N} \beta_j,$$

and put $H = 1 + \Re(B/A) - (N-1)/2A$. Perelli proved (Theorem 4 of [13]) that, if $H > 0$, then, for any $\epsilon > 0$ and $1/2 \le \sigma \le 1$, it holds that

$$\begin{aligned} \int_0^T |L(\sigma+it)|^2 dt \ \ll\ & T(QT)^{\omega(1/2)\epsilon} + (QT^A)^{2(1-\sigma)+\epsilon}\Big\{1 + Q^{-1}T^{1-A} \\ & + (1+G(Q)^2)(QT^A)^{-2H}\Big\}, \end{aligned} \tag{3.1}$$

where $G(Q)$ is a certain quantity depending on Q, and $\omega(1/2) = 1$ or 0 according as $\sigma = 1/2$ or $1/2 < \sigma \le 1$, respectively. Actually, in Perelli's statement, the term $T(QT)^{\omega(1/2)\epsilon}$ is missing. This mistake is caused by the estimate written on l.5, p.300 of [13], which is stated as

$$\int_{-T}^{T}\left|\sum_{n\le x}a_n n^{-s}\right|^2 dt \ll (QT^A)^{2(1-\sigma)+\epsilon}\left(1+\frac{1}{QT^{A-1}}\right), \tag{3.2}$$

but the term $T(QT)^{\omega(1/2)\epsilon}$ should be added to the right-hand side.

In [13] it is indicated that (3.2), and the estimate

$$\int_{-T}^{T}\left|WQ^{1-2s}\frac{\Delta(1-s)}{\Delta(s)}\sum_{n\le x}a_n^* n^{-1+s}\right|^2 dt$$
$$\ll (QT^A)^{2(1-\sigma)+\epsilon}\left(1+\frac{1}{QT^{A-1}}\right), \tag{3.3}$$

which is stated on the next line l.6, p.300 of [13], can be shown by using the Montgomery-Vaughan inequality (see (5.1) below). However, since $x = A''Qt^A$ depends on t, it is better to apply the method of the proof of Theorem 7.3 of Titchmarsh [20]. In fact, by that method we can prove

$$\int_{T}^{2T}\left|\sum_{n\le x}a_n n^{-s}\right|^2 dt$$
$$= TS + O\left((QT^A)^{2(1-\sigma)+\epsilon}\left(1+\frac{1}{QT^{A-1}}\right)\right), \tag{3.4}$$

where

$$S = \begin{cases} O((QT)^\epsilon) & \text{if } \sigma = 1/2, \\ \sum_{n=1}^{\infty}|a_n|^2 n^{-2\sigma} & \text{if } 1/2 < \sigma \le 1. \end{cases}$$

Using (3.3), (3.4) and the Cauchy-Schwarz inequality, we can easily deduce the following sharpening of Perelli's (3.1):

Proposition 1 *Let $L(s)$ be Perelli's "general L-function". Then, for $1/2 \le \sigma \le 1$ and any $\epsilon > 0$, we have*

$$\int_0^T |L(\sigma+it)|^2 dt$$

$$= TS + O\left((QT^A)^{2(1-\sigma)+\epsilon}\left(1 + \frac{1}{QT^{A-1}} + \frac{1+G(Q)^2}{(QT^A)^{2H}}\right)\right)$$

$$+O\left(T^{1/2}(QT^A)^{1-\sigma+\epsilon}\left(1 + \left(\frac{1}{QT^{A-1}}\right)^{1/2} + \frac{1+G(Q)}{(QT^A)^H}\right)\right).$$

The Rankin-Selberg L-function $Z(s)$ is an example of general L-function in the sense of Perelli, especially satisfies the functional equation

$$\Gamma(s+\kappa-1)\Gamma(s)Z(s) = (2\pi)^{4s-2}\Gamma(\kappa-s)\Gamma(1-s)Z(1-s). \tag{3.5}$$

Hence, applying Proposition 1 to $Z(\sigma+it)$, we obtain

$$\int_0^T |Z(\sigma+it)|^2 dt = O\left(T^{4-4\sigma+\epsilon}\right) \tag{3.6}$$

for $1/2 \le \sigma \le 3/4$, and

$$\int_0^T |Z(\sigma+it)|^2 dt = T\sum_{n=1}^{\infty} c_n^2 n^{-2\sigma} + O\left(T^{5/2-2\sigma+\epsilon}\right) \tag{3.7}$$

for $3/4 < \sigma \le 1$. These results imply Theorem 2 for $1/2 \le \sigma \le (12+\sqrt{19})/20$, up to ϵ-factors. Moreover, it is not difficult to replace those ϵ-factors by certain log-powers. But in any case, Perelli's argument depends on Lavrik's approximate functional equation [9], whose proof is very long and complicated. We will describe more self-contained approaches in the following sections.

