Some problems on mean values and the universality of zeta and multiple zeta-functions

Kohji Matsumoto

Abstract

Here I propose and discuss several problems on analytic properties of some zeta-functions and multiple zeta-functions, which are related to my talk or to talks of other participants at the conference.

1 Mean values of multiple zeta-functions

Recall the definition of generalized multiple zeta-functions:

$$\begin{aligned}&\zeta_r((s_1,\ldots,s_r);(\alpha_1,\ldots,\alpha_r),(w_1,\ldots,w_r))\\ &\quad=\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}(\alpha_1+m_1w_1)^{-s_1}(\alpha_2+m_1w_1+m_2w_2)^{-s_2}\\ &\qquad\times\cdots\times(\alpha_r+m_1w_1+\cdots+m_rw_r)^{-s_r},\end{aligned}\tag{1.1}$$

where $s_1,\ldots,s_r$ are complex variables, and $\alpha_1,\ldots,\alpha_r$, $w_1,\ldots,w_r$ are complex parameters. The series (1.1) was first introduced in [30] [31], and then in [32] [33], under certain assumptions on $\alpha_1,\ldots,\alpha_r$, $w_1,\ldots,w_r$, it has been shown that (1.1) can be continued meromorphically to the whole $\mathbf{C}^r$ space. Moreover, if $\alpha_1,\ldots,\alpha_r$, $w_1,\ldots,w_r$ are positive real, then (1.1) is polynomial order with respect to $\Im s_r$ (Theorem 4 of [32]). The order of the magnitude of (1.1) has been further studied by [35] [8], but how is the real order? Does the analogue of the Lindelöf hypothesis hold?

A standard way of studying those problems is to consider the mean value of (1.1). Is it possible to obtain an asymptotic formula, or at least a non-trivial upper bound, of the mean value

$$\begin{aligned}&\int_0^{T_1}\cdots\int_0^{T_r}|\zeta_r((\sigma_1+it_1,\ldots,\sigma_r+it_r);\\ &\qquad(\alpha_1,\ldots,\alpha_r),(w_1,\ldots,w_r))|^2dt_1\cdots dt_r\end{aligned}\tag{1.2}$$

in a certain "critical" region? Let us consider the simplest case $r=2$. I proved a functional equation for

$$\zeta_2(s_1,s_2)=\zeta_2((s_1,s_2);(1,2),(1,1))=\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}m_1^{-s_1}(m_1+m_2)^{-s_2}$$

by using a certain contour integral ([34]). By modifying the argument in [34] suitably, it is probably possible to deduce a certain approximate functional equation for $\zeta_2(s_1, s_2)$, which would produce various mean value results (analogous to the classical case of the Riemann zeta-function $\zeta(s)$ or Dirichlet L-functions $L(s,\chi)$) for $\zeta_2(s_1, s_2)$. It is desirable to carry out this program. For $r \geq 3$, even the functional equation is not known; to search for such an equation is clearly a big problem.

In the case of $\zeta(s)$, there are two major approaches to the mean square problem; the approximate functional equation, and the Atkinson method (see Ivić [9] [10]). How to generalize Atkinson's method to the multiple case? An essential idea of Atkinson is to consider the product $\zeta(s_1)\zeta(s_2)$, and divide it as

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1 + s_2) + \zeta_2(s_1, s_2) + \zeta_2(s_2, s_1).$$

Then the problem is reduced to the study of the double zeta-function $\zeta_2(s_1, s_2)$. Similarly, we can write the product $\zeta_2(s_1, s_2)\zeta_2(s_3, s_4)$ as a sum of several multiple zeta-functions ζ_r with $r \leq 4$. But we have already developed some basic theory of ζ_r ([32] [33] etc.). Hence we may expect that Atkinson's treatment for $\zeta_2(s_1, s_2)$ could be extended to ζ_r ($r \leq 4$) in the frame of our theory, and an explicit formula (of Atkinson type) for the mean square of ζ_2 could be deduced. Moreover the mean square problem for ζ_r ($r \geq 3$) can be reduced similarly to the study of analytic properties of ζ_k ($k \leq 2r$).

