Recent Developments in the Mean Square Theory of the Riemann Zeta and Other Zeta-Functions

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The purpose of the present article is to survey some mean value results obtained recently in zeta-function theory. We do not mention other important aspects of the theory of zeta-functions, such as the distribution of zeros, value-distribution, and applications to number theory. Some of them are probably treated in the articles of Professor Apostol and Professor Ramachandra in the present volume.

Even in the mean value theory, we do not discuss many important recent topics. Those include: Recent progress in the theory of large values and fractional moments made by Heath-Brown [55] and the Indian school (Ramachandra, Balasubramanian, Sankaranarayanan and others, see Ramachandra [172]); mean values taken at the zeros or at the points near the zeros (Gonek [30] [31], Fujii [23]-[26] and others); the mean square of the product of the zeta-function and a Dirichlet polynomial (see Conrey-Ghosh-Gonek [19] and the papers quoted there). All of these three topics are closely connected with the distribution of zeros of zeta-functions, hence the full account of them would require too many pages. We will only discuss the theory of Titchmarsh series very briefly in Section 7. In the fourth power moment theory there have been remarkable developments which may be characterized by the use of the spectral theory of Maass wave forms. We mention this theory occasionally, but only in connection with the mean square problems. For the full details of this theory, see Chapters 4 and 5 of Ivić [68], Motohashi's book [155], and Jutila's series of papers.

In the present article we only discuss the mean square theory of zeta-functions. This is a rather restricted topic, but still it is impossible to mention all the relevant results because the recent progress in this area is very big. The main tools appearing in this article are the approximate functional equations and Atkinson methods, emphasis are laid on the latter. The readers will find, however, that these two tools are not irrelevant (see Sections 4 and 6). Efforts are made to explain the mutual connections among various methods and results.

In Section 1, we summarize the results on the mean square $I_\sigma(T)$ of the Riemann zeta-function, obtained by applying various approximate functional equations. In Sections 3 and 5, $I_\sigma(T)$ is studied ¿from the viewpoint of the method of Atkinson. The background of Atkinson's method is the divisor problem, which is mentioned in Sections 2 and 6. Then, after a brief discussion on some short interval results in Section 7, we proceed to survey the results on more general zeta and L-functions. Sections 8, 9, 10 and 11 are devoted, respectively, to the mean square theory of Dedekind zeta-functions, L-functions attached to cusp forms, Dirichlet L-functions, and Hurwitz zeta and other related zeta-functions.

Throughout this article, $s = \sigma + it$ is a complex variable, $\zeta(s)$ the Riemann zeta-function, $\Gamma(s)$ the gamma-function, γ the Euler constant, $\phi(n)$ the Euler function, $d(n)$ the number of positive divisors of n, and $\sigma_a(n) = \sum_{0<d|n} d^a$. When x tends to infinity, $f(x) \sim g(x)$ means $\lim_{x\to\infty} f(x)/g(x) = 1$, $f(x) = O(g(x))$ or $f(x) \ll g(x)$ means $|f(x)| \leq Cg(x)$ with a certain $C > 0$, $f(x) = \Omega_+(g(x))$ (resp. $f(x) = \Omega_-(g(x))$) means that $f(x_n) > Cg(x_n)$ (resp.$f(x_n) < -Cg(x_n)$) holds for infinitely many x_n such that $x_n \to \infty$, with a certain $C > 0$, and $f(x) = \Omega(g(x))$ means that $|f(x)| = \Omega_+(g(x))$. The letter ε denotes an arbitrarily small positive number, C, C_1, C_2, $\cdots$ denote certain constants, which are not necessarily the same at each occurrence. The references are by no means complete.

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1 The approximate functional equations

A classical problem in the mean value theory of $\zeta(s)$ is to search for the asymptotic formula of the mean square

$$I_\sigma(T) = \int_0^T \left|\zeta(\sigma + it)\right|^2 dt,$$

where $T \geq 2$. (If $\sigma = 1$, we replace the interval of integration by $[1, T]$.) In view of the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$, where

$$\chi(s) = 2(2\pi)^{s-1} \sin\left(\frac{1}{2}\pi s\right)\Gamma(1-s),$$

we may restrict our consideration to the case $\sigma \geq 1/2$. When $\sigma > 1$, the asymptotic formula

$$I_\sigma(T) \sim \zeta(2\sigma)T \tag{1.1}$$

is an easy consequence of the definition of $\zeta(s)$. It was proved by Landau [127, §228, p.816] and Schnee [180] that (1.1) holds for any $\sigma > 1/2$. To prove this fact, the simple approximate formula

$$\zeta(s) = \sum_{n\leq\xi} n^{-s} - \frac{\xi^{1-s}}{1-s} + O(\xi^{-\sigma}) \qquad (|t| \leq \pi\xi) \tag{1.2}$$

is enough (see Titchmarsh [190, Theorem 7.2]). The most difficult case $\sigma = 1/2$ was settled in 1918 by Hardy-Littlewood [47],who proved

$$I_{\frac{1}{2}}(T) \sim T\log T, \tag{1.3}$$

by using the Mellin transform. Five years later, Hardy-Littlewood [49] gave an alternative proof of (1.3). It is based on the approximate functional equation

$$\zeta(s) = \sum_{n\leq\xi} n^{-s} + \chi(s)\sum_{n\leq\eta} n^{s-1} + R_1(s;\xi,\eta), \tag{1.4}$$

which is a refinement of (1.2). Here ξ, η are positive, $2\pi\xi\eta = t$, and

$$R_1(s;\xi,\eta) = O\left(\xi^{-\sigma} + \eta^{\sigma-1} t^{\frac{1}{2}-\sigma}\right). \tag{1.5}$$

The formula (1.4) first appeared in Hardy-Littlewood [48], with a slightly weaker error estimate than (1.5), and the main instrument of the proof in [48] is the Poisson summation formula

$$\sum_{a\le n\le b}{}' f(n) = \int_a^b f(u)du + 2\sum_{n=1}^{\infty}\int_a^b f(u)\cos(2\pi nu)du \tag{1.6}$$

(for $f \in C^1[a,b]$; the symbol $\sum'$ indicates that $\frac{1}{2}f(a)$ and $\frac{1}{2}f(b)$ are to be taken instead of $f(a)$ and $f(b)$, respectively). In [49], Hardy-Littlewood presented an alternative complex-analytic proof of (1.4) and (1.5).

The formula (1.4) and its relatives are really useful, and dominated the next sixty years of the mean value theory. Littlewood [134] announced that

$$I_{\frac{1}{2}}(T) = T\log T - (1 + \log 2\pi - 2\gamma)T + E(T) \tag{1.7}$$

with $E(T) = O(T^{3/4+\varepsilon})$ (actually the term 2γ is missing in [134]). Ingham [63] improved it to $E(T) = O(T^{1/2}\log T)$. Further improvement was done by Titchmarsh [188], who proved

$$E(T) = O(T^{\frac{5}{12}}\log^2 T). \tag{1.8}$$

Titchmarsh succeeded because he could use the Riemann-Siegel formula, proved by Siegel [182], which gives the very precise asymptotic expansion of $R_1(s; \sqrt{t/2\pi}, \sqrt{t/2\pi})$.

Hardy-Littlewood [49] also studied the fourth power moment

$$I_{2,\sigma}(T) = \int_0^T \left|\zeta(\sigma+it)\right|^4 dt,$$

and they showed

$$I_{2,\sigma}(T) \sim \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)}T \qquad (\frac{1}{2} < \sigma < 1) \tag{1.9}$$

and

$$I_{2,\frac{1}{2}}(T) = O(T\log^4 T). \tag{1.10}$$

Ingham [63] improved (1.10) to

$$I_{2,\frac{1}{2}}(T) = (2\pi^2)^{-1}T\log^4 T + O(T\log^3 T), \tag{1.11}$$

by using the approximate functional equation of $\zeta^2(s)$ due to Hardy-Littlewood [50], that is

$$\zeta^2(s) = \sum_{n\le x}\frac{d(n)}{n^s} + \chi^2(s)\sum_{n\le y}\frac{d(n)}{n^{1-s}} + R_2(s;x,y) \tag{1.12}$$

for positive x, y with $4\pi^2 xy = t^2$, where

$$R_2(s;x,y) = O\left(x^{\frac{1}{2}-\sigma}\left(\frac{x+y}{t}\right)^{\frac{1}{4}}\log t\right). \tag{1.13}$$

In [189], Titchmarsh gave a different proof of (1.12) with

$$R_2(s;x,y) = O(x^{\frac{1}{2}-\sigma}\log t). \tag{1.14}$$

Recall that the original proof of (1.4) by Hardy-Littlewood is based on (1.6). Hardy-Littlewood deduced

$$\zeta(s) = \sum_{n\le x} n^{-s} - \frac{x^{1-s}}{1-s} + 2\sum_{n=1}^{\infty}\int_x^{\infty} u^{-s}\cos(2\pi nu)du + O(x^{-\sigma}) \tag{1.15}$$

¿from (1.6) in the first stage of their argument. As an analogue of (1.15), Titchmarsh [189] proved

$$\begin{aligned}\zeta^2(s) &= \sum_{n\le x}\frac{d(n)}{n^s} - x^{-s}\sum_{n\le x} d(n) + \frac{2s-s^2}{(s-1)^2}x^{1-s} + \frac{s}{s-1}x^{1-s}(2\gamma+\log x) + \frac{1}{4}x^{-s}\\ &\quad -2^{4s}\pi^{2s-2}s\sum_{n=1}^{\infty}\frac{d(n)}{n^{1-s}}\int_{4\pi\sqrt{nx}}^{\infty}\frac{K_1(u)+\frac{\pi}{2}Y_1(u)}{u^{2s}}du,\end{aligned} \tag{1.16}$$

where K_1 and Y_1 are Bessel functions. This is the basis of Titchmarsh's proof of (1.14).

A climax of applications of approximate functional equations to the mean value problems came in the late 1970s, with the works of Balasubramanian, Good and Heath-Brown. A very careful analysis based on the Riemann-Siegel formula enabled Balasubramanian [4] to obtain the explicit formula

$$E(T) = 2\sum\sum_{\substack{m,n\le K\\ m\ne n}}\frac{\sin\left(T\log\left(\frac{n}{m}\right)\right)}{(mn)^{\frac{1}{2}}\log\left(\frac{n}{m}\right)} + 2\sum\sum_{\substack{m,n\le K\\ m\ne n}}\frac{\sin(2\theta_1 - T\log mn)}{(mn)^{\frac{1}{2}}(2\theta_1' - \log mn)} + O(\log^2 T), \tag{1.17}$$

where $\theta_1 = \theta_1(T) = \frac{1}{2}T\log(T/2\pi) - \frac{1}{2}T - \frac{1}{8}\pi$, $\theta_1{}'$ is the derivative of θ_1, and $K = [(T/2\pi)^{1/2}]$. Then, applying his own idea of multiple integration process to (1.17), Balasubramanian obtained the estimate of the form

$$E(T) = O(T^{\alpha+\varepsilon}) \tag{1.18}$$

with a certain $\alpha < 1/3$. (In [4], the value $\alpha = 27/82$ was given.) Good [34] proved an explicit formula of $E(T)$ similar to (1.17) but with certain smoothing factors, and from this formula he [35] proved

$$E(T) = \Omega\left(T^{\frac{1}{4}}\right). \tag{1.19}$$

Heath-Brown's work [53] gives an improvement on (1.11). He proved a new type of approximate functional equation, from which he deduced

$$I_{2,\frac{1}{2}}(T) = T\sum_{j=0}^{4} a_j\log^j T + E_2(T),$$

where a_j's are constants, $a_4 = (2\pi^2)^{-1}$, and

$$E_2(T) = O\left(T^{\frac{7}{8}+\varepsilon}\right). \tag{1.20}$$

Heath-Brown's paper also includes an alternative proof of (1.18).

Inspired by these papers, strong interests in the mean value problems revived. Really big progress has been made since 1980, which is the main theme of the present article. But first, we will discuss a closely related problem concerning the behaviour in mean of the divisor function in the next section.

2 The Dirichlet divisor problem

The title of this section means the problem of evaluating the error term $\Delta(x)$ defined by

$$\sum_{n\le x}{}' d(n) = x\log x + (2\gamma - 1)x + \frac{1}{4} + \Delta(x) \tag{2.1}$$

for $x \ge 2$, where $\sum'$ indicates that the last term is to be halved if x is an integer. As can be observed by comparing (2.1) with (1.7), there is a strong analogy between $\Delta(x)$ and $E(T)$. Usually the study of $\Delta(x)$ is easier than that of $E(T)$, hence the results on $\Delta(x)$ are quite suggestive of guessing the behaviour of $E(T)$. Here we quote several known facts on $\Delta(x)$.

Dirichlet himself proved $\Delta(x) = O(x^{1/2})$, and Voronoï [195] improved it to obtain

$$\Delta(x) = O\left(x^{\frac{1}{3}}\log x\right). \tag{2.2}$$

The explicit formula

$$\begin{aligned} \Delta(x) &= -\frac{2}{\pi}x^{\frac{1}{2}}\sum_{n=1}^{\infty}\frac{d(n)}{n^{\frac{1}{2}}}\left\{K_1(4\pi\sqrt{nx}) + \frac{\pi}{2}Y_1(4\pi\sqrt{nx})\right\} \\ &= \frac{x^{\frac{1}{4}}}{\pi\sqrt{2}}\sum_{n=1}^{\infty}d(n)n^{-\frac{3}{4}}\cos\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right) \\ &\quad -\frac{3}{32\sqrt{2}\pi^2}x^{-\frac{1}{4}}\sum_{n=1}^{\infty}d(n)n^{-\frac{5}{4}}\sin\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right) + O\left(x^{-\frac{3}{4}}\right) \end{aligned} \tag{2.3}$$

is due to Voronoï [196]. Sometimes the truncated form

$$\Delta(x) = \frac{x^{\frac{1}{4}}}{\pi\sqrt{2}}\sum_{n\le N}d(n)n^{-\frac{3}{4}}\cos\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right) + O\left(x^{\varepsilon} + x^{\frac{1}{2}+\varepsilon}N^{-\frac{1}{2}}\right) \tag{2.4}$$

is useful. For instance, the formula

$$\int_2^X \Delta^2(x)dx = \frac{\zeta^4(\frac{3}{2})}{6\pi^2\zeta(3)}X^{\frac{3}{2}} + \delta(X) \tag{2.5}$$

with $\delta(X) = O(X^{5/4+\varepsilon})$, due originally to Cramér [21], can be proved by substituting (2.4) into the left-hand side and squaring them out. Tong [191] obtained the improved estimate $\delta(X) =$

$O(X\log^5 X)$, and an alternative simple proof was given by Meurman [142]. The best estimate at present is

$$\delta(X) = O(X\log^4 X) \tag{2.6}$$

due to Preissmann [164].