4 The reflection principle

Ivić stated, as (7.31) of his paper [4], that

$$\int_T^{2T} |Z(\sigma+it)|^2 dt \ll T^{2+\epsilon} \tag{4.1}$$

for $1/2 \le \sigma \le 1$. He did not give the details of the proof, just indicated that (4.1) may be obtained by using the mean value theorem for Dirichlet polynomials and the technique of the reflection principle. Actually, following his indication, we can reduce the exponent on the right-hand side of (4.1) when $\sigma > 1/2$. Also it is possible to replace the factor T^ϵ by $(\log T)^{1+\epsilon}$, and the consequences are

$$\int_T^{2T} |Z(\sigma+it)|^2 dt = O\left(T^{4-4\sigma}(\log T)^{1+\epsilon}\right) \tag{4.2}$$

for $1/2 \le \sigma \le 3/4$ and

$$\int_T^{2T} |Z(\sigma+it)|^2 dt = O(T) \tag{4.3}$$

for $3/4 < \sigma \le 1$. In particular, (4.2) is the conclusion (1.5) of Theorem 2 in the case of $1/2 \le \sigma \le 3/4$. In this and the next section, we will present the proof of the above two results, basically following Ivić's indication.

First, in this section, we apply the reflection principle to obtain a certain approximate functional equation for $Z(s)$. The method is an analogue of the argument described in Section 4.4 of Ivić's book [3], but we show the details for the convenience of readers.

We begin with the well-known formula

$$e^{-x} = \frac{1}{2\pi i}\int_{(c)} \Gamma(z) x^{-z} dz = \frac{1}{2\pi i}\int_{(c)} \Gamma(1+z) x^{-z}\frac{dz}{z}, \tag{4.4}$$

where $c > 1$, $x > 0$, and the path of integration is the vertical line from $c - i\infty$ to $c + i\infty$. Let $h = \log^2 T$ and $Y \ge 1$. Putting $x = (n/Y)^h$, $z = w/h$ in (4.4), multiplying the both sides by $c_n n^{-s}$ and summing up, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} &\exp(-(n/Y)^h) c_n n^{-s} \\ &= \frac{1}{2\pi i}\int_{(c)} \Gamma\left(1+\frac{w}{h}\right) Y^w Z(s+w)\frac{dw}{w}\end{aligned} \tag{4.5}$$

for $0 \le \sigma \le 1$. Shifting the path of integration to $\Re w = -(1/2) - \sigma$, we find that the right-hand side of (4.5) is equal to

$$Z(s) + \Gamma\left(1+\frac{1-s}{h}\right)\frac{A_0}{1-s}Y^{1-s} + I_0,$$

where A_0 is the same as in Section 2 and

$$I_0 = \frac{1}{2\pi i}\int_{(-(1/2)-\sigma)} \Gamma\left(1+\frac{w}{h}\right) Y^w Z(s+w)\frac{dw}{w}.$$

Substituting the functional equation (3.5) into the above and using (1.1), we have

$$\begin{aligned} I_0 &= \frac{1}{2\pi i}\int_{(-(1/2)-\sigma)} F(s,w)\sum_{n\le M} c_n n^{-1+s+w} w^{-1} dw \\ &\quad + \frac{1}{2\pi i}\int_{(-(1/2)-\sigma)} F(s,w)\sum_{n>M} c_n n^{-1+s+w} w^{-1} dw \\ &= I_1 + I_2, \end{aligned} \tag{4.6}$$

say, where $M \ge 1$ and

$$F(s,w) = (2\pi)^{4(s+w)-2} Y^w \Gamma\left(1+\frac{w}{h}\right)\frac{\Gamma(\kappa-s-w)\Gamma(1-s-w)}{\Gamma(s+w+\kappa-1)\Gamma(s+w)}.$$

Assume $h^4 \le t \le 2T$ and $Y \ll T^C$ for a certain positive C. Then by using (2.1) and partial summation it follows that

$$\begin{aligned} &\sum_{n>2Y} \exp\left(-(n/Y)^h\right) c_n n^{-s} \\ &\qquad \ll hY^{-h}\int_{2Y}^{\infty} \xi^{h-\sigma+\epsilon}\exp\left(-(\xi/Y)^h\right) d\xi \\ &\qquad \ll T^{-A} \end{aligned}$$

for any large positive A. Also using Stirling's formula of the form

$$|\Gamma(x+iy)| = O\left((|y|+1)^{x-1/2}\exp(-\pi|y|/2)\right), \tag{4.7}$$

which is valid uniformly in any fixed vertical strip, we have

$$\Gamma\left(1+\frac{1-s}{h}\right)\frac{A_0}{1-s}Y^{1-s} = O(T^{-A}).$$

Summarizing the above results, we obtain

$$\sum_{n\le 2Y} \exp\left(-(n/Y)^h\right) c_n n^{-s} = Z(s) + I_1 + I_2 + O(T^{-A}). \tag{4.8}$$

Next consider I_2. Shift the path of integration of I_2 to the vertical line $\Re w = -\sigma - (h/2)$. Using $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$, we have

$$\begin{aligned} &F(s,w) = \pi^{-2}(2\pi)^{4(s+w)-2} Y^w \sin(\pi(s+w+\kappa-1))\sin(\pi(s+w)) \\ &\times\Gamma\left(1+\frac{w}{h}\right)\Gamma(\kappa-s-w)\Gamma(1-s-w)\Gamma(2-s-w-\kappa)\Gamma(1-s-w). \end{aligned}$$

