THE UNIVERSALITY OF ZETA-FUNCTIONS ATTACHED TO CERTAIN CUSP FORMS

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§1. Introduction

The universality property was first discovered by Voronin [21] in the case of the Riemann zeta-function. Denote by $\mathbb{C}$ the complex plane, $s = \sigma + it$ a complex variable, $\zeta(s)$ the Riemann zeta-function, and $\mathrm{meas}\{A\}$ the Lebesgue measure of the set A. We use the notation

$$\nu_T(\ldots) = T^{-1}\,\mathrm{meas}\{\tau \in [0,T]; \ldots\}$$

for $T > 0$, where in place of dots we write a condition satisfied by τ. The modern statement of Voronin's universality theorem is as follows (see Chapter 6 of [12]):

Let K be a compact subset of the strip $\{s \in \mathbb{C}; 1/2 < \sigma < 1\}$ with connected complement. Let $f(s)$ be a non-vanishing continuous function on K which is analytic in the interior of K. Then for any $\varepsilon > 0$,

$$\liminf_{T\to\infty} \nu_T\left(\sup_{s\in K}\left|\zeta(s+i\tau) - f(s)\right| < \varepsilon\right) > 0.$$

This remarkable result raised much attention among specialists, and Reich [18], [19], Gonek [4], Good [5], Bagchi [1], [2], and the first author [8]-[12], [14] improved and generalized the Voronin theorem to various other Dirichlet series including Dirichlet L, and Dedekind, Hurwitz, and Lerch zeta-functions.

It is the purpose of the present paper to prove the universality theorem for zeta-functions attached to certain cusp forms. Let $F(z)$ be a holomorphic cusp form of weight κ for the full modular group $SL(2,\mathbb{Z})$, and assume that $F(z)$ is a normalized eigenform. Then $F(z)$ has the Fourier series expansion

$$F(z) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n z}, \qquad c(1) = 1.$$

A classical result of Hecke [6] says that the Dirichlet series

$$\varphi(s,F) = \sum_{n=1}^{\infty} c(n) n^{-s}$$

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is absolutely convergent in $\sigma > (\kappa+1)/2$, and can be continued analytically to an entire function. Moreover it satisfies the functional equation

$$(2\pi)^{-s}\Gamma(s)\varphi(s,F) = (-1)^{\kappa/2}(2\pi)^{s-\kappa}\Gamma(\kappa-s)\varphi(\kappa-s,F),$$

which implies that the critical strip for $\varphi(s,F)$ is $(\kappa-1)/2 \leqslant \sigma \leqslant (\kappa+1)/2$. Let $D = \{s \in \mathbb{C};\ \kappa/2 < \sigma < (\kappa+1)/2\}$. Then we shall prove

Theorem. *Let $F(z)$ be a normalized eigenform of weight κ for $SL(2,\mathbb{Z})$. Let K be a compact subset of D with connected complement, and let $f(s)$ be a non-vanishing continuous function on K which is analytic in the interior of K. Then for any $\varepsilon > 0$ we have*

$$\liminf_{T\to\infty} \nu_T\left(\sup_{s\in K} \left|\varphi(s+i\tau,F) - f(s)\right| < \varepsilon \right) > 0.$$

Before the present work, the universality of $\varphi(s,F)$ was obtained by Kačėnas-Laurinčikas [7], and also as a special case of the theorem given in [15], but both papers require rather strong assumptions. For instance, the universality theorem of Kačėnas-Laurinčikas [7] is proved under the assumption of the existence of $\eta > 0$ such that

$$\sum_{\substack{p:\ \text{prime} \\ |c_p|<\eta}} p^{-\delta} < \infty \tag{1.1}$$

for $\delta > 1/2$, where $c_p = c(p)p^{(1-\kappa)/2}$. However it seems to be hopeless to verify (1.1). Now, our theorem assures the universality property of $\varphi(s,F)$ unconditionally.

Bagchi [1] gave a new proof of the universality theorem for $\zeta(s)$, which is presented in Chapter 6 of [12]. In this paper we apply Bagchi's method to $\varphi(s,F)$, but some new ideas are necessary to complete the proof. A key lemma of Bagchi's method is Theorem 6.4.14 of [12], whose proof is based on the well-known fact (see, e.g., [16])

$$\sum_{p\leqslant x} \frac{1}{p} = \log\log x + c_1 + O\big(\exp(-c_2\sqrt{\log x})\big) \tag{1.2}$$

for $x > 1$, with some constants c_1 and $c_2 > 0$. If we try to apply Bagchi's method directly to our case, we need the corresponding asymptotic result for the sum

$$\sum_{p\leqslant x} \frac{|c_p|}{p},$$

but it is quite difficult to obtain such a formula. Instead, we use the asymptotic formula