As for the real order of $\delta(X)$, the author (1992) conjectured (cf.[115]) that $\delta(X) \sim CX(\log X)^B$ with a certain $B > 0$. Lau-Tsang [128] proved

$$\int_2^X \delta(x)dx = -\frac{1}{8\pi^2}X^2\log^2 X + C_1X^2\log X + O(X^2), \tag{2.7}$$

which implies $\delta(X) = \Omega_-(X\log^2 X)$, and conjectured

$$\delta(X) = -\frac{1}{4\pi^2}X\log^2 X + C_2X\log X + O(X). \tag{2.8}$$

Moreover, Tsang [193] proved that (2.8) is valid for almost all X in a certain mean value sense. A generalization of the result of Lau-Tsang [128] was recently obtained by Furuya [27].

Another mean value formula for $\Delta(x)$ is

$$\begin{aligned}\int_2^X \Delta(x)dx &= \frac{1}{2\sqrt{2}\pi^2}X^{\frac{3}{4}}\sum_{n=1}^{\infty} d(n)n^{-\frac{5}{4}}\sin\left(4\pi\sqrt{nx}-\frac{\pi}{4}\right)\\ &\quad+\frac{15}{64\sqrt{2}\pi^3}X^{\frac{1}{4}}\sum_{n=1}^{\infty} d(n)n^{-\frac{7}{4}}\cos\left(4\pi\sqrt{nx}-\frac{\pi}{4}\right)+O(1),\end{aligned} \tag{2.9}$$

which is due to Voronoï. Recently the mean value of the above quantity was studied in detail by Furuya-Tanigawa [28]. In several references (2.9) was quoted incorrectly. Note that sometimes $\Delta(x)$ is defined by (2.1) without the term 1/4; then the term $\frac{1}{4}X$ should be added on the right-hand side of (2.9).

The formula (2.5) includes the fact $\Delta(x) = \Omega(x^{1/4})$, and furthermore, it is known that

$$\Delta(x) = \Omega_-\left\{x^{\frac{1}{4}}\exp\left(C(\log\log x)^{\frac{1}{4}}(\log\log\log x)^{-\frac{3}{4}}\right)\right\} \tag{2.10}$$

(Corrádi-Kátai [20]) and

$$\Delta(x) = \Omega_+\left\{(x\log x)^{\frac{1}{4}}(\log\log x)^{\frac{3+\log 4}{4}}\exp\left(-C\sqrt{\log\log\log x}\right)\right\} \tag{2.11}$$

(Hafner [45]). In view of these Ω-results, it is quite plausible that

$$\Delta(x) = O(x^{\frac{1}{4}+\varepsilon}). \tag{2.12}$$

This is indeed a classical conjecture, but is believed to be extremely difficult. At present, the best known upper-bound is

$$\Delta(x) = O\left(x^{\frac{23}{73}+\varepsilon}\right) \tag{2.13}$$

due to Huxley [59]. This is just a small improvement on (2.2), but such a kind of improvement requires quite hard analysis on exponential sums. That is, we should use the theory of exponent pairs, created by van der Corput, and refined by many authors. For instance, Kolesnik [120] proved $\Delta(x) = O(x^{35/108+\varepsilon})$ by using his own elaborated version of the theory of exponent pairs, and later he [121] improved the exponent to $139/429 + \varepsilon$.

Bombieri-Iwaniec [13] [14] invented a new method of treating exponential sums, which gives an essentially new exponent pairs (see Huxley-Watt [62]). Combining this method with the expression

$$\Delta(x) = -2 \sum_{n \le \sqrt{x}} \psi\left(\frac{x}{n}\right) + O(1), \tag{2.14}$$

where $\psi(x) = x - [x] - \frac{1}{2}$, Iwaniec-Mozzochi [80] obtained the estimate $\Delta(x) = O(x^{7/22+\varepsilon})$. Huxley achieved to prove (2.13) by a further refinement of the method of Bombieri-Iwaniec. For the details of the theory of exponent pairs, the readers are referred to Graham-Kolesnik [39] or Huxley [61].

3 The Atkinson formula and the recent results on $E(T)$

As we mentioned in the previous section, there is an analogy between $\Delta(x)$ and $E(T)$. Therefore it is natural to search a formula analogous to Voronoï's (2.3) or (2.4). This was carried out in 1949 by Atkinson [3]. To state his result, we prepare several notations. Let $X \asymp T$ (i.e. $T \ll X \ll T$), $\operatorname{arsinh} x = \log\left(x + \sqrt{1+x^2}\right)$, and define

$$\begin{aligned} e(T,n) &= \left(1 + \frac{\pi n}{2T}\right)^{-\frac{1}{4}} \left(\frac{2T}{\pi n}\right)^{-\frac{1}{2}} \left(\operatorname{arsinh} \sqrt{\frac{\pi n}{2T}}\right)^{-1}, \\ f(T,n) &= 2T \operatorname{arsinh} \sqrt{\frac{\pi n}{2T}} + (\pi^2 n^2 + 2\pi n T)^{\frac{1}{2}} - \frac{\pi}{4}, \\ g(T,n) &= T \log\left(\frac{T}{2\pi n}\right) - T + \frac{\pi}{4}, \\ B(T,\xi) &= \frac{T}{2\pi} + \frac{1}{2}\xi^2 - \xi\left(\frac{T}{2\pi} + \frac{1}{4}\xi^2\right)^{\frac{1}{2}}, \\ \textstyle\sum_{1,\sigma}(T,X) &= \sqrt{2}\left(\frac{T}{2\pi}\right)^{\frac{3}{4}-\sigma} \sum_{n \le X} (-1)^n \sigma_{1-2\sigma}(n) n^{\sigma - \frac{5}{4}} e(T,n) \cos\left(f(T,n)\right) \end{aligned}$$

and

$$\textstyle\sum_{2,\sigma}(T,X) = 2\left(\frac{T}{2\pi}\right)^{\frac{1}{2}-\sigma} \displaystyle\sum_{n \le B(T,\sqrt{X})} \sigma_{1-2\sigma}(n) n^{\sigma-1} \left(\log \frac{T}{2\pi n}\right)^{-1} \cos\left(g(T,n)\right).$$

Then Atkinson's explicit formula can be stated as

$$E(T) = \textstyle\sum_{1,\frac{1}{2}}(T,X) - \sum_{2,\frac{1}{2}}(T,X) + O(\log^2 T). \tag{3.1}$$

The starting point of Atkinson's proof of (3.1) is the product $\zeta(u)\zeta(v)$, where u and v are independent complex variables. At first assume $\operatorname{Re} u > 1$, $\operatorname{Re} v > 1$. Then

$$\zeta(u)\zeta(v) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{-u} n^{-v}.$$

Divide this double sum into three parts according to the conditions $m = n$, $m > n$ and $m < n$ (Atkinson's dissection). The part corresponding to $m = n$ is clearly $\zeta(u+v)$. By using the Poisson summation formula (1.6), Atkinson showed an integral expression of the remaining parts, which enables the analytic continuation. Then he transformed this expression by applying Voronoï's formula (2.3), and evaluated the resulting integrals by his own saddle-point lemma, which gives an asymptotic formula for the integral of the type

$$\int_a^b g(x) \exp\left(2\pi i\Big(f(x) + kx\Big)\right) dx \tag{3.2}$$

with real k and certain functions $f(x)$ and $g(x)$. The details of the proof are rather long and complicated.

When Atkinson published (3.1), no one noticed its usefulness. After the disregard during about thirty years, Heath-Brown first gave attention to Atkinson's paper. In [51], Heath-Brown proved

$$\int_2^T E^2(t)dt = \frac{2\zeta^4(\frac{3}{2})}{3(2\pi)^{\frac{1}{2}}\zeta(3)} T^{\frac{3}{2}} + F(T) \tag{3.3}$$

with

$$F(T) = O\left(T^{\frac{5}{4}} \log^2 T\right), \tag{3.4}$$

as an application of (3.1). This is the analogy of (2.5), and implies (1.19) of Good. Heath-Brown's another paper [52] deduced the estimate

$$\int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{12} dt = O(T^2 \log^{17} T) \tag{3.5}$$

¿from a certain estimate of the mean square of $|\zeta(1/2 + it)|$ in short intervals. Atkinson's formula (3.1) was used to prove the latter estimate. (See also Chapter 7 of Ivić [66] for a different proof.)

These works of Heath-Brown showed the fruitfulness of Atkinson's formula (3.1), but it was Jutila [81I] who noticed the real value lying in Atkinson's method. He sketched in [81I] how easily can (1.18) be obtained from (3.1). Following Jutila's idea, and combining with Kolesnik's technique [120], Ivić described a proof of

$$E(T) = O\left(T^{\frac{35}{108}+\varepsilon}\right) \tag{3.6}$$

in Section 15.5 of [66].

The main theme of Jutila's aforementioned paper [81I] is the analogy between $E(T)$ and a modification of $\Delta(x)$, that is

$$\Delta^*(x) = -\Delta(x) + 2\Delta(2x) - \frac{1}{2}\Delta(4x).$$

A consequence of his analysis is the hypothetical bound

$$E(T) = O\left(T^{\frac{5}{16}+\varepsilon}\right) \tag{3.7}$$

under the assumption of the conjecture (2.12). The exponent 5/16 in (3.7) was later improved to 3/10 by Jutila [81II]. In [82], Jutila proved

$$\int_2^T \left(E(t) - 2\pi\Delta^*\left(\frac{t}{2\pi}\right)\right)^2 dt = O\left(T^{\frac{4}{3}}\log^3 T\right). \tag{3.8}$$

The transformation method for Dirichlet polynomials was created by Jutila [83]. The basic tool of this method is Voronoï's summation formula

$$\begin{aligned} \sum_{a\le n\le b}{}' d(n)f(n) &= \int_a^b f(u)(\log u + 2\gamma)du \\ &\quad + \sum_{n=1}^{\infty} d(n)\int_a^b f(u)\Big(4K_0\left(4\pi\sqrt{nu}\right) - 2\pi Y_0\left(4\pi\sqrt{nu}\right)\Big)du, \end{aligned} \tag{3.9}$$

valid for $f \in C^2[a,b]$, where K_0 and Y_0 are Bessel functions. Jutila's idea is to transform the Dirichlet polynomial

$$S(M_1, M_2; t) = \sum_{M_1\le m\le M_2} d(m)m^{-\frac{1}{2}-it} \tag{3.10}$$

by applying (3.9), and then use a lemma of Atkinson's type to evaluate the resulting expressions. One of his results is an explicit formula for $|\zeta(1/2+it)|^2$, whose shape is similar to Atkinson's formula. Several new ideas are included in his argument. One of them is the device of multiplying the original polynomial (3.10) by trivial factors $e^{2\pi irm}$, where r is an integer. Another novelty is the multiple-averaged version of Atkinson's saddle-point lemma, which gives an asymptotic formula for the integral of the form

$$U^{-J}\int_0^U du_1 \cdots \int_0^U du_J \int_{a+u_1+\cdots+u_J}^{b-u_1-\cdots-u_J} g(x)\exp\left(2\pi i\Big(f(x)+kx\Big)\right)dx \tag{3.11}$$

instead of (3.2). This point was fully developed in Jutila [87]. We will encounter the transformation method again in Sections 7 and 9.

By using the above averaged saddle-point lemma of Jutila, Meurman [142] improved (3.4) to

$$F(T) = O(T\log^5 T). \tag{3.12}$$

In fact, Meurman proved an averaged version of Atkinson's formula, which can be stated as

$$E(T) = {\textstyle\sum}^*_{1,\frac{1}{2}}(T) - {\textstyle\sum}^*_{2,\frac{1}{2}}(T) + \pi + O(T^{-\frac{1}{4}}\log T), \tag{3.13}$$

where $\sum^*_{j,\frac{1}{2}}(T)$ is a certain weighted sum similar to $\sum_{j,\frac{1}{2}}(T,X)$ $(j=1,2)$. The deduction of (3.12) from (3.13) is basically analogous to the argument of Heath-Brown [51].

The estimate (3.12) was proved also by Motohashi [149IV] [150] independently. From his asymptotic formula for $R_2(s;t/2\pi)$ (see the next section), Motohashi deduced another version of Atkinson's formula, and from which he otained (3.12). Motohashi's argument includes an alternative proof of the original formula (3.1) with a slightly better error term $O(\log T)$. Note that another different proof of (3.1) was obtained by Jutila [92]. His argument is based on the Laplace transform of $|\zeta(1/2+it)|^2$, and does not appeal to Atkinson's dissection device.

Inspired by Preissmann's proof of (2.6) (which is an application of the inequality (8.12) below due to Montgomery-Vaughan), Preissmann himself [165] and Ivić [68, (2.100)] independently of each other proved

$$F(T) = O(T \log^4 T), \tag{3.14}$$

which is the best at present. There is a conjecture of the author that $F(T) \sim CT(\log T)^B$ would hold with a certain $B > 0$. Probably $B = 2$.