We estimate this quantity on the line $\Re w = -\sigma - (h/2)$ by Stirling's formula. Apply (4.7) to the first gamma-factor, and estimate the other four gamma-factors by

$$\Gamma(z+b) = (2\pi)^{1/2} z^{z+b-1/2} e^{-z}(1 + O(|z|^{-1})), \tag{4.9}$$

which is valid uniformly in the region $|\arg z| \leq \pi - \delta$ $(\delta > 0)$, except for the neighbourhoods of $z = 0$ and poles of $\Gamma(z+b)$, where b is a constant (cf. Ivić [3], (A.33)). Put $w = -\sigma - (1/2)h + iv$, and apply (4.9) with $z = (1/2)h - i(t+v)$. We obtain

$$F(s,w) \ll Y^{-\sigma-(h/2)} \left(\left|\frac{v}{h}\right| + 1\right)^{-\sigma/h} \left(\frac{z}{2\pi e}\right)^{2h+2}$$
$$\times \exp\left(-\frac{\pi}{2}\left|\frac{v}{h}\right| + 2\pi|t+v| + 4(t+v)\arg z\right).$$

In the case of $|t+v| \leq h^2$, we have

$$F(s,w) = O(\exp(-c_1 h^3))$$

with a certain $c_1 > 0$, because $t \geq h^4$. Hence the contribution of this case to I_2 is negligible. In the case of $|t+v| > h^2$, we have

$$2\pi|t+v| + 4(t+v)\arg z = 2h + O(h^{-1}),$$

hence the contribution of this case to I_2 is

$$\ll Y^{-\sigma-(h/2)} \left(\sum_{n>M} c_n n^{-1-(h/2)}\right) \int_{|t+v|>h^2} \left(\left|\frac{v}{h}\right| + 1\right)^{-\sigma/h} \exp\left(-\frac{\pi}{2}\left|\frac{v}{h}\right|\right)$$
$$\times \left(\frac{|t+v|+h}{2\pi}\right)^{2h+2} \frac{dv}{|-\sigma-(h/2)+iv|}. \tag{4.10}$$

From (2.2) it follows by partial summation that

$$\sum_{n>M} c_n n^{-1-(h/2)} = O\left(M^{-h/2}\right).$$

Also it is easy to see that the integral on the right-hand side of (4.10) is $O(T^{2h+2})$. (Divide the integral into three parts according as $|v| \leq h$, $h < |v| \leq t$ and $|v| > t$, and estimate each part separately.) Hence we obtain

$$I_2 \ll Y^{-\sigma} T^2 (T^4/YM)^{h/2},$$

which implies $I_2 = O(T^{-A})$ under the assumption

$$YM \geq T^{4+(B/h)}, \tag{4.11}$$

where B is a constant satisfying $B \geq 2A + 4$. Combining this with (4.8), we now obtain the following approximate functional equation.

Proposition 2 *Let $T \geq 2$, $h = \log^2 T$, $1 \leq Y \ll T^C$, $M \geq 1$. Fix a large positive number A, and assume (4.11) with $B \geq 2A+4$. Then, for $0 \leq \sigma \leq 1$ and $h^4 \leq t \leq 2T$, we have*

$$Z(s) = \sum_{n \leq 2Y} \exp\left(-(n/Y)^h\right) c_n n^{-s} - I_1 + O(T^{-A}), \tag{4.12}$$

where I_1 is defined by (4.6).

5 Upper bounds for the mean square

In this section we deduce from Proposition 2 the estimates (4.2) and (4.3), by using (2.4) and the following mean value theorem of Dirichlet polynomials due to Montgomery-Vaughan [11]: It holds that

$$\int_0^T \left| \sum_{n \leq N} a_n n^{it} \right|^2 dt = T \sum_{n \leq N} |a_n|^2 + O\left(\sum_{n \leq N} n |a_n|^2 \right) \tag{5.1}$$

for any complex numbers $a_1, \ldots, a_N$.

Let $1/2 \leq \sigma \leq 1$. From Proposition 2 we have

$$\begin{aligned} \int_T^{2T} |Z(\sigma + it)|^2 dt \quad &\ll \quad \int_T^{2T} \left| \sum_{n \leq 2Y} \exp\left(-(n/Y)^h\right) c_n n^{-\sigma - it} \right|^2 dt \\ &\qquad + \int_T^{2T} |I_1|^2 dt + O(T^{1-2A}). \end{aligned} \tag{5.2}$$

To evaluate the first term on the right-hand side, we apply (5.1) with $N = 2Y$ and $a_n = \exp(-(n/Y)^h) c_n n^{-\sigma}$. By (2.4) and partial summation, we have

$$\sum_{n \leq 2Y} |a_n|^2 \leq \sum_{n \leq 2Y} c_n^2 n^{-2\sigma} \ll (\log Y)^{\omega(1/2)(2+\epsilon)},$$

where $\omega(1/2)$ is the same as in (3.1). The sum $\sum_{n\le 2Y} n|a_n|^2$ can be estimated in a similar way, and then we obtain

$$\int_T^{2T}\left|\sum_{n\le 2Y}\exp\left(-(n/Y)^h\right)c_n n^{-\sigma-it}\right|^2 dt$$
$$\ll T(\log Y)^{\omega(1/2)(2+\epsilon)}+Y^{2-2\sigma}(\log Y)^{1+\omega(1)+\epsilon}, \qquad (5.3)$$

where $\omega(1)=1$ or 0 according as $\sigma=1$ or $1/2\le\sigma<1$, respectively.