Of course various other mean values different from (1.2) may also be considered. Concerning the Hurwitz zeta-function

$$\zeta(s,\alpha) = \sum_{n=0}^{\infty} (n+\alpha)^{-s} \qquad (0 < \alpha \leq 1), \tag{1.3}$$

mean square values with respect to the parameter α have been studied in detail by Katsurada and Matsumoto [14] [15] [16] [17] and Katsurada [13]. Dr. J. Furuya pointed out that, similarly, the mean square of the multiple zeta-function (1.1) with respect to the parameters $\alpha_1, \ldots, \alpha_r$ may be considered, and it will be more accessible than (1.2).

2 Universality of multiple zeta-functions

Another interesting problem on multiple zeta-functions is to consider the universality property. Detailed accounts on the universality of $\zeta(s)$ can be found in Karatsuba-Voronin [12] or Laurinčikas [18]. Probably an accessible case is the Barnes multiple zeta-function, that is the special case $s_1 = s_2 = \cdots = s_{r-1} = 0$ in (1.1):

$$\zeta_r(s_r; \alpha_r, (w_1, \ldots, w_r)) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} (\alpha_r + m_1 w_1 + \cdots + m_r w_r)^{-s_r}. \tag{2.1}$$

This is a function of single variable s_r, and is a generalization of the Hurwitz zeta-function (1.3). It is known that (1.3) satisfies the universality property, when α is transcendental or rational (Bagchi [1], Gonek [7]). By generalizing the proof of such universality results, it is likely that we could obtain a universality theorem for the Barnes multiple zeta-function (2.1).

More challenging is to consider the universality for general multi-variable series (1.1). In this case, the statement of the result should be formulated on the $\mathbf{C}^r$ space. Again it is natural to begin with the case $r = 2$. In [34], I introduced the double Hurwitz-Lerch zeta-function of the form

$$\zeta_2((s_1, s_2); \alpha, \beta, w) = \sum_{m_1=0}^{\infty} (\alpha + m_1)^{-s_1} \sum_{m_2=1}^{\infty} e^{2\pi i m_2 \beta} (\alpha + m_1 + m_2 w)^{-s_2}. \quad (2.2)$$

Various limit theorems and universality theorems are known for the classical Lerch zeta-function

$$\zeta(s, \alpha, \beta) = \sum_{n=0}^{\infty} e^{2\pi i n \beta} (n + \alpha)^{-s}$$

(see [2] [3] [4] [5] [6] [19] [20] [21] [22] [25] [26] [28], due to Garunkštis, Laurinčikas and myself). Is it possible to prove analogous results for (2.2), by generalizing the methods developed in those papers?

3 Further problems on the universality

The theory of universality for usual (single variable, single sum) zeta and L-functions is now a very active research field. At the conference several talks were devoted to the study of various discrete limit theorems. Hence it is natural to go further and consider discrete universality theorems. In particular, it is an interesting problem to prove discrete universality theorem for L-functions attached to cusp forms. This should be achieved by modifying the idea given in [29].

Discrete limit theorems in further general situations have been proved by Kačinskaitė [11] and her subsequent papers (partly with Laurinčikas). It is desirable to obtain discrete universality theorems for this general class of zeta-functions, under suitable assumptions. A possible way is to modify the papers [23] [24] of Laurinčikas, but by combining with the idea in [29] it is likely that we could weaken the assumptions considerably (cf. [27]).

Another direction is to study discrete limit theorems or discrete universality for multiple zeta-functions. For instance, is it possible to prove such results for the Barnes multiple zeta-function (2.1), or the double Hurwitz-Lerch zeta-function (2.2)? Can one generalize the recent work of J. Ignatavičiūtė?