Higher power moments of $E(t)$ were first studied by Ivić [64], who proved

$$\int_2^T \left|E(t)\right|^\alpha dt = O(T^{1+\frac{\alpha}{4}+\varepsilon}) \qquad \left(0 \le \alpha \le \frac{35}{4}\right) \tag{3.15}$$

and

$$\int_2^T \left|E(t)\right|^\alpha dt = O(T^{\frac{38+35\alpha}{108}+\varepsilon}) \qquad \left(\alpha \ge \frac{35}{4}\right). \tag{3.16}$$

Heath-Brown [56] proved the existence of the limit

$$\lim_{T\to\infty} T^{-1-\frac{\alpha}{4}} \int_2^T \left|E(t)\right|^\alpha dt \tag{3.17}$$

for $0 \le \alpha < 28/3$. Tsang [192] obtained

$$\int_2^T E(t)^k dt = C(k) T^{1+\frac{k}{4}} + O\left(T^{1+\frac{k}{4}-\delta}\right) \tag{3.18}$$

for $k = 3$ or 4, where $C(k)$ is an explicitly written positive constant, and $\delta > 0$. Recently Ivić [73] showed, using (3.8), that one can take $\delta = 1/14$ for $k = 3$ and $\delta = 1/23$ for $k = 4$.

In view of the Ω-result (1.19), it is plausible that

$$E(T) = O\left(T^{\frac{1}{4}+\varepsilon}\right), \tag{3.19}$$

as an analogue of (2.12). The above results on higher power moments can also be regarded as supporting facts of this conjecture. The best known upper-bound is, however, still far from this conjecture. Heath-Brown and Huxley [57] applied the methods of Bombieri, Iwaniec and Mozzochi (mentioned in Section 2) and some lemmas proved in [53] to obtain $E(T) = O\left(T^{7/22}(\log T)^{111/22}\right)$, which is better than (3.6), and this was further improved to

$$E(T) = O\left(T^{\frac{72}{227}}(\log T)^{\frac{679}{227}}\right) \tag{3.20}$$

by Huxley [60].

The Ω-result (1.19) was refined by Hafner-Ivić [46], who showed, analogously to (2.10) and (2.11), that

$$E(T) = \Omega_-\left\{T^{\frac{1}{4}} \exp\left(C(\log\log T)^{\frac{1}{4}}(\log\log\log T)^{-\frac{3}{4}}\right)\right\} \tag{3.21}$$

and

$$E(T) = \Omega_+\left\{(T\log T)^{\frac{1}{4}}(\log\log T)^{\frac{3+\log 4}{4}} \exp\left(-C\sqrt{\log\log\log T}\right)\right\}. \tag{3.22}$$

The local behaviour of sign-changes of $E(T)$ was studied by Ivić [67], Ivić-te Riele [78], and (independently) Heath-Brown and Tsang [58]. Ivić [67] showed that there exist positive constants C_1 and C_2, such that every interval $[T, T + C_1\sqrt{T}]$ (for $T \geq T_0$) contains numbers t_1, t_2 for which

$$E(t_1) > C_2 t_1^{1/4}, \qquad E(t_2) < -C_2 t_2^{1/4}$$

hold. This result is also included in Heath-Brown and Tsang [58].

¿From Atkinson's formula Hafner-Ivić [46] deduced, as an analogue of (2.9), that

$$\int_2^T E(t)dt = \pi T + \frac{1}{2}\left(\frac{2T}{\pi}\right)^{\frac{3}{4}} \sum_{n=1}^{\infty} (-1)^n \frac{d(n)}{n^{\frac{5}{4}}} \sin\left(2\sqrt{2\pi n T} - \frac{\pi}{4}\right) + O\left(T^{\frac{2}{3}} \log T\right). \tag{3.23}$$

This implies that the function $E(t)$ has the mean value π. Hence it is natural to consider the zeros of the function $E(t) - \pi$, which we denote by t_n $(2 \leq t_1 < t_2 < \cdots)$. Then the above mentioned result implies that

$$t_{n+1} - t_n \ll t_n^{1/2}. \tag{3.24}$$

Let $\kappa = \inf\{c \geq 0 \ ; \ t_{n+1} - t_n \ll t_n^c\}$. Then (3.24) implies that $\kappa \leq 1/2$. Ivić-te Riele [78] studied $\{t_n\}$ both theoretically and numerically, and proposed the conjecture that $\kappa = 1/4$. This conjecture is very strong because it would lead to (3.19) (see Theorem 1 of [78]). However, this conjecture was disproved by Heath-Brown and Tsang [58]. They showed that for any $\delta > 0$ and any $T \geq T_0(\delta)$, there are at least $C_1 \delta T^{1/2} \log^5 T$ disjoint subintervals of length $C_2 \delta T^{1/2} (\log T)^{-5}$ in $[T, 2T]$ such that

$$\left|E(t)\right| > (B_0 - \delta) t^{\frac{1}{4}}, \qquad B_0 = \frac{\zeta^2(\frac{3}{2})}{2(2\pi)^{\frac{1}{4}} \zeta(3)^{\frac{1}{2}}}$$

whenever t lies in any of these subintervals. In particular $E(t)$ does not change sign in any of these subintervals. Therefore the local behaviour of $E(t)$ is much more mysterious than was expected by Ivić and te Riele.

4 The remainder term in the approximate functional equation for $\zeta^2(s)$

In the middle of 1980s, new light was shed on the remainder term $R_2(s; x, y)$ in (1.12). Jutila [83] pointed out that (1.12) with (1.14) can be deduced from the Voronoï summation formula (3.9); this should be compared with the fact, mentioned in Section 1, that (1.4) can be deduced from the Poisson summation formula (1.6). The details are presented in Ivić [65] [66] (see Section 4.2 of [66]).

In the special case $x = y = t/2\pi$ ("the symmetric case"), Motohashi [148I] showed that the estimate (1.14) for $R_2(s; t/2\pi, t/2\pi)$ (which we abbreviate as $R_2(s; t/2\pi)$) follows from the estimate (1.5) for $R_1(s; \sqrt{t/2\pi}, \sqrt{t/2\pi})$ by the Dirichlet device. However, we know much more precise information on $R_1(s; \sqrt{t/2\pi}, \sqrt{t/2\pi})$, that is the Riemann-Siegel formula. What can we obtain if we combine Motohashi's argument with the Riemann-Siegel formula? This idea was pursued by Motohashi himself, and he [148II] [150] [153] obtained a very precise asymptotic formula of $R_2(s;t/2\pi)$. His result includes

$$\chi(1-s)R_2\Big(s\ ;\ \frac{t}{2\pi}\Big)=\Big(\frac{t}{2\pi}\Big)^{-\frac14}\sum_{n=1}^{\infty}d(n)h(n)n^{-\frac14}\sin\Big(2\sqrt{2\pi tn}+\frac{\pi}{4}\Big)+O\left(t^{-\frac12}\log t\right), \tag{4.1}$$

where

$$\begin{aligned} h(n) &= \Big(\frac{2}{\pi}\Big)^{\frac12}\int_0^{\infty}(u+n\pi)^{-\frac12}\cos\Big(u+\frac{\pi}{4}\Big)du \\ &= -\frac{1}{\pi}n^{-\frac12}+O\left(n^{-\frac32}\right). \end{aligned}$$

Actually Motohashi's formula is more precise and complicated; the error term in (4.1) is replaced by some more explicit terms and a smaller error $O(t^{-1}\log t)$. A simple consequence of (4.1) and (2.3) is the relation

$$\chi(1-s)R_2\Big(s\ ;\ \frac{t}{2\pi}\Big)=-\sqrt{2}\Big(\frac{t}{2\pi}\Big)^{-\frac12}\Delta\Big(\frac{t}{2\pi}\Big)+O\left(t^{-\frac14}\right), \tag{4.2}$$

which was announced in [148I]. A formula of the same type was already given long before in Taylor's posthumous article [186], but Motohashi [153] pointed out that Taylor's argument was incorrect.

Jutila [85] gave an alternative proof of (4.2). His starting point is Titchmarsh's explicit formula (1.16). His idea is to smooth the right-hand side of (1.16) by using multiple integration, and then apply his own saddle-point lemma mentioned in Section 3. So far Jutila's method cannot give a proof of Motohashi's precise formula ((4.1) and more). An advantage of Jutila's approach is that it can be applied to many other Dirichlet series, satisfying a certain functional equation. In [85], Jutila presented analogous results on

$$\varphi(s,F)=\sum_{n=1}^{\infty}a(n)n^{-s}, \tag{4.3}$$

where $a(n)$'s are the Fourier coefficients of a holomorphic cusp form $F(z)=\sum_{n=1}^{\infty}a(n)\exp(2\pi inz)$ of weight κ (an even integer) for the full modular group $SL(2,\mathbf{Z})$. There are many analogous properties shared by $\zeta^2(s)$ and $\varphi(s,F)$, but a big difference is that $\zeta^2(s)$ has the good square-root function (i.e. $\zeta(s)$), while $\varphi(s,F)$ does not. This is why Motohashi's approach cannot be applied to $\varphi(s,F)$. Recently, Guthmann [42] [43] [44] has developed another unified approach to the remainder terms in the approximate functional equations for $\zeta^2(s)$ and $\varphi(s,F)$.

The formula (4.2) tells that a strong analogy between $R_2(s\ ;\ t/2\pi)$ and $\Delta(t/2\pi)$ should exist (cf. Ivić [69]). Kiuchi-Matsumoto [114] proved, as an analogue of (2.5), that

$$\int_2^T\left|R_2\Big(\frac12+it\ ;\ \frac{t}{2\pi}\Big)\right|^2dt=\sqrt{2\pi}C_0T^{\frac12}+K(T), \tag{4.4}$$

where

$$C_0=\sum_{n=1}^{\infty}d^2(n)h^2(n)n^{-\frac12} \tag{4.5}$$

and $K(T) = O(T^{\frac{1}{4}} \log T)$. The proof is based on (4.1). Using a more precise form of Motohashi's formula, Kiuchi [111] gave the improved bound $K(T) = O(\log^5 T)$, and suggested the conjecture

$$K(T) \sim C \log^3 T. \tag{4.6}$$

The hitherto best upper-bound is

$$K(T) = O(\log^4 T) \tag{4.7}$$

due to Kiuchi [113II]. On the other hand, Ivić [72] proved

$$\int_2^T K(t)dt = C_1 T \log^3 T + C_2 T \log^2 T + O(T \log T) \tag{4.8}$$

with $C_1 < 0$, analogously to (2.7). This implies

$$K(T) = \Omega_-(\log^3 T), \tag{4.9}$$

which supports the conjecture (4.6).

Motohashi's formula has been obtained in the symmetric case $x = y = t/2\pi$. How is the non-symmetric case? Motohashi [148III] [150] proved a formula when $x = \alpha t/2\pi$, $y = t/2\pi\alpha$ with a rational number α, but the result is not so precise as in the symmetric case. Jutila [85] considered the general situation, and in some non-symmetric cases his bound is better than (1.14). See also Jutila [86]. Mean-value results in the non-symmetric case were discussed by Kiuchi[112].

5 The mean square of $\zeta(s)$ in the critical strip

Now we return to the problem of evaluating $I_\sigma(T)$, and discuss the case $1/2 < \sigma \le 1$. After the classical result (1.1) of Landau and Schnee, the development in this direction had been very slow (cf. Ingham [63] and (8.112) of Ivić [66]). In 1989, the author [135] published the analogue of Atkinson's formula in the strip $1/2 < \sigma < 3/4$. It is stated as

$$E_\sigma(T) = \sum\nolimits_{1,\sigma}(T, X) - \sum\nolimits_{2,\sigma}(T, X) + O(\log T), \tag{5.1}$$

where $E_\sigma(T)$ is defined, for $1/2 < \sigma < 1$, by

$$I_\sigma(T) = \zeta(2\sigma)T + (2\pi)^{2\sigma-1} \frac{\zeta(2-2\sigma)}{2-2\sigma} T^{2-2\sigma} + E_\sigma(T). \tag{5.2}$$

It can be easily seen that $E_\sigma(T) \to E(T)$ as $\sigma \to 1/2 + 0$. In the same paper [135], as applications of (5.1), the author showed that

$$E_\sigma(T) = O\left(T^{\frac{1}{1+4\sigma}} \log^2 T\right) \qquad \left(\frac{1}{2} < \sigma < \frac{3}{4}\right) \tag{5.3}$$

(the O-constant may depend on σ) and

$$\int_2^T E_\sigma(t)^2 dt = A_1(\sigma) T^{\frac{5}{2}-2\sigma} + F_\sigma(T) \qquad \left(\frac{1}{2} < \sigma < \frac{3}{4}\right) \tag{5.4}$$

with $F_\sigma(T) = O(T^{7/4-\sigma}\log T)$, where

$$A_1(\sigma) = \frac{2}{5-4\sigma}(2\pi)^{2\sigma-\frac{3}{2}}\frac{\zeta(\frac{3}{2})^2}{\zeta(3)}\zeta\Big(\frac{5}{2}-2\sigma\Big)\zeta\Big(\frac{1}{2}+2\sigma\Big).$$

Independently of [135], Laurinčikas [129I] obtained the analogue of Atkinson's formula near the critical line, that is an explicit formula for the error term in the asymptotic formula of the integral

$$\int_0^T \Big|\zeta(\sigma_T+it)\Big|^2 dt,$$

where $\sigma_T = 1/2 + l_T^{-1}$, $0 < l_T \ll \log T$, and l_T tends to infinity as $T \to \infty$. See also Laurinčikas [129II] [131] [132] [133]; in [132], he proved that (5.3) (with a slightly different log-factor) holds uniformly in σ.

An asymptotic formula over short intervals was obtained by Sankaranarayanan-Srinivas [179] by a quite different method. They proved

$$\frac{1}{H}\int_T^{T+H}|\zeta(\sigma+it)|^2 dt = \zeta(2\sigma) + O\Big(\exp\Big(-\frac{C_1(\log T)^{2-2\sigma}}{\log\log T}\Big)\Big)$$

for $\exp((\log T)^{2-2\sigma}) \le H \le T$ and $(1/2) + C_2(\log\log T)^{-1} \le \sigma \le 1 - C_3$ under the assumption of the Riemann hypothesis. It should be noted that their method can be applied to much more general Dirichlet series.