Next consider I_1. Shifting the path of integration of I_1 to $\Re w=1+\delta-\sigma$, we have

$$\begin{aligned}I_1 &= -F(s,0)\sum_{n\le M}c_n n^{-1+s}-R(1-s)\sum_{n\le M}c_n(1-s)^{-1}\\ &\quad+\frac{1}{2\pi i}\int_{(1+\delta-\sigma)}F(s,w)\sum_{n\le M}c_n n^{-1+s+w}w^{-1}dw,\end{aligned}$$

where $R(1-s)$ is the residue of $F(s,w)$ at $w=1-s$. Evaluating the factors $F(s,w)$, $F(s,0)$ and $R(1-s)$ by Stirling's formula (4.7), we have

$$\begin{aligned}I_1 &\ll t^{2-4\sigma}\left|\sum_{n\le M}c_n n^{-1+s}\right|\\ &\quad+\left(\frac{t}{h}\right)^{(1/2)+(1-\sigma)/h}\exp\left(-\frac{\pi t}{2h}\right)Y^{1-\sigma}|1-\sigma-it|^{-1}\left|\sum_{n\le M}c_n\right|\\ &\quad+\int_{-\infty}^{\infty}\left(\frac{|v|}{h}+1\right)^{(1/2)+(1+\delta-\sigma)/h}\exp\left(-\frac{\pi|v|}{2h}\right)Y^{1+\delta-\sigma}\\ &\quad\times(|t+v|+1)^{-2-4\delta}\left|\sum_{n\le M}c_n n^{\delta+i(t+v)}\right|\frac{dv}{|1+\delta-\sigma+iv|}\\ &= I_{11}+I_{12}+I_{13},\end{aligned}$$

say. We assume $M\ll T^C$ with a certain $C>0$. Then we have clearly

$$\int_T^{2T}|I_{12}|^2dt=O(T^{-A}) \qquad (5.4)$$

for any $A>0$. Applying (2.4) and partial summation, we have

$$\begin{aligned}\int_T^{2T}|I_{11}|^2dt &\ll T^{5-8\sigma}M^{2\sigma-1}(\log M)^{1+\omega(1/2)+\epsilon}\\ &\quad+T^{4-8\sigma}M^{2\sigma}(\log M)^{1+\epsilon}. \qquad (5.5)\end{aligned}$$

As for I_{13}, we further divide it as

$$I_{13} = \int_{|v|\le h} + \int_{h<|v|\le h^2} + \int_{|v|>h^2} = I_{13}^{(1)} + I_{13}^{(2)} + I_{13}^{(3)},$$

say. Then, using the Cauchy-Schwarz inequality, for $T \le t \le 2T$ we have

$$\begin{aligned} I_{13}^{(1)} &\ll Y^{1+\delta-\sigma}T^{-2-4\delta}\int_{-h}^{h}\left|\sum_{n\le M} c_n n^{\delta+i(t+v)}\right|\frac{dv}{1+|v|} \\ &\le Y^{1+\delta-\sigma}T^{-2-4\delta}\left(\int_{-h}^{h}\frac{dv}{1+|v|}\right)^{1/2}\left(\int_{-h}^{h}\left|\sum_{n\le M} c_n n^{\delta+i(t+v)}\right|^2\frac{dv}{1+|v|}\right)^{1/2} \end{aligned}$$

and

$$\begin{aligned} I_{13}^{(2)} &\ll Y^{1+\delta-\sigma}T^{-2-4\delta}\int_{h}^{h^2}\left(\frac{v}{h}\right)^{1/2} e^{-\pi v/2h}\left|\sum_{n\le M} c_n n^{\delta+i(t+v)}\right|\frac{dv}{v} \\ &\ll Y^{1+\delta-\sigma}T^{-2-4\delta}h^{-1/2}\left(\int_{h}^{h^2} e^{-\pi v/h}dv\right)^{1/2} \\ &\qquad \times\left(\int_{h}^{h^2}\left|\sum_{n\le M} c_n n^{\delta+i(t+v)}\right|^2\frac{dv}{v}\right)^{1/2}. \end{aligned}$$

Since $I_{13}^{(3)}$ is clearly small, we get

$$\begin{aligned} \int_{T}^{2T}|I_{13}|^2dt &\ll Y^{2(1+\delta-\sigma)}T^{-4-8\delta}(\log h) \\ &\quad\times \int_{-h^2}^{h^2}\int_{T}^{2T}\left|\sum_{n\le M} c_n n^{\delta+iv}n^{it}\right|^2\frac{dtdv}{1+|v|} \;+\; T^{-A}, \end{aligned}$$

which is

$$\ll Y^{2(1-\sigma)}T^{-4}(T+M)M\left(\frac{YM}{T^4}\right)^{2\delta}(\log M)^{1+\epsilon}(\log\log T)^2 \;+\; T^{-A}$$

by applying (5.1) and (2.4) again. This estimate with (5.4) and (5.5) gives an upper bound of

$$\int_{T}^{2T}|I_1|^2dt,$$

and, substituting it and (5.3) into (5.2), we obtain

$$\int_T^{2T} |Z(\sigma+it)|^2 dt \ll T(\log Y)^{\omega(1/2)(2+\epsilon)} + Y^{2-2\sigma}(\log Y)^{1+\omega(1)+\epsilon}$$
$$+T^{5-8\sigma}M^{2\sigma-1}(\log M)^{1+\omega(1/2)+\epsilon} + T^{4-8\sigma}M^{2\sigma}(\log M)^{1+\epsilon}$$
$$+Y^{2(1-\sigma)}T^{-4}(T+M)M\left(\frac{YM}{T^4}\right)^{2\delta}(\log M)^{1+\epsilon}(\log\log T)^2$$
$$+T^{-A} \tag{5.6}$$

for $1/2 \le \sigma \le 1$, where $1 \le Y \ll T^C$, $1 \le M \ll T^C$, and the condition (4.11) is required. From (5.6) with the choice $Y = M = T^{2+(B/2h)}$, the estimates (4.2) and (4.3) immediately follow.