$$\sum_{p\leqslant x} c_p^2 = \pi(x)\big(1+o(1)\big), \qquad x\to\infty \tag{1.3}$$

which is equivalent to Theorem 2 of Rankin [17]. Here, $\pi(x)$ denotes the number of primes up to x. From (1.3), we can deduce a vanishing lemma. This is Lemma 6 stated in Section 3, and plays an essential role in our argument. The proof of Lemma 6 will be given in Section 4, and this is the most novel part of the present paper. From Lemma 6 we can obtain Lemma 2, which corresponds to Lemma 6.5.4 of [12]. The deduction of our theorem from Lemma 2 is essentially the same as Bagchi's argument.

§2 A limit theorem for the function $\varphi(s,F)$.

Since $F(z)$ is a normalized eigenform, the function $\varphi(s,F)$ for $\sigma > (\kappa+1)/2$ has the Euler product expansion

$$\varphi(s,F) = \prod_p \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)}{p^s}\right)^{-1},$$

with

$$c(p) = \alpha(p) + \beta(p) \tag{2.1}$$

and

$$\big|\alpha(p)\big| \leqslant p^{(\kappa-1)/2}, \qquad \big|\beta(p)\big| \leqslant p^{(\kappa-1)/2} \tag{2.2}$$

(Deligne [3]). From (2.1) and (2.2) we have

$$|c_p| \leqslant 2. \tag{2.3}$$

Let $N > 0$, $D_N = \{s \in \mathbb{C}; \kappa/2 < \sigma < (\kappa+1)/2, |t| < N\}$, and denote by $H(D_N)$ the space of analytic on D_N functions equipped with the topology of uniform convergence on compacta. Let $\mathcal{B}(S)$ stand for the class of Borel sets of the space S. Define on $\big(H(D_N), \mathcal{B}(H(D_N))\big)$ the probability measure

$$P_T(A) = \nu_T\big(\varphi(s+i\tau, F) \in A\big), \qquad A \in \mathcal{B}(H(D_N)).$$

For our purpose we need a limit theorem in the sense of the weak convergence of probability measures for P_T as $T \to \infty$, with an explicit form of the limit measure. Let $\gamma = \{s \in \mathbb{C}; |s| = 1\}$, and let

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for all primes p. The infinite dimensional torus Ω is a compact topological Abelian group. Denote by m_H the probability Haar measure on $\big(\Omega, \mathcal{B}(\Omega)\big)$; thus we obtain the probability space $\big(\Omega, \mathcal{B}(\Omega), m_H\big)$. Let $\omega(p)$ be the projection of $\omega \in \Omega$ to the coordinate space γ_p , and define the $H(D_N)$-valued random element $\varphi(s,\omega,F)$ on $\big(\Omega,\mathcal{B}(\Omega),m_H\big)$ by the formula

$$\varphi(s,\omega,F)=\prod_p\left(1-\frac{\alpha(p)\omega(p)}{p^s}\right)^{-1}\left(1-\frac{\beta(p)\omega(p)}{p^s}\right)^{-1}$$

for $s\in D_N$. Denote by P_φ the distribution of the random element $\varphi(s,\omega,F)$, i.e.

$$P_\varphi(A)=m_H\big(\omega\in\Omega;\ \varphi(s,\omega,F)\in A\big),\qquad A\in\mathcal{B}(H(D_N)).$$

Then we have

Lemma 1. *The probability measure* P_T *converges weakly to* P_φ *as* $T\to\infty$ *.*

Kačėnas-Laurinčikas [7] proved this limit theorem on the space $H(\tilde{D})$, where $\tilde{D}=\{s\in\mathbb{C};\ \sigma>\kappa/2\}$, and from which Lemma 1 follows immediately. Lemma 1 can also be regarded as a special case of the result proved in [13].

§3. A denseness lemma.

Let, for $|z|<1$,

$$\log(1+z)=z-\frac{z^2}{2}+\frac{z^3}{3}-\dots,$$

and define

$$f_p(s)=f_p(s;a_p)=-\log\left(1-\frac{\alpha(p)a_p}{p^s}\right)-\log\left(1-\frac{\beta(p)a_p}{p^s}\right)$$

for $s\in D_N$ and $a_p\in\gamma$. We shall prove

Lemma 2. *The set of all convergent series*

$$\sum_p f_p(s;a_p)$$

is dense in $H(D_N)$ *.*

In the proof of this lemma we will use the following three lemmas.