The basic tool of the author [135] is Oppenheim's Voronoï-type formula [161] for the error term $\Delta_{1-2\sigma}(x)$ defined by

$$\sum_{n\le x}{}^{'}\sigma_{1-2\sigma}(n) = \zeta(2\sigma)x + \frac{\zeta(2-2\sigma)}{2-2\sigma}x^{2-2\sigma} - \frac{1}{2}\zeta(2\sigma-1) + \Delta_{1-2\sigma}(x). \tag{5.5}$$

The series in Oppenheim's formula is convergent only for $\sigma < 3/4$, which is the reason why the restriction $1/2 < \sigma < 3/4$ exists. It was pointed out in [135] that the coefficient $A_1(\sigma)$ tends to infinity when $\sigma \to 3/4 - 0$, which suggests some singular situation occurring at $\sigma = 3/4$. Now we know that the behaviour of $E_\sigma(T)$ in fact transposes at $\sigma = 3/4$, which can be well observed by the following refinement of (5.4):

$$\int_2^T E_\sigma(t)^2 dt = \begin{cases} A_1(\sigma)T^{\frac{5}{2}-2\sigma} + O(T) & (\frac{1}{2} < \sigma < \frac{3}{4}) \\ A_0 T\log T + O(T) & (\sigma = \frac{3}{4}) \\ O(T) & (\frac{3}{4} < \sigma < 1), \end{cases} \tag{5.6}$$

where $A_0 = \zeta^2(3/2)\zeta(2)/\zeta(3)$. These are due to Matsumoto-Meurman [140II] $(1/2 < \sigma < 3/4)$, Lam [125] $(\sigma = 3/4)$, and Matsumoto-Meurman [140III] $(3/4 < \sigma < 1)$, respectively. (In [140III], the formula for $\sigma = 3/4$ was given with a slightly weaker error term $O(T(\log T)^{1/2})$.)

To prove the result for $1/2 < \sigma < 3/4$, that is $F_\sigma(T) = O(T)$, Matsumoto-Meurman [140II] gave a new averaged version (somewhat similar to (3.13)) of Atkinson-type formula, which is proved by combining the methods of Meurman [142] and Preissmann [165] with some additional new idea. In the same paper [140II], the conjecture

$$F_\sigma(T) \sim 4\pi^2\zeta(2\sigma-1)^2 T \qquad \Big(\frac{1}{2} < \sigma < \frac{3}{4}\Big) \tag{5.7}$$

was proposed. There are several heuristic arguments which may suggest (5.7) (see [115] [136]). The reason presented in [140II] is the fact that $E_\sigma(T)$ has the mean value $-2\pi\zeta(2\sigma-1)$. This fact was discovered independently by Ivić [68]; he proved

$$\int_2^T E_\sigma(t)dt = B(\sigma)T + O\left(T^{\frac{5}{4}-\sigma}\right) \qquad \left(\frac{1}{2}<\sigma<\frac{3}{4}\right) \tag{5.8}$$

((3.39) of [68]). The expression of $B(\sigma)$ given in [68] is complicated, but it is actually equal to $-2\pi\zeta(2\sigma-1)$ (see Appendix of Matsumoto-Meurman [140II]). The above (5.8) is a direct consequence of

$$\begin{aligned}\int_2^T E_\sigma(t)dt &= B(\sigma)T + 2^{\sigma-\frac{3}{4}}\left(\frac{T}{\pi}\right)^{\frac{5}{4}-\sigma}\sum_{n=1}^{\infty}(-1)^n\sigma_{1-2\sigma}(n)n^{\sigma-\frac{7}{4}}\sin\left(\sqrt{8\pi nT}-\frac{\pi}{4}\right)\\ &\quad +O\left(T^{1-\frac{2}{3}\sigma}\log T\right)\end{aligned} \tag{5.9}$$

((3.30) of Ivić [68]), which is the analogue of (3.23). We mention here that it might be better to define the "real" error term in (5.2) (resp. (1.7)) as $E_\sigma(T)+2\pi\zeta(2\sigma-1)$ (resp. $E(T)-\pi$). The constant $-2\pi\zeta(2\sigma-1)$ (resp. π) corresponds to $-(1/2)\zeta(2\sigma-1)$ in (5.5) (resp. 1/4 in (2.1)).

Matsumoto-Meurman [140III] proved that the formula (5.1) is valid for all σ satisfying $1/2<\sigma<1$. When $\sigma\geq 3/4$ the Voronoï-type formula for the Riesz mean of $\sigma_{1-2\sigma}(n)$ is applied in [140III], because Oppenheim's series is divergent. It is again a certain averaged version of Atkinson-type formula from which the case $3/4<\sigma<1$ of (5.6) was deduced in [140III]. (Here we note that in the statement of Lemma 4 of [140III], σ should be deleted. The author would like to thank Dr. Hideki Nakaya who pointed out this mistake.)

As an extension of (5.7), the author proposed the conjecture that the error terms $O(T)$ in (5.6) could be replaced by $A_2(\sigma)T+o(T)$ for $1/2<\sigma<1$, with a certain constant $A_2(\sigma)$ (see [137]). A refined version is:

Conjecture 1 *The error terms $O(T)$ in (5.6) could be replaced by*

$$A_2(\sigma)T + O\left(T^{2-2\sigma}(\log T)^C\right) \tag{5.10}$$

for $1/2<\sigma\leq 3/4$, where $C\geq 0$, and by

$$A_2(\sigma)T + A_3(\sigma)T^{\frac{5}{2}-2\sigma} + O\left(T^{2-2\sigma}(\log T)^C\right) \tag{5.11}$$

with a certain $A_3(\sigma)$ for $3/4<\sigma<1$.

The reason of the error estimates $O(T^{2-2\sigma}(\log T)^C)$ is the result (6.6) mentioned in the next section. The author proposed (5.10) first in correspondence, which is mentioned in Ivić-Kiuchi [74]. The conjecture (5.11) first appeared in [115] (though the term $A_3(\sigma)T^{5/2-2\sigma}$ is missing there). Even the weaker form of the above conjecture is still open.

The formula (5.4) obviously implies $E_\sigma(T)=\Omega(T^{3/4-\sigma})$ for $1/2<\sigma<3/4$. Ivić [68] improved this to $\Omega_\pm(T^{3/4-\sigma})$ with some information about local sign-changes of $E_\sigma(T)$. The best known Ω-results at present are

$$E_\sigma(T)=\Omega_-\left\{T^{\frac{3}{4}-\sigma}\exp\left(C(\log\log T)^{\sigma-\frac{1}{4}}(\log\log\log T)^{\sigma-\frac{5}{4}}\right)\right\} \qquad \left(\frac{1}{2}<\sigma<\frac{3}{4}\right) \tag{5.12}$$

(Ivić-Matsumoto [75]), exactly corresponding to (3.21), and

$$E_\sigma(T) = \Omega_+\left(T^{\frac{3}{4}-\sigma}(\log T)^{\sigma-\frac{1}{4}}\right) \qquad \left(\frac{1}{2} < \sigma < \frac{3}{4}\right) \tag{5.13}$$

(Matsumoto-Meurman [140III]). It is much more difficult to obtain any Ω-result in the strip $3/4 \le \sigma < 1$. The only known result is

$$E_{\frac{3}{4}}(T) = \Omega\left((\log T)^{\frac{1}{2}}\right), \tag{5.14}$$

a direct consequence of the case $\sigma = 3/4$ of (5.6).

What is the real order of $E_\sigma(T)$? In view of (5.6) and the above Ω-results, we may formulate the conjecture

$$E_\sigma(T) \ll \begin{cases} T^{\frac{3}{4}-\sigma+\varepsilon} & \left(\frac{1}{2} < \sigma < \frac{3}{4}\right) \\ T^{\varepsilon} & \left(\frac{3}{4} \le \sigma < 1\right). \end{cases} \tag{5.15}$$

In Ivić-Matsumoto [75] this conjecture is stated, and also it is pointed out that if we assume the very strong conjecture that $(\varepsilon, 1/2+\varepsilon)$ would be an exponent pair for any $\varepsilon > 0$, then (5.15) would follow.

The critical behaviour of $E_\sigma(T)$ at $\sigma = 3/4$ is again clear in (5.15); it might suggest some unexpected properties of $\zeta(s)$. In connection with this observation, an interesting discussion concerning the Lindelöf hypothesis is given in Ivić [71]. See also the final section of [136].

The proof of the conjecture (5.15) seems to be out of reach now. As for the upper bound of $E_\sigma(T)$, Motohashi (unpublished) proved that (5.3) holds for any σ satisfying $1/2 < \sigma < 1$. His idea, inspired by his own work [151] on the fourth power mean of $\zeta(s)$, is to use the weighted integral

$$\frac{1}{\Delta\sqrt{\pi}}\int_{-\infty}^{\infty}\left|\zeta(\sigma+i(T+t))\right|^2 e^{-(\frac{t}{\Delta})^2}dt \qquad (\Delta > 0). \tag{5.16}$$

Ivić [68] combined Motohashi's idea with the theory of exponent pairs, and obtained various improved upper bounds of $E_\sigma(T)$. This direction was further studied by Ivić-Matsumoto [75] and Kačėnas [94] [95]; for instance, we have

$$E_\sigma(T) = O\left(T^{\frac{2(1-\sigma)}{3}}(\log T)^{\frac{2}{9}}\right) \qquad \left(\frac{1}{2} < \sigma < 1\right)$$

(Ivić-Matsumoto [75]) and

$$E_\sigma(T) = O\left(T^{\frac{72}{227}-\frac{1}{2}\delta+\varepsilon}\right) \qquad \left(\frac{1}{2} < \sigma < \frac{51}{100}\right)$$

with $\delta = \sigma - 1/2$ (Kačėnas [95]). The latter is uniform in σ, and exactly corresponds to Huxley's bound (3.20).

We conclude this section with mentioning the case $\sigma = 1$. No analogue of Atkinson's formula is known in this case. Starting from the simple approximate formula (1.2), Balasubramanian-Ivić-Ramachandra [7] proved the asymptotic formula

$$I_1(T) = \zeta(2)T - \pi\log T + \widetilde{E_1}(T) \tag{5.17}$$

with

$$\widetilde{E_1}(T) = O\left((\log T)^{\frac{2}{3}}(\log\log T)^{\frac{1}{3}}\right). \tag{5.18}$$

The connection between $E_\sigma(T)$ and $\widetilde{E_1}(T)$ is given by

$$\lim_{\sigma\to 1-0}\left\{\zeta(2\sigma)T + (2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{2-2\sigma}(T^{2-2\sigma}-1)\right\} = \zeta(2)T - \pi\log T$$

(Ivić [70]). In [7], they also proved the mean value results

$$\int_2^T \widetilde{E_1}(t)dt = O(T),$$
$$\int_2^T \widetilde{E_1}(t)^2 dt = O\Big(T(\log\log T)^4\Big)$$

and conjectured that the latter integral would be asymptotically equal to CT.

To show the estimate (5.18), the method of I.M.Vinogradov and Korobov, based on the deep theory of I.M.Vinogradov on the estimation of exponential sums, is applied. In fact, it is noted in [7] that from (5.18) one can deduce the estimate $\zeta(1+it) = O\left((\log t)^{2/3}(\log\log t)^{1/3}\right)$, which is very close to the sharpest known bound $\zeta(1+it) = O((\log t)^{2/3})$, obtained by the Vinogradov-Korobov theory (see Chapter 6 of Ivić [66]).

6 Mean values of $\Delta_{1-2\sigma}(x)$ and $R_2(\sigma+it; t/2\pi)$

In Section 3 we explained that a guiding principle of the study of $E(T)$ is to pursue the analogy with $\Delta(x)$. Similarly, it is useful to study the behaviour of $\Delta_{1-2\sigma}(x)$, defined by(5.5), which is the object analogous to $E_\sigma(T)$.

We already mentioned in Section 5 that the Voronoï-type formula for $\Delta_{1-2\sigma}(x)$ due to Oppenheim [161] was used in the proof of (5.1). By using the truncated Voronoï-type formula, Kiuchi [109] proved that

$$\int_2^X \Delta_{1-2\sigma}(x)^2 dx = B_1(\sigma)X^{\frac{5}{2}-2\sigma} + O\left(X^{\frac{7}{4}-\sigma+\varepsilon}\right) \qquad \left(\frac{1}{2}<\sigma<\frac{3}{4}\right) \tag{6.1}$$

with

$$B_1(\sigma) = \frac{\zeta(\frac{3}{2})^2}{2\pi^2(5-4\sigma)\zeta(3)}\zeta\Big(\frac{5}{2}-2\sigma\Big)\zeta\Big(\frac{1}{2}+2\sigma\Big).$$

It was already mentioned by Cramér [22] that the left-hand side of (6.1) is asymptotically equal to $B_1(\sigma)X^{5/2-2\sigma}$ for $1/2<\sigma<3/4$. Meurman [143] refined (6.1) to obtain

$$\int_2^X \Delta_{1-2\sigma}(x)^2 dx = \begin{cases} B_1(\sigma)X^{\frac{5}{2}-2\sigma} + O(X) & (\frac{1}{2}<\sigma<\frac{3}{4}) \\ B_0 X\log X + O(X) & (\sigma = \frac{3}{4}) \\ O(X) & (\frac{3}{4}<\sigma<1) \end{cases} \tag{6.2}$$

with $B_0 = \zeta^2(3/2)/24\zeta(3)$. The formula (6.2) gives the complete analogue of (5.6). Hence the following analogue of Conjecture 1 can be formulated.

Conjecture 2 *The error term $O(X)$ in (6.2) could be replaced by*

$$B_2(\sigma)X + O\left(X^{2-2\sigma}(\log X)^C\right) \tag{6.3}$$

for $1/2 < \sigma \leq 3/4$, and by

$$B_2(\sigma)X + B_3(\sigma)X^{\frac{5}{2}-2\sigma} + O\left(X^{2-2\sigma}(\log X)^C\right) \tag{6.4}$$

for $3/4 < \sigma < 1$, with certain $B_2(\sigma)$, $B_3(\sigma)$ and $C \geq 0$.