6 Asymptotic formulas for the mean square

The purpose of this section is to prove the second half of Theorem 2. First we show that

$$\int_2^T |Z(\sigma+it)|^2 dt = T\sum_{n=1}^{\infty} c_n^2 n^{-2\sigma} + O(T^{(5/2)-2\sigma+\epsilon}) \tag{6.1}$$

holds for $3/4 < \sigma \le 1$.

This can be proved by the same method as the proof of Lemma 8.4 of Ivić [3]. The outline is as follows. The starting point is

$$\int_2^T |Z(\sigma+it)|^2 dt = \int_2^T \left|\sum_{n\le L} c_n n^{-\sigma-it}\right|^2 dt + O\left(\int_2^T |F(\sigma+it)|dt\right), \tag{6.2}$$

where $L \ge 1$ and

$$F(s) = Z^2(s) - \left(\sum_{n\le L} c_n n^{-s}\right)^2.$$

By using (5.1) and (2.4) we have that the first term on the right-hand side of (6.2) is equal to

$$T\sum_{n=1}^{\infty} c_n^2 n^{-2\sigma} + O((T+L)L^{1-2\sigma}(\log L)^{1+\omega(1)+\epsilon}). \tag{6.3}$$

As for the second term, we apply a convexity lemma (Lemma 8.3 of Ivić [3]) to obtain

$$\int_2^T |F(\sigma+it)|dt \le \left(\int_1^{2T} \left|F\left(\frac{3}{4}+it\right)\right| dt + 1\right)^{(4(1-\sigma)+4\delta)/(1+4\delta)}$$
$$\times \left(\int_1^{2T} |F(1+\delta+it)|dt + 1\right)^{(4\sigma-3)/(1+4\delta)}$$

where $\delta > 0$. The first factor on the right-hand side can be estimated by using (4.2) with $\sigma = 3/4$ and (5.1), while the second factor can be estimated by the Cauchy-Schwarz inequality and (5.1), with noticing

$$F(1+\delta+it) = \sum_{n=1}^{\infty} h(n) n^{-1-\delta-it},$$

where $h(n) = O(n^{\epsilon})$ for any positive integer n and $h(n) = 0$ for $n \le L$. The result is that

$$\int_2^T |F(\sigma+it)|dt \ll \left(T(\log T)^{1+\epsilon} + L^{1/2}(\log L)^{1+\epsilon}\right)^{(4(1-\sigma)+4\delta)/(1+4\delta)}$$
$$\times \left(T^{1/2}L^{-1/2}(T^{1/2}+L^{1/2})L^{-\delta+\epsilon}\right)^{(4\sigma-3)/(1+4\delta)}.$$

From (6.2), (6.3) and the above estimate, with the choice $L = T^2$, we obtain (6.1).

When $\sigma > 1$, using (5.1) we can easily see that

$$\int_2^T |Z(\sigma+it)|^2 dt = T\sum_{n=1}^{\infty} c_n^2 n^{-2\sigma} + O(1).$$

Therefore it is desirable to improve the error estimate in (6.1) to obtain the error term whose exponent tends to zero (up to the ϵ-factor) when $\sigma \to 1-0$. In the rest of this section we prove such a formula, that is

$$\int_T^{2T} |Z(\sigma+it)|^2 dt = T\sum_{n=1}^{\infty} c_n^2 n^{-2\sigma} + O\left(T^{60(1-\sigma)/(29-20\sigma)+\epsilon}\right) \qquad (6.4)$$

for $31/40 \le \sigma \le 1$.

Let $\sigma > 1$, $X \geq 1$, and assume $\log T \ll X^{\epsilon}$. We divide $Z(s)$ as

$$Z(s) = \sum_{n \leq X} c_n n^{-s} + \sum_{n > X} c_n n^{-s},$$

and denote the second sum by $f(s)$. By (2.2) and partial summation we have

$$\begin{aligned} f(s) &= \int_X^{\infty} \{A_0(\eta - X) + \Delta(\eta) - \Delta(X)\} s\eta^{-s-1} d\eta \\ &= \frac{A_0}{s-1} X^{1-s} - \Delta(X) X^{-s} + s \int_X^{\infty} \Delta(\eta)\eta^{-s-1} d\eta, \end{aligned} \tag{6.5}$$

where we abbreviate $\Delta(x;\phi)$ to $\Delta(x)$. In view of (2.3), this gives the analytic continuation of $f(s)$, hence $Z(s)$, to the region $\sigma > 3/5$.