As a generalization of the Euler-Zagier multiple sum, multiple Dirichlet series of the form

$$\Phi_r((s_1, \ldots, s_r); (\varphi_1, \ldots, \varphi_r))$$

$$= \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{a_1(m_1)}{m_1^{s_1}} \frac{a_2(m_2)}{(m_1+m_2)^{s_2}} \cdots \frac{a_r(m_r)}{(m_1+\cdots+m_r)^{s_r}} \tag{3.1}$$

has been introduced in [35], where

$$\varphi_j(s) = \sum_{n=1}^{\infty} a_j(n) n^{-s} \qquad (1 \le j \le r)$$

are usual Dirichlet series. Assume $\varphi_j(s)$ is (for all j) convergent absolutely in the region $\Re s > \alpha_j (> 0)$, can be continued meromorphically to the whole plane, holomorphic except for the pole (of order at most 1) at $s = \alpha_j$, and of polynomial order with respect to $\Im s$. Then in [35] it is shown that the multiple series (3.1) can be continued to the whole $\mathbf{C}^r$ space. Consider the special case that $\varphi_j(s)$ $(1 \le j \le r)$ are L-functions attached to cusp forms. Then, is it possible to apply the idea in [29] to obtain some universality (or discrete universality) result for this multiple series? We may also consider various other interesting special cases of (3.1). In which case can one prove a kind of universality?

References

[1] B. Bagchi, The statistical behaviour and universality properties of the Riemann zeta function and other allied Dirichlet series, Thesis, Indian Statistical Institute, Calcutta, 1981.

[2] R. Garunkštis, An explicit form of limit distribution with weight for the Lerch zeta-function in the space of analytic functions, Liet. Mat. Rink. **37** (1997), 309-326 (= Lith. Math. J. **37** (1997), 230-242).

[3] R. Garunkštis, The universality theorem with weight for the Lerch zeta-function, in "Analytic and Probabilistic Methods in Number Theory", Proc. 2nd Palanga Intern Conf., A. Laurinčikas et al. (eds.), New Trends in Probab. and Statist. Vol. 4, VSP/TEV, 1997, pp. 59-67.

[4] R. Garunkštis and A. Laurinčikas, On the Lerch zeta-function, Liet. Mat. Rink. **36** (1996), 423-434 (= Lith. Math. J. **36** (1996), 337-346).

[5] R. Garunkštis and A. Laurinčikas, Limit theorem with weight for the Lerch zeta function in the space of analytic functions, Trudy Mat. Inst. Steklov. **218** (1997), 109-121 (= Proc. Steklov Inst. Math. **218** (1997), 104-116).

[6] R. Garunkštis and A. Laurinčikas, On one Hilbert's problem for the Lerch zeta-function, Publ. Inst. Math. Beograd **65**(79) (1999), 63-68.

[7] S. M. Gonek, Analytic properties of zeta and L-functions, Thesis, Univ. of Michigan, 1979.

[8] H. Ishikawa, K. Matsumoto and Y. Tanigawa, in preparation.

[9] A. Ivić, The Riemann Zeta-Function, Wiley, 1985.

[10] A. Ivić, Mean Values of the Riemann Zeta Function, Lectures on Math. and Phys. Vol. 82, Tata Inst. Fund. Res., Springer, 1991.

[11] R. Kačinskaitė, A discrete limit theorem for the Matsumoto zeta-function on the complex plane, Liet. Mat. Rink. **40** (2000), 475-492 (= Lith. Math. J. **40** (2000), 364-378).

[12] A. A. Karatsuba and S. M. Voronin, The Riemann Zeta-Function, Walter de Gruyter, 1992.

[13] M. Katsurada, An application of Mellin-Barnes' type integrals to the mean square of Lerch zeta-functions, Collect. Math. **48** (1997), 137-153.

[14] M. Katsurada and K. Matsumoto, Disctere mean values of Hurwitz zeta-functions, Proc. Japan Acad. **69A** (1993), 164-169.

[15] M. Katsurada and K. Matsumoto, Explicit formulas and asymptotic expansions for certain mean square of Hurwitz zeta-functions I, Math. Scand. **78** (1996), 161-177.