Meurman first proposed (6.3) in correspondence, while (6.4) appeared in Kiuchi-Matsumoto [115], though the term $B_3(\sigma)X^{5/2-2\sigma}$ is missing there.

In Section 4, we discussed the analogy between $\Delta(x)$ and $R_2(1/2+it;t/2\pi)$. We can find that there also exists an analogy between $\Delta_{1-2\sigma}(x)$ and $R_2(\sigma+it;t/2\pi)$ for $1/2 < \sigma < 1$. Kiuchi [113I] proved that

$$\int_2^T \left|R_2(\sigma+it;t/2\pi)\right|^2 dt = \begin{cases} C_1(\sigma)T^{\frac{3}{2}-2\sigma} + O(1) & (\frac{1}{2} < \sigma < \frac{3}{4}) \\ \pi C_0 \log T + O(1) & (\sigma = \frac{3}{4}) \\ O(1) & (\frac{3}{4} < \sigma \leq 1), \end{cases} \tag{6.5}$$

where C_0 is defined by (4.5) and $C_1(\sigma) = (2\pi)^{2\sigma-\frac{1}{2}}C_0/(3-4\sigma)$. This precisely corresponds to (5.6) and (6.2).

A remarkable fact is that we can go further in this case. Now it is known that the terms $O(1)$ in (6.5) can be replaced by

$$\begin{cases} C_2(\sigma) + O\Big(T^{1-2\sigma}(\log T)^4\Big) & (\frac{1}{2} < \sigma \leq \frac{3}{4}) \\ C_2(\sigma) + C_1(\sigma)T^{\frac{3}{2}-2\sigma} + O\Big(T^{1-2\sigma}(\log T)^4\Big) & (\frac{3}{4} < \sigma \leq 1) \end{cases} \tag{6.6}$$

with a certain constant $C_2(\sigma)$. The author [137] showed (6.6) in the case of $1/2 < \sigma \leq 3/4$, and in the same paper the weaker result with the error estimate $O(T^{1/4-\sigma})$ was given for $3/4 < \sigma \leq 1$. The result of the form (6.6) for $3/4 < \sigma \leq 1$ is due to Kiuchi [113II]. The above (6.6) implies that the facts corresponding to Conjectures 1 and 2 are indeed true for $R_2(\sigma+it;t/2\pi)$.

Higher moments of $R_2(\sigma+it;t/2\pi)$ have also been discussed. The results analogous to (3.15)-(3.18) for $R_2(1/2+it;t/2\pi)$ were obtained by Kiuchi [110] and Ivić [69]. The k-th power moment of $R_2(\sigma+it;t/2\pi)$, where k is a positive even integer and $0 \leq \sigma \leq 1$, was studied by Kiuchi-Matsumoto [115]. Their results especially imply that the transposing line for the k-th power moment is $\sigma = 1/4+1/k$, unconditionally for k =2,4,6 and 8, and under a certain plausible assumption for any even k.

In the case $3/4 < \sigma < 1$, the bound $O(X)$ in (6.2) is not the best known result. Already in 1932, Chowla [18] proved the asymptotic formula

$$\int_2^X \Delta_{1-2\sigma}(x)^2 dx = \frac{1}{2\pi^2}\left\{\sum_{n=1}^{\infty}\left(\frac{\sigma_{2-2\sigma}(n)}{n}\right)^2\right\} X + O\left(X^{\frac{5}{2}-2\sigma}\log X\right) \qquad \left(\frac{3}{4} < \sigma < 1\right), \tag{6.7}$$

which gives the partial solution of the case $3/4 < \sigma < 1$ of Conjecture 2. Recently, Yanagisawa [198] rediscovered (6.7) and also obtained more general results. The basic tool of both Chowla and Yanagisawa is a generalization of (2.14), that is

$$\Delta_{1-2\sigma}(x) = -G_{1-2\sigma}(x) - x^{1-2\sigma}G_{2\sigma-1}(x) + O\left(x^{\frac{1}{2}-\sigma}\right), \tag{6.8}$$

where

$$G_a(x) = \sum_{n\le\sqrt{x}} n^a \psi\Big(\frac{x}{n}\Big).$$

(As for (6.8), see Kanemitsu [98].) An asymptotic formula for the mean square of $\Delta_{-1}(x)$ was given by Walfisz [197].

On the other hand, as an analogue of (2.7), Lam-Tsang [126] proved

$$\int_2^X \delta_\sigma(x)dx = C(\sigma)X^2 + O\left(X^{2+\frac{(1-2\sigma)(3-4\sigma)}{2(3-2\sigma)}}\log X\right) \qquad \Big(\frac{1}{2} < \sigma < \frac{3}{4}\Big), \tag{6.9}$$

where

$$\delta_\sigma(X) = \int_2^X \Delta_{1-2\sigma}(x)^2 dx - B_1(\sigma)X^{\frac{5}{2}-2\sigma}$$

and

$$C(\sigma) = -\frac{\zeta(2\sigma)^2\zeta(3-4\sigma)}{12(2\pi)^{3-4\sigma}\zeta(4\sigma)}\Gamma(3-4\sigma)\sin(2\pi\sigma).$$

This result clearly implies the fact

$$\delta_\sigma(X) = \Omega_-(X) \qquad \Big(\frac{1}{2} < \sigma < \frac{3}{4}\Big), \tag{6.10}$$

which may be regarded as a support for the case $1/2 < \sigma < 3/4$ of Conjecture 2. The order of $\delta_\sigma(X)$ for $1/2 < \sigma < 3/4$ is completely determined by (6.2) and (6.10).

It is an interesting problem to prove the analogue of Lau-Tsang's (2.7) or Lam-Tsang's (6.9) for $F(T)$ or $F_\sigma(T)$. Another attractive problem is to search the analogue of the method of Chowla and Yanagisawa for the function $E_\sigma(T)$ in the case $3/4 < \sigma < 1$; or at least, to find the analogue of (6.8) for $E_\sigma(T)$. The last type of problem was sometimes mentioned by S. Kanemitsu in correspondence and oral communication.

7 Some mean value results in short intervals

We mentioned in Section 1 that Good [35] showed $E(T) = \Omega(T^{1/4})$. He actually proved an asymptotic formula for the integral

$$\int_0^T \Big(E(t+U) - E(t)\Big)^2 dt \qquad (1 \le U \ll T^{\frac{1}{2}}),$$

and the Ω-result is its corollary. The same formula is also used in the proof of Heath-Brown and Tsang [58] mentioned in Section 3.

Next, Jutila [84] studied a similar problem, but for short intervals, by using Atkinson's formula. His result is

$$\begin{aligned}\int_T^{T+H} \Big(E(t+U) - E(t)\Big)^2 dt &= \frac{1}{\sqrt{2\pi}} \sum_{n\le\frac{T}{U}} d(n)^2 n^{-\frac{3}{2}} \int_T^{T+H} t^{\frac{1}{2}} \left|\exp\left(i\Big(\frac{2\pi n}{t}\Big)^{\frac{1}{2}} U\right) - 1\right|^2 dt \\ &\quad + O(T^{1+\varepsilon}) + O\left(HU^{\frac{1}{2}}T^{\varepsilon}\right)\end{aligned} \tag{7.1}$$

for $T \geq 2$, $1 \leq U \ll T^{1/2} \ll H \leq T$. Note that the right-hand side can be estimated as

$$\ll (HU + T)T^{\varepsilon}. \tag{7.2}$$

Jutila did not give the details of the proof; instead, he described the proof (based on the truncated Voronoï formula (2.4)) of the corresponding formula for $\Delta(x)$, that is

$$\begin{aligned}\int_X^{X+H} \Big(\Delta(x+U) - \Delta(x)\Big)^2 dx &= \frac{1}{4\pi^2} \sum_{n \leq \frac{X}{2U}} d(n)^2 n^{-\frac{3}{2}} \int_X^{X+H} x^{\frac{1}{2}} \left| \exp\left(2\pi i \Big(\frac{n}{x}\Big)^{\frac{1}{2}} U\right) - 1 \right|^2 dx \\ &\quad + O(X^{1+\varepsilon}) + O\left(HU^{\frac{1}{2}} X^{\varepsilon}\right) \end{aligned} \tag{7.3}$$

for $X \geq 2$, $1 \leq U \ll X^{1/2} \ll H \leq X$. Moreover, Jutila raised the problem of extending (7.1) and (7.3) to higher power moments. In particular, he pointed out that if one could prove

$$\int_2^T \Big(E(t+U) - E(t)\Big)^4 dt = O(T^{1+\varepsilon} U^2),$$

then the very important conjectural bound

$$\int_0^T \left|\zeta\Big(\frac{1}{2} + it\Big)\right|^6 dt = O(T^{1+\varepsilon})$$

would follow.

A formula for $\Delta_{1-2\sigma}(x)$ $(1/2 < \sigma < 1)$, analogous to (7.3), was recently obtained by Kiuchi-Tanigawa [116]; they actually treated a more general quantity which involves exponential factors. Yanagisawa [199] studied the same problem by the method similar to his another work [198] mentioned in the preceding section. The analogy of (7.1) for $E_\sigma(T)$ $(1/2 < \sigma < 1)$ was given by Kiuchi-Tanigawa [117]. In [118], they studied the same type of short interval mean square of $R_2(\sigma + it; \alpha t/2\pi, t/2\pi\alpha)$ for rational α.

In [88I], Jutila proved the estimate

$$\int_T^{T+H} \Big(E(t+U) - E(t)\Big)^2 dt = O\left((HU + T^{\frac{2}{3}} U^{\frac{4}{3}})T^{\varepsilon}\right) \tag{7.4}$$

for $1 \leq H$, $U \leq T$, which improves (7.2) when $U \ll T^{1/4}$. Jutila noted that (7.4) implies the estimate

$$\int_T^{T+T^{\frac{2}{3}}} \left|\zeta\Big(\frac{1}{2} + it\Big)\right|^4 dt = O\left(T^{\frac{2}{3}+\varepsilon}\right), \tag{7.5}$$

due originally to Iwaniec [79]. In fact, since

$$\int_t^{t+U} \left|\zeta\Big(\frac{1}{2} + iu\Big)\right|^2 du$$

can be approximated by $E(t+U) - E(t)$, (7.5) easily follows ¿from (7.4) with $H = T^{2/3}$, $U = T^{\varepsilon}$, by applying Lemma 7.1 of Ivić [66].

Iwaniec's proof [79] of (7.5) was really epoch-making, because it was the first successful application of Kuznetsov's trace formula [122] to the mean value theory of zeta-functions, and was followed by many important works of Zavorotnyi, Motohashi, Ivić, Jutila and others in the fourth power moment theory. The full account of this theory is out of the scope of this article, but here we should mention Jutila's alternative proof [89] of (7.5). The basic idea of Jutila is to transform a certain relevant exponential sum by using (3.9), hence it is under the same philosophy as [83] [87]. The remarkable feature of Jutila's proof is that it only uses classical means, without the fancy tools of spectral theory. In Jutila's proof, a lemma due to Bombieri-Iwaniec [13] plays an important role. This lemma contains the arithmetic essence of (7.5), which is included in Kuznetsov's formula (or Kloosterman's sum) in Iwaniec's original proof.

Extending the above idea, Jutila [88I, II] studied the integral of the type

$$J = \sum_{r=1}^{R} \int_0^V \Big| \sum_{M \le m \le M'} d(m) g(m,v,y_r) \exp\Big(2\pi i f(m,v,y_r)\Big)\Big|^2 dv, \tag{7.6}$$

where M is a large positive number, $M < M' \le 2M$, $V > 0$, the functions f and g satisfy certain regularity conditions, and y_r $(1 \le r \le R)$ runs over a well-spaced set of numbers lying in [0,1]. Jutila [88I] proved a certain upper-bound of J, and from which he deduced (7.4) as well as its analogue for $\Delta(x)$. A further development of this method, with applications to Dirichlet L-functions, can be found in Jutila [90].

Lastly in this section we mention briefly the theory of Titchmarsh series developed by Ramachandra and his colleagues. Here we do not give the definition of general Titchmarsh series. They are elements of a certain class of Dirichlet series, including $\zeta(s)^k$ (for any positive integer k) as an example. In [169I], Ramachandra raised a conjecture on the lower bound of the mean square of Titchmarsh series over short intervals. Ramachandra (partly with Balasubramanian) wrote many papers ([9], [168]-[170]) on this topic, and finally, Balasubramanian-Ramachandra [12] (and Ramachandra [171]) solved completely the conjecture in a more precise form. This solution especially implies

$$\frac{1}{H}\int_T^{T+H} \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt \ge C_k (\log H)^{k^2} + O\Big(\frac{\log\log T}{H}(\log H)^{k^2}\Big) + O\left((\log H)^{k^2-1}\right) \tag{7.7}$$

for $\log\log T \ll H \le T$, where

$$C_k = \frac{1}{\Gamma(k^2+1)} \prod_p \left\{ (1-p^{-1})^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(k+m)}{m!\Gamma(k)}\right)^2 p^{-m} \right\}.$$

Ramachandra [171] includes an interesting lower bound of the mean square of $\zeta(1+it)$ over short intervals. Upper and lower bounds of the mean value

$$\frac{1}{H}\int_T^{T+H} \left| \frac{d^l}{ds^l}\zeta(s)^{2k} \right|_{s=\frac{1}{2}+it} dt$$

were studied by Ramachandra [168].

In the present article we do not discuss the full details of the theory of Titchmarsh series. This theory includes the treatment of the mean value of $|\zeta(s)^{2k}|$ with non-integral complex values of k, various Ω-results, sign-change theorems on $\arg\zeta(s)$, and generalizations etc. The readers are referred to Ramachandra's lecrure note [172].