Now let $3/4 < \sigma \leq 1$. We have

$$\begin{aligned} \int_T^{2T} |Z(\sigma + it)|^2 dt &= \int_T^{2T} \left| \sum_{n \leq X} c_n n^{-\sigma - it} \right|^2 dt + \int_T^{2T} |f(\sigma + it)|^2 dt \\ &+ O\left(\left(\int_T^{2T} \left| \sum_{n \leq X} c_n n^{-\sigma - it} \right|^2 dt \right)^{1/2} \left(\int_T^{2T} |f(\sigma + it)|^2 dt \right)^{1/2} \right), \end{aligned} \tag{6.6}$$

and the first term on the right-hand side is equal to

$$T \sum_{n=1}^{\infty} c_n^2 n^{-2\sigma} + O((T + X) X^{1-2\sigma} (\log X)^{1+\omega(1)+\epsilon}) \tag{6.7}$$

by (5.1) and (2.4). Also, using the expression (6.5) and the estimate (2.3), we have

$$\int_T^{2T} |f(\sigma + it)|^2 dt \ll T^{-1} X^{2-2\sigma} + T X^{6/5 - 2\sigma} + J, \tag{6.8}$$

where

$$\begin{aligned} J &= \int_T^{2T} \left| (\sigma + it) \int_X^{\infty} \Delta(\eta) \eta^{-\sigma - it - 1} d\eta \right|^2 dt \\ &= \int_X^{\infty} \int_X^{\infty} \Delta(\eta)\Delta(\xi)(\eta\xi)^{-\sigma-1} d\eta d\xi \int_T^{2T} (\sigma^2 + t^2) \left(\frac{\xi}{\eta} \right)^{it} dt. \end{aligned}$$

The innermost integral is trivially $O(T^3)$, while by integration by parts we can see that it is also estimated as $O(|\log(\xi/\eta)|^{-1}T^2)$. Hence

$$\begin{aligned} J &\ll T^3\int_X^\infty |\Delta(\eta)|\eta^{-\sigma-1}\int_{\eta-(\eta/T)}^{\eta+(\eta/T)}|\Delta(\xi)|\xi^{-\sigma-1}d\xi d\eta \\ &\quad +T^2\int_X^\infty |\Delta(\eta)|\eta^{-\sigma-1}\int_{I(\eta)}|\Delta(\xi)|\xi^{-\sigma-1}\frac{d\xi d\eta}{|\log(\xi/\eta)|} \\ &= J_1+J_2, \end{aligned}$$

say, where

$$I(\eta)=\{\xi\,|\,\xi\geq X,\,\xi\notin(\eta-(\eta/T),\eta+(\eta/T))\}.$$

Using (2.3) we have

$$\begin{aligned} J_1 &\ll T^3\int_X^\infty |\Delta(\eta)|\eta^{-\sigma-1}\int_{\eta-(\eta/T)}^{\eta+(\eta/T)}\xi^{-\sigma-2/5}d\xi d\eta \\ &\ll T^2\int_X^\infty |\Delta(\eta)|\eta^{-2\sigma-2/5}d\eta. \end{aligned}$$

The upper bound

$$\int_Y^{2Y}|\Delta(\eta)|^2 d\eta \ll Y^{2+\epsilon} \tag{6.9}$$

is contained in Theorem 5 of Ivić [4]. (An alternative proof is mentioned in the remark at the end of Section 2 of [5].) Hence

$$\begin{aligned} \int_Y^{2Y}|\Delta(\eta)|\eta^{-2\sigma-2/5}d\eta &\leq \left(\int_Y^{2Y}|\Delta(\eta)|^2 d\eta\right)^{1/2}\left(\int_Y^{2Y}\eta^{-4\sigma-4/5}d\eta\right)^{1/2} \\ &\ll Y^{-2\sigma+(11/10)+\epsilon}, \end{aligned} \tag{6.10}$$

which gives

$$J_1=O(T^2X^{-2\sigma+(11/10)+\epsilon}). \tag{6.11}$$

Consider J_2. We evaluate the inner integral by using $|\log(\xi/\eta)|\sim\eta^{-1}|\xi-\eta|$ if $\eta/T\leq|\xi-\eta|\leq\eta/2$ and $|\log(\xi/\eta)|\gg 1$ if $|\xi-\eta|>\eta/2$. Also we use (2.3) in the former case. Then the inner integral is

$$\begin{aligned}
&\ll \int\limits_{\eta/T\le|\xi-\eta|\le\eta/2} \xi^{-\sigma-2/5}\eta\frac{d\xi}{|\xi-\eta|} + \int\limits_{\substack{\xi\ge X\\ |\xi-\eta|>\eta/2}} |\Delta(\xi)|\xi^{-\sigma-1}d\xi \\
&\ll \eta^{-\sigma+3/5}\log T + \int_X^\infty |\Delta(\xi)|\xi^{-\sigma-1}d\xi,
\end{aligned}$$

and similarly to (6.10) we see that the second term on the right-hand side is $O(X^{1/2-\sigma+\epsilon})$. Hence

$$\begin{aligned}
J_2 &\ll T^2\int_X^\infty |\Delta(\eta)|\eta^{-2\sigma-2/5}\log T d\eta \\
&\quad +T^2X^{1/2-\sigma+\epsilon}\int_X^\infty |\Delta(\eta)|\eta^{-\sigma-1}d\eta.
\end{aligned}$$

These integrals can be estimated again similarly to (6.10), and the result is

$$J_2 = O(T^2X^{-2\sigma+(11/10)+\epsilon}). \tag{6.12}$$

From (6.8), (6.11) and (6.12) we get

$$\int_T^{2T} |f(\sigma+it)|^2dt \ll T^{-1}X^{2-2\sigma} + TX^{6/5-2\sigma} + T^2X^{-2\sigma+(11/10)+\epsilon}. \tag{6.13}$$