Lemma 3. *Let* $\{z_m\}$ *be a sequence of complex numbers such that*

$$\sum_{m=1}^{\infty}|z_m|^2<\infty.$$

Let $\{\varepsilon_m\}$ *be a sequence of independent random variables on a certain probability space* $\big(S,\mathcal{B}(S),\mathbb{P}\big)$ *, such that*

$$\mathbb{P}(\varepsilon_m=1)=\mathbb{P}(\varepsilon_m=-1)=\frac{1}{2}$$

for any m *. Then the series*

$$\sum_{m=1}^{\infty} \varepsilon_m z_m$$

converges almost surely.

The assertion of this lemma is included in the proof of Lemma 6.5.3 of [12].

Lemma 4. *Let* $\{f_m\}$ *be a sequence in* $H(D_N)$ *which satisfies:*

(a) If μ *is a complex measure on* $\big(\mathbb{C}, \mathcal{B}(\mathbb{C})\big)$ *with compact support contained in* D_N *such that*

$$\sum_{m=1}^{\infty} \left| \int\limits_{\mathbb{C}} f_m \, d\mu \right| < \infty,$$

then

$$\int\limits_{\mathbb{C}} s^r \, d\mu(s) = 0$$

for any non-negative integer r *.*

(b) The series

$$\sum_{m=1}^{\infty} f_m$$

converges in $H(D_N)$ *.*

(c) For any compact $K \subset D_N$ *,*

$$\sum_{m=1}^{\infty} \sup_{s \in K} \big|f_m(s)\big|^2 < \infty.$$

Then the set of all convergent series

$$\sum_{m=1}^{\infty} a_m f_m, \qquad a_m \in \gamma,$$

is dense in $H(D_N)$ *.*

Lemma 5. *Let* μ *be a complex measure on* $\big(\mathbb{C}, \mathcal{B}(\mathbb{C})\big)$ *with compact support contained in the half-plane* $\sigma > \sigma_0$ *, and let*

$$f(z) = \int\limits_{\mathbb{C}} e^{sz} \, d\mu(s).$$

If $f(z) \not\equiv 0$ *, then*

$$\limsup_{r \to \infty} \frac{\log |f(r)|}{r} > \sigma_0.$$

Both of Lemmas 4 and 5 are due to Bagchi [1]. For the proofs, see Theorem 6.3.10 and Lemma 6.4.10, respectively, of [12].

Now we start the proof of Lemma 2, which we divide into three steps.

The first step. Let

$$\tilde{f}_p = \tilde{f}_p(s) = -\log\left(1 - \frac{\alpha(p)}{p^s}\right) - \log\left(1 - \frac{\beta(p)}{p^s}\right),$$

and let $p_0 > 0$. Define

$$\widehat{f}_p = \widehat{f}_p(s) = \begin{cases} \tilde{f}_p(s) & \text{if } p > p_0, \\ 0 & \text{if } p \leqslant p_0. \end{cases}$$

We claim that there exists a sequence $\{\widehat{a}_p; \widehat{a}_p \in \gamma\}$ such that the series

$$\sum_p \widehat{a}_p \widehat{f}_p \tag{3.1}$$

converges in $H(D_N)$.

To prove this claim, we observe that in view of (2.1) and (2.2)

$$\tilde{f}_p(s) = \frac{\alpha(p) + \beta(p)}{p^s} + r_p(s) = \frac{c(p)}{p^s} + r_p(s)$$

with

$$r_p(s) = O(p^{\kappa - 2\sigma - 1}). \tag{3.2}$$

The series

$$\sum_p r_p(s) \tag{3.3}$$

converges uniformly on any compact subset of D_N. Next, let $\{\sigma(j)\}$ be a sequence of real numbers, $\sigma(1) > \sigma(2) > \dots$ and $\sigma(j) \to \kappa/2$ as $j \to \infty$. For each j, the series

$$\sum_p \varepsilon_p c(p) p^{-\sigma(j)}$$

converges almost surely by Lemma 3. Hence we can find a sequence $\{\widehat{a}_p; \widehat{a}_p = \pm 1\}$ such that

$$\sum_p \widehat{a}_p c(p) p^{-\sigma(j)}$$

converges for any j. By a well-known property of Dirichlet series,

$$\sum_p \widehat{a}_p c(p) p^{-s}$$

converges uniformly on any compact subset of D_N . This and the convergence of (3.3) imply our claim on the series (3.1).

The second step. We now claim that the set of all convergent series

$$\sum_p a_p \widehat{f}_p, \qquad a_p \in \gamma \tag{3.4}$$

is dense in $H(D_N)$. For this purpose we apply Lemma 4. Obviously it suffices to show that the set of all convergent series

$$\sum_p a_p g_p, \qquad a_p \in \gamma, \tag{3.5}$$

is dense in $H(D_N)$, where $g_p = \widehat{a}_p \widehat{f}_p$.