8 Several general principles and the mean square of Dedekind zeta-functions

In the previous sections we discussed mainly the mean square of the Riemann zeta-function and related problems. A very important direction of research is to generalize the obtained results to various other zeta-functions. From this viewpoint, it is useful to find some general principles to obtain mean value results. One of them is the following classical theorem of Carlson [15], which is a generalization of (1.1) due to Landau and Schnee. Let

$$f(s)=\sum_{n=1}^{\infty}a_n n^{-s}$$

be a Dirichlet series, convergent in a certain half-plane. Assume that $f(s)$ can be continued to a holomorphic (or with possible poles included in a fixed compact set) function of finite order in the region $\sigma\geq\alpha+\varepsilon>\alpha$. Moreover suppose that

$$\int_{-T}^{T}\left|f(\sigma+it)\right|^2dt=O(T) \tag{8.1}$$

for $\sigma\geq\sigma_0>\alpha$. Then Carlson's theorem asserts that

$$\int_{-T}^{T}\left|f(\sigma+it)\right|^2dt\sim 2\left(\sum_{n=1}^{\infty}|a_n|^2n^{-2\sigma}\right)T \tag{8.2}$$

for $\sigma>\sigma_0$. (The part of the range of integration near the poles is omitted.) Potter [163I, II] studied this matter further. Potter's results are especially useful when $f(s)$ can be continued to the whole plane and satisfies a certain functional equation.

Carlson's theorem can be applied to the mean square of the Dedekind zeta-function $\zeta_K(s)$ attached to an algebraic number field K, and the result is

$$\int_1^T\left|\zeta_K(\sigma+it)\right|^2dt\sim\left(\sum_{n=1}^{\infty}a_K(n)^2n^{-2\sigma}\right)T\qquad(\sigma>1-l^{-1}),$$

where $l=[K:\mathbf{Q}]\geq 2$ and $a_K(n)$ is the number of integral ideals in K with norm n. But this is not the best known result. Chandrasekharan-Narasimhan [16] developed a general theory of approximate functional equations, and the following is a consequence of their theory:

$$\int_1^T\left|\zeta_K(\sigma+it)\right|^2dt=\left(\sum_{n=1}^{\infty}a_K(n)^2n^{-2\sigma}\right)T+O\left(T^{\frac{l-l\sigma+1}{2}}(\log T)^{\frac{l}{2}}\right) \tag{8.3}$$

if $\sigma>1-l^{-1}$, and

$$\int_1^T\left|\zeta_K(\sigma+it)\right|^2dt=O\Big(T^{l(1-\sigma)}(\log T)^l\Big) \tag{8.4}$$

if $1/2\leq\sigma\leq 1-l^{-1}$. When $l=2$, that is the case that K is a quadratic field, (8.3) gives the asymptotic formula for $\sigma>1/2$. In the case of $\sigma=1/2$, (8.4) gives the upper-bound $O(T\log^2 T)$. At the end of their paper [16], Chandrasekharan-Narasimhan conjectured

$$\int_1^T\left|\zeta_K\Big(\frac{1}{2}+it\Big)\right|^2dt\sim C_2T\log^2 T \tag{8.5}$$

for any real quadratic field K, with a certain constant C_2.

Let D be the discriminant of a quadratic field K, χ_D the Dirichlet character defined as the Kronecker symbol $\chi_D(n) = (\frac{D}{n})$, and $L(s, \chi_D)$ the corresponding Dirichlet L-function. It is well-known that $\zeta_K(s) = \zeta(s)L(s, \chi_D)$, hence the mean square of $\zeta_K(1/2+it)$ is a generalization of the fourth power moment of $\zeta(1/2+it)$. As for the latter problem, Titchmarsh [187] proved

$$\int_0^\infty \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 e^{-\delta t} dt \sim \frac{1}{2\pi^2\delta}\log^4\frac{1}{\delta} \tag{8.6}$$

as $\delta \to 0$. This immediately implies

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt \sim \frac{1}{2\pi^2} T\log^4 T, \tag{8.7}$$

because there is the general principle that if $f(t) \geq 0$ for all t and

$$\int_0^\infty f(t)e^{-\delta t}dt \sim \frac{1}{\delta}\log^m\frac{1}{\delta} \tag{8.8}$$

as $\delta \to 0$, then

$$\int_0^T f(t)dt \sim T\log^m T \tag{8.9}$$

(see Section 7.12 of Titchmarsh [190]). This principle is, in a sense, a kind of Tauberian theorem. Following the idea of Titchmarsh, Motohashi [147] proved that

$$\int_0^\infty \left|\zeta_K\left(\frac{1}{2}+it\right)\right|^2 e^{-\delta t}dt = C_2\frac{1}{\delta}\log^2\frac{1}{\delta} + O\left(\frac{1}{\delta}\log\frac{1}{\delta}\right), \tag{8.10}$$

thereby established the conjecture (8.5). He found that

$$C_2 = \frac{6}{\pi^2}L^2(1,\chi_D)\prod_{p|D}\left(1+\frac{1}{p}\right)^{-1}.$$

Another useful general principle is the mean value theorem for Dirichlet polynomials. For any complex numbers $a_1, \ldots, a_N$, we have

$$\int_0^T \Big|\sum_{n\leq N} a_n n^{it}\Big|^2 dt = T\sum_{n\leq N}|a_n|^2 + O\Big(\sum_{n\leq N} n|a_n|^2\Big). \tag{8.11}$$

(This remains valid for $N = \infty$, if the series on the right-hand side converge.) The formula (8.11) is due to Montgomery-Vaughan [146], and the key of their proof is the following generalization of Hilbert's inequality: Let $\lambda_1, \ldots, \lambda_R$ be distinct real numbers and $\delta_n = \min_{m\neq n}|\lambda_m - \lambda_n|$. Then, for any complex numbers $a_1, \ldots, a_R$, we have

$$\left|\sum\sum_{m\neq n}\frac{a_m\overline{a}_n}{\lambda_m-\lambda_n}\right| \leq \frac{3\pi}{2}\sum_n |a_n|^2\delta_n^{-1}. \tag{8.12}$$

This inequality has a close connection with the theory of large sieve inequalities; see Montgomery [144] [145].

Using (8.11), Ramachandra [166] gave a simple proof of Ingham's (1.11). Applying the same idea to $\zeta_K(1/2+it)$, it is possible to prove

$$\int_0^T \left|\zeta_K\left(\frac{1}{2}+it\right)\right|^2 dt = C_2 T \log^2 T + O(T\log T)$$

for a quadratic field K.

A further refinement was done by Müller [156], who generalized Heath-Brown's proof of (1.20). His result is that

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)L\left(\frac{1}{2}+it,\chi\right)\right|^2 dt = \widetilde{C_2} T\log^2 T + C_1 T\log T + C_0 T + O\left(q^{\frac{35}{16}+\varepsilon}T^{\frac{7}{8}+\varepsilon}\right), \qquad (8.13)$$

where χ is a primitive Dirichlet character (mod $q \geq 2$), and $\widetilde{C_2}$ is similar to C_2, just replacing $L^2(1,\chi_D)$ by $|L(1,\chi)|^2$.

Another possible way is to generalize the recent spectral-theoretic developments of the fourth power moment theory of $\zeta(s)$. See Motohashi [154], in which a certain explicit formula is given.

9 L-functions attached to cusp forms

In the preceding section we discussed the analogy between $\zeta_K(s)$ for quadratic fields and $\zeta^2(s)$. Another class of Dirichlet series, which may be regarded as an analogue of $\zeta^2(s)$, is L-functions $\varphi(s,F)$ attached to holomorphic cusp forms, defined by (4.3). The function $\varphi(s,F)$ is convergent absolutely for $\sigma > (\kappa+1)/2$, and can be continued to an entire function. The critical strip is $(\kappa-1)/2 \leq \sigma \leq (\kappa+1)/2$. In this section we survey the results on the mean square of $\varphi(s,F)$. Some of the quoted papers actually study more general cases (e.g. congruence subgroups), but here we restrict ourselves to the case of the full modular group for simplicity. Also we assume that $F(z)$ is a normalized eigenform (i.e. a simultaneous eigenfunction of Hecke operators with $a(1)=1$).

The connection between $F(z)$ and $\varphi(s,F)$ was established by Hecke in 1936-37. Just a few years later, the mean square of $\varphi(s,F)$ was already studied by Potter [163I, II]. Let

$$I_\sigma(T,F) = \int_0^T \left|\varphi(\sigma+it,F)\right|^2 dt.$$

As a consequence of his general theorem, Potter [163I] proved

$$I_\sigma(T,F) \sim \left(\sum_{n=1}^{\infty} a^2(n)n^{-2\sigma}\right)T \qquad \left(\sigma > \frac{\kappa}{2}\right), \qquad (9.1)$$

and then he [163II] proved that $I_{\kappa/2}(T,F) = O(T\log T)$.

In the middle of 1970s, Good began deeper investigations of $I_\sigma(T,F)$. First, Good [32] applied Titchmarsh's idea of using the Tauberian principle (8.8)-(8.9) to the present case, and obtained the asymptotic formula

$$I_{\kappa/2}(T,F) \sim 2\kappa A_0 T\log T, \qquad (9.2)$$

where

$$A_0 = \frac{12(4\pi)^{\kappa-1}}{\Gamma(\kappa+1)}\int\int_{\mathcal{F}} \left|F(x+iy)\right|^2 y^{\kappa-2}dxdy,$$

the integral being taken over a fundamental domain $\mathcal{F}$ of $SL(2, \mathbf{Z})$. The constant A_0 appears in Rankin's celebrated formula

$$\sum_{n \le x} a^2(n) = A_0 x^\kappa + O\left(x^{\kappa - \frac{2}{5}}\right),$$

which is essentially used in Good's proof of (9.2).

The next paper [33] of Good gives a certain approximate functional equation for $\varphi(s, F)$. Let $\omega : [0, \infty) \to \mathbf{R}$ be a (fixed) C^∞-function such that $\omega(\rho) = 1$ if $0 \le \rho \le 1/2$ and $\omega(\rho) = 0$ if $\rho \ge 2$. Define $\omega_0(\rho) = 1 - \omega(1/\rho)$. Then, a useful form of Good's formula can be stated as

$$\begin{aligned} \varphi(s, F) \;\; = \;\; & \sum_{j=0}^{l} \gamma_j(s, |t|^{-1}) \sum_{n=1}^{\infty} a(n) n^{-s} \omega^{(j)}\left(\frac{n}{y_1}\right)\left(-\frac{n}{y_1}\right)^j \\ & + (-1)^{\frac{\kappa}{2}} (2\pi)^{2s-\kappa} \frac{\Gamma(\kappa - s)}{\Gamma(s)} \sum_{j=0}^{l} \gamma_j(\kappa - s, |t|^{-1}) \sum_{n=1}^{\infty} a(n) n^{s-\kappa} \omega_0^{(j)}\left(\frac{n}{y_2}\right)\left(-\frac{n}{y_2}\right)^j \\ & + O\left(\left\|\omega^{(l+1)}\right\|_1 y_1^{\frac{\kappa+1}{2} - \sigma} |t|^{-\frac{l}{2}}\right) + O\left(\left\|\omega_0^{(l+1)}\right\|_1 y_1^{\frac{\kappa}{2} - \sigma} y_2^{\frac{1}{2}} |t|^{-\frac{l}{2}}\right) \end{aligned} \tag{9.3}$$

for $l \ge (\kappa + 1)/2$ and $4\pi^2 y_1 y_2 = t^2$, where $\| \cdot \|_1$ means the L^1-norm and $\gamma_j(s, |t|^{-1})$ is a quantity defined by a certain integral. Note that $\gamma_0(s, |t|^{-1}) \equiv 1$. It is possible to deduce the approximate functional equation of classical type (like (1.4)) from (9.3), but the above form is more effective in applications. Using(9.3), Good [33] proved

$$I_\sigma(T, F) = \begin{cases} 2\kappa A_0 T \log T + O(T) & (\sigma = \frac{\kappa}{2}) \\ \left(\sum_{n=1}^{\infty} a^2(n) n^{-2\sigma}\right) T + O\left(T^{\kappa+1-2\sigma}\right) & (\frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}) \\ \left(\sum_{n=1}^{\infty} a^2(n) n^{-2\sigma}\right) T + O(\log^2 T) & (\sigma = \frac{\kappa+1}{2}). \end{cases} \tag{9.4}$$

The formula (9.3) and its relatives are fundamental in Good's theory. In [34], Good proved a formula of the same type for $\zeta(s)$, and from which he [34] [35] deduced several new facts on $E(T)$ mentioned in Sections 1 and 7. A discrete mean square of $\varphi(s, F)$ was studied by Good [36], as an application of (9.3). As for $I_\sigma(T, F)$, the next step of Good's research is [37], in which he proved

$$I_{\kappa/2}(T, F) = 2\kappa A_0 T \log T + A_1 T + E(T, F) \tag{9.5}$$

where A_1 is a constant, with

$$E(T, F) = O\left(T^{\frac{5}{6}} (\log T)^{\frac{13}{6}}\right). \tag{9.6}$$

Moreover he showed that a certain non-vanishing assumption would lead to

$$E(T, F) = \Omega\left(T^{\frac{1}{2}}\right). \tag{9.7}$$

To prove these results, Good applied (9.3) to the integral

$$J_F(T, U) = \int_{-\infty}^{\infty} \left|\varphi\left(\frac{\kappa}{2} + it, F\right)\right|^2 \Psi_U\left(\frac{t}{T}\right) dt$$

with a certain weight function Ψ_U, and obtained an explicit formula for $J_F(T,U)$, in which a sum involving the factor $a(l)a(l+n)(l+n/2)^{-\kappa}$ appears. To analyze this sum, Good studied the behaviour of the Dirichlet series

$$\sum_{l=1}^{\infty} a(l)a(l+n)\left(l+\frac{n}{2}\right)^{-s} \tag{9.8}$$

by using the spectral theory. He found an explicit expression of $J_F(T,U)$ written in terms of non-analytic Poincaré series and Fourier coefficients of Maass wave forms, from which he derived the above results (9.5)-(9.7).