Hence, with (6.6) and (6.7), we now obtain

$$\begin{aligned}
\int_T^{2T} |Z(\sigma+it)|^2dt = T\sum_{n=1}^\infty c_n^2 n^{-2\sigma} \\
+O(X^{2-2\sigma}(\log X)^{1+\omega(1)+\epsilon} + TX^{3/5-\sigma} \\
+T^{3/2}X^{(11/20)-\sigma+\epsilon} + T^2X^{(11/10)-2\sigma+\epsilon}).
\end{aligned} \tag{6.14}$$

The best way is to choose the value of X for which the first error term is (up to ϵ-factors) dominant in the above; this requires

$$X \gg T^{\max\{30/(29-20\sigma),\, 20/9\}}.$$

If $\sigma \ge 31/40$, then $30/(29-20\sigma) \ge 20/9$, hence the best choice is

$$X = T^{30/(29-20\sigma)}.$$

Under this choice, (6.14) implies the desired result (6.4).

Since $60(1-\sigma)/(29-20\sigma) \leq 5/2 - 2\sigma$ when $\sigma \geq (12+\sqrt{19})/20$, the proof of Theorem 2 is now complete.

7 Proof of the universality

In this final section we prove Theorem 1. Our proof is analogous to the argument described in Laurinčikas [6], due originally to Bagchi [1][2], which gives an alternative proof of Voronin's universality theorem for $\zeta(s)$. We mention here that universality theorems for various zeta and L-functions have recently been shown by Bagchi's method. Among them we quote the paper [8] by Laurinčikas and the author, in which the universality theorem for the cusp form L-function

$$L(s;\phi) = \sum_{n=1}^{\infty} a(n) n^{-s}$$

attached to $\phi(z)$ has been proved.

In Bagchi's proof of the universality for $\zeta(s)$, the well-known asymptotic formula

$$\sum_{p\leq x} \frac{1}{p} = \log\log x + A_1 + O\left(\exp(-B_1\sqrt{\log x}\,)\right), \tag{7.1}$$

where p runs over primes and A_1, B_1 are certain constants, is used essentially. The corresponding formula for $L(s;\phi)$ is not known, hence we need some additional technical argument in [8]. The present case of $Z(s)$ is actually simpler, because the corresponding formula

$$\sum_{p\leq x} \frac{c_p}{p} = \log\log x + A_2 + O\left(\exp(-B_2\sqrt{\log x}\,)\right), \tag{7.2}$$

with certain constants A_2 and B_2, can be easily deduced from Perelli's result([14])

$$\sum_{n\leq x} a(n)^2 \Lambda(n) = \frac{x^\kappa}{\kappa} + O\left(x^\kappa \exp(-B_2\sqrt{\log x}\,)\right).$$

Here $\Lambda(n)$ denotes the von Mangoldt function.

Another important fact on $Z(s)$ is that it has the Euler product expansion

$$\begin{aligned} Z(s) &= \prod_p (1-\lambda_p^2 p^{-s})^{-1}(1-\lambda_p\bar{\lambda}_p p^{-s})^{-2}(1-\bar{\lambda}_p^2 p^{-s})^{-1} \\ &= \prod_p (1-\lambda_p^2 p^{-s})^{-1}(1-p^{-s})^{-2}(1-\bar{\lambda}_p^2 p^{-s})^{-1} \qquad (7.3) \end{aligned}$$

for $\sigma > 1$, where λ_p's are complex numbers satisfying

$$\lambda_p + \bar{\lambda}_p = a(p)p^{(1-\kappa)/2}, \quad |\lambda_p| = 1$$

(here $\bar{\lambda}_p$ is the complex conjugate). Since we have the facts (1.4), (7.2) and (7.3) for $Z(s)$, the whole proof of our theorem can be developed analogously to Bagchi's original argument, so it is enough to give a brief sketch of the proof.

Let $M(D_1)$ be the space of meromorphic functions on $D_1 = \{s \mid \sigma > 3/4\}$ equipped with the topology of uniform convergence on compact subsets. For any space S, we denote by $\mathcal{B}(S)$ the family of all Borel subsets of S. Define a probability measure Q_T on $(M(D_1), \mathcal{B}(M(D_1)))$ by

$$Q_T(A) = T^{-1} m\{\tau \in [0, T] \mid Z(s+i\tau) \in A\}$$

for any $A \in \mathcal{B}(M(D_1))$. Next, let $\mathbf{C}$ be the complex number field, $\gamma = \{z \in \mathbf{C} \mid |z| = 1\}$, and define

$$\Omega = \prod_p \gamma_p,$$

where p runs over all prime numbers and $\gamma_p = \gamma$ for all p. By the product topology Ω may be regarded as a compact Abelian group, hence there is the unique probability Haar measure μ on $(\Omega, \mathcal{B}(\Omega))$. For any $\omega \in \Omega$, we denote by $\omega(p)$ the projection of ω on the coordinate space γ_p. For any positive integer n, we define

$$\omega(n) = \prod_{j=1}^{r} \omega(p)^{\alpha(j)} \quad \text{if} \quad n = \prod_{j=1}^{r} p^{\alpha(j)}.$$