[16] M. Katsurada and K. Matsumoto, Explicit formulas and asymptotic expansions for certain mean square of Hurwitz zeta-functions II, in "Analytic and Probabilistic Methods in Number Theory", Proc. 2nd Palanga Intern Conf., A. Laurinčikas et al. (eds.), New Trends in Probab. and Statist. Vol. 4, VSP/TEV, 1997, pp. 119-134.

[17] M. Katsurada and K. Matsumoto, Explicit formulas and asymptotic expansions for certain mean square of Hurwitz zeta-functions III, Compositio Math., to appear.

[18] A. Laurinčikas, Limit Theorems for the Riemann Zeta-Function, Kluwer, 1996.

[19] A. Laurinčikas, On limit distribution of the Lerch zeta-function, in "Analytic and Probabilistic Methods in Number Theory", Proc. 2nd Palanga Intern Conf., A. Laurinčikas et al. (eds.), New Trends in Probab. and Statist. Vol. 4, VSP/TEV, 1997, pp. 135-148.

[20] A. Laurinčikas, A limit theorem for the Lerch zeta-function in the space of analytic functions, Liet. Mat. Rink. **37** (1997), 191-203 (= Lith. Math. J. **37** (1997), 146-155).

[21] A. Laurinčikas, The universality of the Lerch zeta-function, Liet. Mat. Rink. **37** (1997), 367-375 (= Lith. Math. J. **37** (1997), 275-280).

[22] A. Laurinčikas, On the Lerch zeta-function with rational parameters, Liet. Mat. Rink. **38** (1998), 113-124 (= Lith. Math. J. **38** (1998), 89-97).

[23] A. Laurinčikas, On the Matsumoto zeta-function, Acta Arith. **84** (1998), 1-16.

[24] A. Laurinčikas, On the zeros of linear combinations of the Matsumoto zeta-functions, Liet. Mat. Rink. **38** (1998), 185-204 (= Lith. Math. J. **38** (1998), 144-159).

[25] A. Laurinčikas, On the effectivization of the universality theorem for the Lerch zeta-function, Liet. Mat. Rink. **40** (2000), 172-178 (= Lith. Math. J. **40** (2000), 135-139).

[26] A. Laurinčikas and K. Matsumoto, Joint value-distribution theorems on Lerch zeta-functions, Liet. Mat. Rink. **38** (1998), 312-326 (= Lith. Math. J. **38** (1998), 238-249).

[27] A. Laurinčikas and K. Matsumoto, On estimation of the number of zeros of linear combinations of certain zeta-functions, Liet. Mat. Draug. Mokslo Darbai II Tomas, Vilnius "Technika" (1998), 43-48.

[28] A. Laurinčikas and K. Matsumoto, The joint universality and the functional independence for Lerch zeta-functions, Nagoya Math. J. **157** (2000), 211-227.

[29] A. Laurinčikas and K. Matsumoto, The universality of zeta-functions attached to certain cusp forms, Acta Arith. **98** (2001), 345-359.

[30] K. Matsumoto, Asymptotic expansions of double zeta-functions of Barnes, of Shintani, and Eisenstein series, preprint.

[31] K. Matsumoto, On analytic continuation of various multiple zeta-functions, in "Number Theory for the Millenniam, the Millennial Conference on Number Theory", B. Berndt et al. (eds.), A K Peters, to appear.

[32] K. Matsumoto, The analytic continuation and the asymptotic behaviour of multiple zeta-functions I, preprint.

[33] K. Matsumoto, The analytic continuation and the asymptotic behaviour of multiple zeta-functions II, preprint.

[34] K. Matsumoto, Functional equations for double zeta-functions, preprint.

[35] K. Matsumoto and Y. Tanigawa, The analytic continuation and the order estimate of multiple Dirichlet series, in preparation.

Graduate Schoolof Mathematics
Nagoya University
Chikusa-ku, Nagoya 464-8602
Japan