In [38], Good developed his theory further, and improved (9.6) to

$$E(T,F) = O\left(T^{\frac{2}{3}}(\log T)^C\right) \tag{9.9}$$

with $C = 2/3$. The bound

$$\varphi\left(\frac{\kappa}{2}+it,F\right) = O\left(|t|^{\frac{1}{3}}(\log|t|)^{\frac{5}{6}}\right) \qquad (|t| \geq 2) \tag{9.10}$$

is an immediate consequence of (9.9).

Recently, Kamiya [96] generalized (9.3) to the case of

$$\varphi(s,F,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)a(n)}{n^s} \tag{9.11}$$

with a Dirichlet character χ mod q, and used it to obtain

$$\sum_{\chi \bmod q}{}^{*} \int_{-T}^{T} \left|\varphi(\sigma+it,F,\chi)\right|^2 dt \ll \phi(q)T\log(qT) \tag{9.12}$$

if $q \ll T$, uniformly for $\kappa/2 - 1/\log(qT) \leq \sigma \leq \kappa/2 + 1/\log(qT)$ and q. Here, $\sum^*$ denotes the summation running over primitive characters. This is the result corresponding to Montgomery's estimate [144] for the fourth power moment of $L(s,\chi)$.

Another type of mean square of $\varphi(s,F,\chi)$ (including the non-holomorphic case) was discussed in Stefanicki [185] by a different method. See also Kamiya [97].

Jutila's consistent principle is to develop the theory which may treat the both cases $\zeta^2(s)$ and $\varphi(s,F)$ simultaneously. In Chapter 4 of his lecture note [87], Jutila gave a proof of $\varphi(\kappa/2+it,F) = O(|t|^{1/3+\varepsilon})$, slightly weaker than Good's (9.10), in such a unified way. The estimate

$$\int_0^T \left|\varphi\left(\frac{\kappa}{2}+it,F\right)\right|^6 dt = O(T^{2+\varepsilon}), \tag{9.13}$$

an analogue of (3.5), was also proved in the same chapter. Jutila proved those results by means of his transformation method, hence the arguments are of elementary nature. At the end of [87] Jutila proposed the problem of showing (7.5) and the corresponding estimate

$$\int_T^{T+T^{\frac{2}{3}}} \left|\varphi\left(\frac{\kappa}{2}+it,F\right)\right|^2 dt = O\left(T^{\frac{2}{3}+\varepsilon}\right), \tag{9.14}$$

an obvious corollary of (9.9), in a unified way. This problem was solved by Jutila himself in [88I], mentioned in Section 7. Hence [88I] includes a proof of (9.14) by classical means. An alternative proof of (9.13) is given in [88II].

We sometimes mentioned the recent spectral-theoretic approach to the fourth power moment of $\zeta(s)$. In view of the analogy between $\zeta^2(s)$ and $\varphi(s, F)$, one may expect, suggested by (9.9), that (1.20) could be improved to

$$E_2(T) = O\left(T^{\frac{2}{3}+\varepsilon}\right). \tag{9.15}$$

This was first achieved by Zavorotnyi [201] by using Kuznetsov's convolution formula [124]. Ivić-Motohashi [77] gave an alternative proof, with replacing T^ε by a log-power. In the latter proof, Motohashi's explicit formula [151] for the weighted fourth power mean of $|\zeta(1/2+it)|$ is essentially used. It is worth while noting that the basic idea of [151] is an extension of Atkinson's dissection argument to the fourth power situation.

We already discussed the close connection between $E(T)$ and the Dirichlet divisor problem. Similarly, $E_2(T)$ is closely related to the additive divisor problem, as was first noticed by Atkinson [2]. The additive divisor problem is the problem of evaluating the sum $\sum_{n\leq x} d(n)d(n+r)$, and has a long and rich history. The associated zeta-function is

$$\sum_{n=1}^{\infty} d(n)d(n+r)n^{-s}, \tag{9.16}$$

whose explicit spectral-theoretic expression was obtained by A.I.Vinogradov-Takhtadzhyan [194]. One may notice the similarity between (9.8) and (9.16), both of which were handled by spectral theory. Inspired by those works of Good and Vinogradov-Takhtadzhyan, and also inspired by the classical works of Titchmarsh [187] and Atkinson [2] on the Laplace transform of $|\zeta(1/2+it)|^4$, Jutila [91] developed a new unified approach to $E_2(T)$ and $E(T, F)$. He obtained spectral-theoretic explicit formulas for both $E_2(T)$ and $E(T, F)$, which, in the case of $E_2(T)$, has the same flavour as Motohashi's explicit formula [151]. As a consequence, Jutila reproved (9.9) (with the factor T^ε) and (9.15).

Another approach was given by Motohashi [152]. He sketched the way how to modify the argument in [151] to obtain an explicit formula for the weighted mean square of $\varphi(s, F)$, and to deduce from which the estimate (9.9) as well as the mean square estimate

$$\int_0^T E^2(t, F)dt = O\left(T^2(\log T)^C\right). \tag{9.17}$$

The latter is the analogue of

$$\int_0^T E_2^2(t)dt = O\left(T^2(\log T)^C\right) \tag{9.18}$$

due to Ivić-Motohashi [76]. Jutila [93] pursued his approach via Laplace transforms further, and proved (9.17) and (9.18) in a unified way.

An important advantage of Jutila's method is that it may also treat the non-holomorphic case. The results corresponding to (9.9) and (9.17) (with replacing $(\log T)^C$ by T^ε) for L-functions attached to Maass wave forms are proved in [91] [93]. The former is an improvement of Kuznetsov's result [123], which gives the exponent $6/7+\varepsilon$ in the error term. See also Müller [157] for another approach to the non-holomorphic case.

10 Dirichlet L-functions

Now we return to the $GL(1)$-situation, and in this section we discuss various mean square formulas for Dirichlet L-functions. Let χ be a Dirichlet character mod q, and $L(s,\chi)$ the corresponding Dirichlet L-function. A natural extension of $I_\sigma(T)$ is the mean value

$$I_\sigma(T,q)=\frac{1}{\phi(q)}\sum_{\chi \bmod q}\int_0^T \left|L(\sigma+it,\chi)\right|^2 dt.$$

Serious research on this quantity was started by Ramachandra and his school in 1970s. It was mentioned by Ramachandra [167] that he had obtained the asymptotic formula

$$I_{\frac{1}{2}}(T,q)=\frac{\phi(q)}{q}T\log(qT)+O(T(\log(qT))^\varepsilon).$$

It is easy to see that

$$L(s,\chi)=q^{-s}\sum_{a=1}^{q}\chi(a)\zeta\Big(s,\frac{a}{q}\Big), \tag{10.1}$$

where $\zeta(s,\alpha)$ is the Hurwitz zeta-function defined by the analytic continuation of the Dirichlet series $\sum_{n=0}^\infty(n+\alpha)^{-s}$. Hence we can reduce the problem of evaluating $I_\sigma(T,q)$ to the study of the mean square of $\zeta(s,\alpha)$. The approximate functional equation of $\zeta(s,\alpha)$, corresponding to (1.4), can be stated as

$$\zeta(s,\alpha)=\sum_{0\le n\le\xi}(n+\alpha)^{-s}+\Big(\frac{2\pi}{t}\Big)^{\sigma-\frac{1}{2}+it}e^{i(\frac{\pi}{4}+t)}\sum_{1\le n\le\eta}\frac{e^{-2\pi in\alpha}}{n^{1-s}}+O\left(\xi^{-\sigma}\log t\right), \tag{10.2}$$

valid for $1\le\xi\le\eta$, $2\pi\xi\eta=t$, and $0<\sigma<1$. Rane [173] proved a more precise approximate formula of the Riemann-Siegel type, and used it to prove

$$\int_0^T\left|\zeta\Big(\frac{1}{2}+it,\alpha\Big)\right|^2 dt=T\log T+C(\alpha)T-\frac{1}{\alpha}+O\left(\alpha^{-\frac{1}{2}}T^{\frac{1}{2}}\log T\right), \tag{10.3}$$

where $C(\alpha)$ is a constant depending on α. From (10.1) and (10.3) Rane [173] proved

$$I_{\frac{1}{2}}(T,q)=\frac{\phi(q)}{q}T\Big(\log\frac{qT}{2\pi}+2\gamma-1+\sum_{p|q}\frac{\log p}{p-1}\Big)+E(T,q) \tag{10.4}$$

with

$$E(T,q)\ll\frac{\phi(q)}{q}\left(T^{\frac{1}{2}}\log T+\log q\right). \tag{10.5}$$

Balasubramanian-Ramachandra [8] gave a simpler proof of (10.4) and (10.5). A key lemma in their argument is a short interval mean square estimate of $\zeta(1/2+it,\alpha)$, which is proved by the idea of Ramachandra [166].

The next step was due to Narlikar [160], who improved (10.5) to

$$E(T,q)\ll\frac{\phi(q)}{q}T^{5/12}(\log T)^2.$$

This is based on her refinement [159] of (10.3). Unfortunately Narlikar's argument includes an error which leads to the existence of extra terms of the order $T^{1/2}$ in her statements, which are to be deleted. Zhan [202] mentioned that it is possible to correct this error and justify Narlikar's argument. Zhan [202] himself adopted a different way; he proved the approximate formula of the type of Heath-Brown [53] for $\zeta(1/2+it,\alpha)$, and using it he gave further improvements on the results mentioned above. In particular he obtained $E(T,q)=O(T^{\alpha+\varepsilon})$ with a certain $\alpha<1/3$. Other variants of approximate functional equation for $\zeta(s,\alpha)$ were recently given by Rane [177] [178]. See also Balasubramanian-Ramachandra [10].

Meurman [141] proved a generalization of Atkinson's formula to $E(T,q)$, and from which he deduced

$$E(T,q) \ll \begin{cases} \phi(q)^{-1}\Big((qT)^{\frac{1}{3}+\varepsilon}+q^{1+\varepsilon}\Big) & (q \ll T) \\ \phi(q)^{-1}\Big((qT)^{\frac{1}{2}+\varepsilon}+qT^{-1}\Big) & (q \gg T). \end{cases} \tag{10.6}$$

Laurinčikas [130] proved an analogue of Meurman's formula near the critical line, while the analogue for fixed σ, $1/2<\sigma<1$, was obtained by Nakaya [158]. Meurman's paper [141] includes the short interval estimate

$$\sum_{\chi \bmod q} \int_T^{T+H} \left|L\Big(\frac{1}{2}+it,\chi\Big)\right|^2 dt \ll \Big(qH+(qT)^{\frac{1}{3}}\Big)\Big(q(T+H)\Big)^{\varepsilon}, \tag{10.7}$$

for $H \gg 1$, as a corollary. An alternative proof of (10.7) was obtained by Balasubramanian-Ramachandra [11], in which further improvements by using the theory of exponent pairs were also discussed. It is to be noted that their argument includes, as a special case, a simple proof of the estimate

$$\int_T^{T+T^{1/3}} \left|\zeta\Big(\frac{1}{2}+it\Big)\right|^2 dt \ll T^{1/3+\varepsilon}.$$

The study of the mean square of individual L-functions

$$\int_0^T \Big|L(\sigma+it,\chi)\Big|^2 dt$$

is more difficult. See the recent article [108] of Katsurada and the author, in which the history of this problem is sketched.

Another version of mean square of L-functions is

$$U(s,q)=\frac{1}{\phi(q)}\sum_{\chi \bmod q}\Big|L(s,\chi)\Big|^2.$$

In the case of $s=1/2+it$, $t \geq 2$, Gallagher [29] proved

$$U\Big(\frac{1}{2}+it,q\Big) \ll \frac{1}{\phi(q)}(q+t)\log(qt). \tag{10.8}$$

Improved upper-bounds were obtained by Meurman [141] and Rane [176]. Balasubramanian [6] gave an asymptotic formula with the main term $q^{-1}\phi(q)\log(qt)$, which was further refined by

W.Zhang [205] [214] and Yu [200]. In [214], it is shown that

$$\begin{aligned} U\Big(\frac{1}{2}+it,q\Big) &= \frac{\phi(q)}{q}\Big\{\log\left(\frac{qt}{2\pi}\right)+2\gamma+\sum_{p|q}\frac{\log p}{p-1}\Big\} \\ &\quad +O\left(\frac{1}{\phi(q)}qt^{-1}+\frac{1}{\phi(q)}(qt)^{\frac{1}{2}}\exp\left(\frac{\log(qt)}{\log\log(qt)}\right)\right). \end{aligned} \tag{10.9}$$

In most of the above works, the problem is reduced by (10.1) to the mean square of Hurwitz zeta-functions. And for the latter problem, for example in [214], it is shown that

$$\sum_{a=1}^{q}\left|\zeta\Big(\frac{1}{2}+it,\frac{a}{q}\Big)\right|^2 = q\Big\{\log\left(\frac{qt}{2\pi}\right)+2\gamma\Big\}+O\left(qt^{-1}+(qt)^{\frac{1}{2}}\log t\right) \tag{10.10}$$

(a special case of Lemma 7 of [214]). W.Zhang's papers [206] [215] are devoted to the study of $\sum_{\chi \bmod q}{}^{*}|L(1/2+it,\chi)|^2$, while in [204] he obtained an asymptotic formula for

$$\sum_{q\le Q}\frac{q}{\phi(q)}\sum_{\chi \bmod q}\left|L\Big(\frac{1}{2}+it,\chi\Big)\right|^2.$$

On the other hand, Motohashi [149I] applied Atkinson's dissection argument (see Section 3) to the study of $U(s,q)$, and obtained an asymptotic formula for $U(1/2+it,p)$ for any prime p and fixed t. Motohashi's idea was further developed by Katsurada-Matsumoto [103], who proved the following formula:

$$\begin{aligned} U\Big(\frac{1}{2}+it,q\Big) &= \frac{\phi(q)}{q}\Big\{\log\frac{q}{2\pi}+2\gamma+\sum_{p|q}\frac{\log p}{p-1}+\mathrm{Re}\frac{\Gamma'}{\Gamma}\Big(\frac{1}{2}+it\Big)\Big\} \\ &\quad +\frac{2}{q}\sum_{k|q}\mu\Big(\frac{q}{k}\Big)T\Big(\frac{1}{2}+it;k\Big), \end{aligned} \tag{10.11}$$

where $\mu(\cdot)$ denotes the Möbius function and $T(1/2+it;k)$ satisfies the asymptotic formula

$$\begin{aligned} T\Big(\frac{1}{2}+it;k\Big) &= \mathrm{Re}\left\{\sum_{n=0}^{N-1}\binom{-\frac{1}{2}+it}{n}k^{\frac{1}{2}+it-n}\zeta\Big(\frac{1}{2}+it-n\Big)\zeta\Big(\frac{1}{2}-it+n\Big)\right\} \\ &\quad +O\left(k^{1-N}t^{2N}\right) \end{aligned} \tag{10.12}$$

for any positive integer N, where the O-constant depends only on N. The quantity $T(1/2+it;1)$ can be written down in a closed form, while (10.12) gives the asymptotic expansion of $T(1/2+it;k)$ with respect to k if $k>1$. Hence, if $q=p^m$ is a prime power, then from (10.11) we can deduce the asymptotic expansion of $U(1/2+it,p^m)$ with respect to p. The special case $t=0$ of (10.11) was first obtained by Heath-Brown [54] by a different method, but the coefficients of the expansion are not explicitly given there. J.Zhang-Xing [203] gave an alternative proof of (10.11) by using Hurwitz zeta-functions. Another different proof of (10.11) was recently obtained by Katsurada [101]. Rane [175] also gave a similar expansion, but the coefficients are not explicit.