The series

$$Z(s,\omega) = \sum_{n=1}^{\infty} c_n \omega(n) n^{-s}$$

is convergent almost surely for $\sigma > 1/2$, hence it defines an $M(D_1)$-valued (actually holomorphic) random element. Let P be the distribution of this element, that is the probability measure on $(M(D_1), \mathcal{B}(M(D_1)))$ defined by

$$P(A) = \mu\{\omega \in \Omega \mid Z(s,\omega) \in A\}$$

for $A \in \mathcal{B}(M(D_1))$. We can apply the general limit theorem of Laurinčikas [7] to $Z(s)$, because the assumptions of his theorem are satisfied by (1.4) and (7.3). The theorem of Laurinčikas [7] implies

Proposition 3 *The measure Q_T converges weakly to P as $T \to \infty$.*

This result may be regarded as the convergence in $H(D)$, the space of holomorphic functions on $D = \{s \mid 3/4 < \sigma < 1\}$ with the topology of uniform convergence on compact subsets.

Another key fact to the proof of Theorem 1 is the following "denseness lemma". Let $a_p \in \gamma$, and define

$$\begin{aligned} f_p(s; a_p) &= -\log\left(1 - \frac{\lambda_p^2 a_p}{p^s}\right) - 2\log\left(1 - \frac{a_p}{p^s}\right) \\ &\quad - \log\left(1 - \frac{\bar{\lambda}_p^2 a_p}{p^s}\right) \end{aligned}$$

for $s \in D$. Then

Proposition 4 *The set of all convergent* (*in $H(D)$*) *series of the form*

$$\sum_p f_p(s; a_p)$$

is dense in $H(D)$.

From these two propositions and Mergelyan's approximation theorem [10], we can easily deduce the conclusion of Theorem 1 in a standard way, following the method written in Section 6.5 of Laurinčikas [6].

Therefore the only remaining task is to prove Proposition 4. Now we outline the proof. The details, which are similar to the proof of Lemma 6.5.4 of Laurinčikas [6], are omitted.

First, using Lemma 6.5.3 of [6], we find a sequence $\{\hat{a}_p\}$, $\hat{a}_p \in \gamma$, such that

$$\sum_{p>p_0} \hat{a}_p f_p(s;1)$$

(where $p_0 > 0$ is fixed) converges in $H(D)$. Put $g_p(s) = \hat{a}_p f_p(s;1)$. The most essential part of the proof of Proposition 4 is the claim that the set $\mathcal{S}$, which consists of all convergent (in $H(D)$) series of the form

$$\sum_{p>p_0} a_p g_p(s), \qquad a_p \in \gamma,$$

is dense in $H(D)$. This claim is proved by applying the following general denseness result (Theorem 6.3.10 of Laurinčikas [6], originally due to Bagchi [1]).

Proposition 5 *Let $\{f_m\}$ be a sequence in $H(D)$ which satisfies*

(a) if μ is a complex Borel measure on $(\mathbf{C}, \mathcal{B}(\mathbf{C}))$ with compact support contained in D such that

$$\sum_{m=1}^{\infty} \left| \int_{\mathbf{C}} f_m(s) d\mu(s) \right| < \infty,$$

then

$$\int_{\mathbf{C}} s^r d\mu(s) = 0$$

for any non-negative integer r;

(b) the series $\sum_{m=1}^{\infty} f_m$ converges in $H(D)$;

(c) for any compact subset K of D,

$$\sum_{m=1}^{\infty} \sup_{s \in K} |f_m(s)|^2 < \infty.$$

Then the set of all convergent series of the form $\sum_{m=1}^{\infty} a_m f_m$, $|a_m| = 1$, is dense in $H(D)$.

We apply this proposition with $f_m = g_p$. The assumptions (b) and (c) are clearly satisfied. In order to check (a), it is enough to prove that if

$$\sum_p \left| \int_{\mathbf{C}} g_p(s) d\mu(s) \right| < \infty, \tag{7.4}$$

then

$$\rho(z) = \int_{\mathbf{C}} e^{-sz} d\mu(s)$$

is identically equal to zero. Assume the contrary. Then from Lemma 6.4.10 of [6] we have

$$\limsup_{r \to \infty} \frac{\log |\rho(r)|}{r} > -1. \tag{7.5}$$

Then, similarly to Theorem 6.4.14 of [6], we can show

$$\sum_p c_p |\rho(\log p)| = \infty. \tag{7.6}$$

In the course of the proof of (7.6), we should evaluate the sum $\sum p^{-1} c_p$, running over all primes satisfying

$$(m - 1/4)\beta < \log p \le (m + 1/4)\beta,$$

where m is a positive integer and $\beta > 0$, fixed. This can be achieved by using (7.2), and the result is

$$\sum p^{-1} c_p = \frac{1}{2m} + O\left(\frac{1}{m^2}\right).$$

From this and Theorem 6.4.12 of [6], we can get (7.6). However from the assumption (7.4) it can be easily seen that

$$\sum_p c_p |\rho(\log p)| < \infty,$$

which is a contradiction. Hence $\rho(z) \equiv 0$, and the assumption (a) is verified. Our claim on the denseness of $\mathcal{S}$ now follows from Proposition 5.

Finally, from this claim, with a suitable choice of p_0, we can deduce the assertion of Proposition 4. This completes the proof of Theorem 1.

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