The mean square of $(d/ds)L(s,\chi)$ was considered by W.Zhang [211] [212] and Chen [17]. Katsurada [99III] generalized the method of [103] [99II] to study the case of $(d^k/ds^k)L(s,\chi)$ for any positive integer k.

In [103], the asymptotic expansion formula was proved not only for $\sigma = 1/2$, but for any σ satisfying $0 < \sigma < N+1$, as was pointed out in Katsurada-Matsumoto [106]. The region was further extended by Katsurada [99II] [101]. The formula (10.11) can be derived ¿from this general formula as the limit case $\sigma \to 1/2$. Another important limit case is $\sigma = 1$, and in this way we can deduce a precise formula for

$$V(q) = \sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \left| L(1,\chi) \right|^2,$$

where χ_0 is the principal character mod q. Evaluation of $V(q)$ is a classical problem, and in the case that $q = p$ is a prime, the formula

$$V(p) = \zeta(2)p + O(\log^2 p)$$

goes back to Paley [162] and Selberg [181]. This was refined by Slavutskiĭ [183] [184] and then W.Zhang [209] [210]; the result given in [210] is that

$$V(p) = \zeta(2)p - \log^2 p + C + O\left(\frac{1}{\log p}\right).$$

Following the above mentioned method, Katsurada-Matsumoto [106] proved the asymptotic expansion

$$\begin{aligned} V(p) &= \zeta(2)p - \log^2 p + \left(\gamma^2 - 2\gamma_1 - 3\zeta(2)\right) - \left(\gamma^2 - 2\gamma_1 - 2\zeta(2)\right)\frac{1}{p} \\ &\quad +2\left(1 - \frac{1}{p}\right)\left\{\sum_{n=1}^{N-1}(-1)^n \zeta(1-n)\zeta(1+n)p^{-n} + O(p^{-N})\right\} \end{aligned} \tag{10.13}$$

for any positive integer N, where γ_1 is defined by the following Laurent expansion of $\zeta(s)$ at $s = 1$:

$$\zeta(s) = \frac{1}{s-1} + \sum_{j=0}^{\infty} \gamma_j (s-1)^j, \qquad \gamma_0 = \gamma. \tag{10.14}$$

The formula (10.13) gives the satisfactory answer to the problem of evaluating $V(p)$. Moreover, in [106], a generalization of (10.13) to any composite q is proved. Some explicit expressions of $U(m,q)$, where $m(\neq 1)$ is an integer, are also obtained in [106].

11 Hurwitz zeta and other related zeta-functions

We already mentioned some mean value results on $\zeta(s,\alpha)$ in the preceding section (see (10.3) and (10.10)). In this final section we mainly discuss the approach by Atkinson's method, due to the recent papers of Katsurada and the author.

First, it is easily seen that a simple modification of the method developed in Katsurada-Matsumoto [103] can be applied to the discrete mean square $\sum_{a=1}^{q} |\zeta(s,a/q)|^2$. The result is an asymptotic expansion formula similar to (10.11) and (10.12), proved in Katsurada-Matsumoto [104]. This should be compared with (10.10).

A more interesting problem is to evaluate the integral

$$H(s) = \int_0^1 \left|\zeta_1(s,\alpha)\right|^2 d\alpha,$$

where $\zeta_1(s,\alpha)=\zeta(s,\alpha)-\alpha^{-s}$. In the case of $s=1/2+it$, $t\geq 2$, this problem was first considered by Koksma-Lekkerkerker [119], who showed $H(1/2+it)=O(\log t)$. This result was used in Gallagher's proof of (10.8). Balasubramanian [5] proved the asymptotic formula

$$H\left(\frac{1}{2}+it\right)=\log t+O(\log\log t),$$

and further refinements were done by Rane [174], Sitaramachandrarao (unpublished), and W.Zhang [207] [213], by using the approximate functional equation (10.2). Zhang [213] arrived at the result

$$H\left(\frac{1}{2}+it\right)=\log\left(\frac{t}{2\pi}\right)+\gamma+O\left(t^{-\frac{7}{36}}(\log t)^{\frac{25}{18}}\right),$$

and conjectured that the error estimate could be improved to $O(t^{-1/4})$. Ramachandra independently expressed the same opinion. This conjecture was solved by Andersson [1] and Zhang himself [217], independently of each other, in the following unexpected form:

$$H\left(\frac{1}{2}+it\right)=\log\left(\frac{t}{2\pi}\right)+\gamma-2\mathrm{Re}\frac{\zeta(\frac{1}{2}+it)}{\frac{1}{2}+it}+O(t^{-1}). \tag{11.1}$$

Shortly after their works, Katsurada-Matsumoto [105] [107I] obtained the following asymptotic expansion. For any integer $K\geq 0$, it holds that

$$\begin{aligned}H\left(\frac{1}{2}+it\right) \quad = \quad & \mathrm{Re}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+it\right)+\gamma-\log 2\pi-2\mathrm{Re}\frac{\zeta(\frac{1}{2}+it)-1}{\frac{1}{2}+it}\\ & -2\mathrm{Re}\sum_{k=1}^{K}\frac{(-1)^{k-1}(k-1)!}{(\frac{3}{2}-k+it)(\frac{5}{2}-k+it)\ldots(\frac{1}{2}+it)}\sum_{l=1}^{\infty}l^{-k}(l+1)^{-\frac{3}{2}+k-it}\\ & +O(t^{-K-1}).\end{aligned} \tag{11.2}$$

The starting point of the proof is Atkinson's dissection device (cf. Section 3). For $\mathrm{Re}\,u>1$, $\mathrm{Re}\,v>1$, we have

$$\zeta(u,\alpha)\zeta(v,\alpha)=\zeta(u+v,\alpha)+f(u,v;\alpha)+f(v,u;\alpha),$$

where

$$f(u,v;\alpha)=\sum_{m=0}^{\infty}(m+\alpha)^{-u}\sum_{n=1}^{\infty}(m+n+\alpha)^{-v}. \tag{11.3}$$

By the argument similar to [149I] [103], we can prove a contour-integral expression of $f(u,v;\alpha)$, which gives the analytic continuation. Analyzing this expression further, Katsurada-Matsumoto [107I] obtained the following formula, which is fundamental in their theory. Let N be a positive integer, $(s)_n=\Gamma(s+n)/\Gamma(s)$ the Pochhammer symbol, and define

$$S_N(u,v)=\sum_{n=0}^{N-1}\frac{(u)_n}{(1-v)_{n+1}}\Big(\zeta(u+n)-1\Big)$$

and

$$T_N(u,v)=\frac{(u)_N}{(1-v)_N}\sum_{l=1}^{\infty}l^{1-u-v}\int_l^{\infty}\beta^{u+v-2}(1+\beta)^{-u-N}d\beta.$$

Then it holds that

$$\begin{aligned}\int_0^1 \zeta_1(u,\alpha)\zeta_1(v,\alpha)d\alpha &= \frac{1}{u+v-1}+\Gamma(u+v-1)\zeta(u+v-1)\Big(\frac{\Gamma(1-v)}{\Gamma(u)}+\frac{\Gamma(1-u)}{\Gamma(v)}\Big)\\ &\quad -S_N(u,v)-S_N(v,u)-T_N(u,v)-T_N(v,u) \qquad (11.4)\end{aligned}$$

for $-N+1<\mathrm{Re}\,u<N+1$, $-N+1<\mathrm{Re}\,v<N+1$ and $(u,v)\notin E$, where E is the set of (u,v) at which some factor in (11.4) has a singularity. We can derive a formula for $(u,v)\in E$ as a limit case. For instance, (11.2) follows easily from the case $N=1$ of (11.4), by integrating $T_1(u,v)$ and $T_1(v,u)$ by parts K-times and taking $u\to 1/2+it$ and $v\to 1/2-it$. Explicit expressions of $H(1+it)$, and of $H(m)$ for any integer $m(\neq 1)$, can also be deduced from (11.4) (see [107I] and [107II], respectively). Letting $N\to\infty$, and then $u\to 1/2+it$ and $v\to 1/2-it$ in (11.4), we obtain

$$H\Big(\frac{1}{2}+it\Big)=\mathrm{Re}\frac{\Gamma'}{\Gamma}\Big(\frac{1}{2}+it\Big)+\gamma-\log 2\pi-2\mathrm{Re}\sum_{n=0}^{\infty}\frac{\zeta(\frac{1}{2}+n+it)-1}{\frac{1}{2}+n+it}, \qquad (11.5)$$

due originally to Andersson [1]. (The special case $t=0$ is included also in W.Zhang [217].)

Katsurada [100] presented an alternative proof of (11.4). His proof uses the Mellin-Barnes type of integrals and properties of hypergeometric functions, hence it is under the same principle as his [101]. Actually Katsurada's paper [100] treats a more general situation, that is the mean square of Lerch zeta-functions defined by the analytic continuation of $\sum_{n=0}^{\infty}e^{2\pi i\lambda n}(n+\alpha)^{-s}$, where $\alpha>0$ and λ is real. He obtained various asymptotic expansions, which includes a refinement of W.Zhang's former result [216] [218].

Next we consider the derivative case

$$H_k(s)=\int_0^1\left|\frac{d^k}{ds^k}\zeta_1(s,\alpha)\right|^2 d\alpha.$$

The case $k=1$ was studied by W.Zhang [208] and Guo [40] [41], and it is shown that

$$H_1\Big(\frac{1}{2}+it\Big)=\frac{1}{3}\log^3\left(\frac{t}{2\pi}\right)+\gamma\log^2\left(\frac{t}{2\pi}\right)+2\gamma_1\log\left(\frac{t}{2\pi}\right)+2\gamma_2+O(t^{-1}\log^2 t) \qquad (11.6)$$

in [40] [41]. On the other hand, as was first noticed by Katsurada [99III], the method based on Atkinson's dissection device is suitable to study the mean square of higher derivatives. The idea is, roughly speaking, to differentiate (11.4) k-times with respect to both u and v and analyze the resulting expression carefully. The result is that

$$\begin{aligned}H_k\Big(\frac{1}{2}+it\Big) &= \frac{1}{2k+1}\log^{2k+1}\left(\frac{t}{2\pi}\right)+\sum_{j=0}^{2k}\frac{(2k)!}{(2k-j)!}\gamma_j\log^{2k-j}\left(\frac{t}{2\pi}\right)\\ &\quad -2\mathrm{Re}\left(\frac{k!\zeta^{(k)}(\frac{1}{2}+it)}{(\frac{1}{2}+it)^{k+1}}\right)+O\Big(t^{-2}(\log t)^{2k}\Big) \qquad (11.7)\end{aligned}$$

for any $k\geq 1$, where γ_j is defined by (10.14). The case $k=1$ gives a refinement of (11.6). The formula (11.7) was announced in [107II], and the detailed proof is described in [107III].

Finally we mention the author's work [139] (already announced in [138]) on the double zeta-function

$$\zeta_2(s;\alpha,w)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha+m+nw)^{-s} \qquad (11.8)$$

of Barnes, where $\alpha > 0$, $w > 0$ be parameters. Inspired by the similarity of (11.3) and (11.8), the author introduced the generalized double zeta-function

$$\widetilde{\zeta}_2(u, v; \alpha, w) = \sum_{m=0}^{\infty} (\alpha + m)^{-u} \sum_{n=1}^{\infty} (\alpha + m + nw)^{-v},$$

and applied the method similar to that in [107I] to $\widetilde{\zeta}_2(u, v; \alpha, w)$. Then we put $u = 0$, $v = s$ to obtain the asymptotic expansion of $\zeta_2(s; \alpha, w)$ with respect to w. Certain asymptotic expansions for double gamma-functions, and for the value at $s = 1$ of Hecke L-functions of real quadratic fields, were also obtained in [139]. Recently, Katsurada [102] introduced another generalization

$$\sum_{m=0}^{\infty} (m + \alpha)^{-u} \sum_{n=0}^{\infty} (m + n + \alpha + \beta)^{-v},$$

where α, β are positive, and studied its properties by using the Mellin-Barnes type of integrals. He actually considered a more general series involving exponential factors.

The results mentioned in the last two sections show that Atkinson's method is indeed useful in a much wider area than was expected before.

The readers probably find that the recent developments in the mean square theory are really impressive. However, the mean square theory has been by no means exhausted; there remain many unsolved problems and uncultivated areas. It will still be one of the main streams in zeta-function theory, and fascinating new methods and results will surely appear in the coming century.

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