We have already shown that the series

$$\sum_p g_p$$

converges in $H(D_N)$. Also it is easy to see that

$$\sum_p \sup_{s \in K} \left| g_p(s) \right|^2 < \infty$$

for any compact subset $K \subset D_N$. Thus it remains to verify the condition (a) of Lemma 4.

Let μ be a complex measure on $\big(\mathbb{C}, \mathcal{B}(\mathbb{C})\big)$ with compact support contained in D_N such that

$$\sum_p \left| \int_{\mathbb{C}} g_p(s)\, d\mu(s) \right| < \infty. \tag{3.6}$$

We put $h_p(s) = \widehat{a}_p c(p) p^{-s}$. Then in virtue of (3.2) we have

$$\sum_p \sup_{s \in K} \left| g_p(s) - h_p(s) \right| < \infty.$$

From this and (3.6)

$$\sum_p \left| \int_{\mathbb{C}} h_p(s)\, d\mu(s) \right| < \infty,$$

so

$$\sum_p |c(p)| \left| \int_{\mathbb{C}} p^{-s}\, d\mu(s) \right| < \infty. \tag{3.7}$$

Let $D_{1,N} = \{s \in \mathbb{C};\ 1/2 < \sigma < 1, |t| < N\}$ and let $h(s) = s - (\kappa - 1)/2$. Then

$$\mu h^{-1}(A) = \mu\big(h^{-1}(A)\big), \qquad A \in \mathcal{B}(\mathbb{C}),$$

is a complex measure with compact support contained in $D_{1,N}$. From (3.7) it follows that

$$\sum_p |c_p| \left| \int_{\mathbb{C}} p^{-s}\, d\mu h^{-1}(s) \right| < \infty. \tag{3.8}$$

(Recall $c_p = c(p)p^{(1-\kappa)/2}$.) Define

$$\varrho(z) = \int_{\mathbb{C}} e^{-sz}\, d\mu h^{-1}(s), \qquad z \in \mathbb{C}.$$

Then (3.8) can be written as

$$\sum_p |c_p| \big|\varrho(\log p)\big| < \infty. \tag{3.9}$$

From (3.9) we can deduce

Lemma 6. $\varrho(z) \equiv 0$.

The proof of this fact is the most novel part of the present paper, and will be given in the next section.

Let r be a non-negative integer. Differentiating r-times the equality $\varrho(z) \equiv 0$ with respect to z, and then putting $z = 0$, we see that

$$\int_{\mathbb{C}} s^r\, d\mu h^{-1}(s) = 0,$$

hence

$$\int_{\mathbb{C}} s^r\, d\mu(s) = 0.$$

Consequently we find that all hypotheses of Lemma 4 are satisfied, and we obtain the denseness of the set of all convergent series (3.5), hence (3.4).

The third step. Let $x_0(s) \in H(D_N)$, K be a compact subset of D_N, and $\varepsilon > 0$. We choose a p_0 for which

$$\sup_{s \in K} \left(\sum_{p > p_0} \sum_{l=2}^{\infty} \frac{|\alpha(p)|^l + |\beta(p)|^l}{l p^{l\sigma}} \right) < \frac{\varepsilon}{4} \tag{3.10}$$

holds. By the claim proved in the second step we find a sequence $\{\tilde{a}_p; \tilde{a}_p \in \gamma\}$ such that

$$\sup_{s \in K} \left| x_0(s) - \sum_{p \leqslant p_0} \tilde{f}_p(s) - \sum_{p > p_0} \tilde{a}_p \tilde{f}_p(s) \right| < \frac{\varepsilon}{2}. \tag{3.11}$$

We put

$$a_p = \begin{cases} 1 & \text{if } p \leqslant p_0, \\ \tilde{a}_p & \text{if } p > p_0. \end{cases}$$

Then (3.10) and (3.11) yield

$$\begin{aligned} &\sup_{s\in K}\left| x_0(s) - \sum_p f_p(s;a_p)\right| \\ &\leqslant \sup_{s\in K}\left| x_0(s) - \sum_{p\leqslant p_0} \tilde{f}_p(s) - \sum_{p>p_0} \tilde{a}_p \tilde{f}_p(s)\right| \\ &+ \sup_{s\in K}\left| \sum_{p>p_0} \tilde{a}_p \tilde{f}_p(s) - \sum_{p>p_0} f_p(s;a_p)\right| \\ &< \frac{\varepsilon}{2} + 2\sup_{s\in K}\left(\sum_{p>p_0}\sum_{l=2}^{\infty} \frac{|\alpha(p)|^l + |\beta(p)|^l}{lp^{l\sigma}} \right) < \varepsilon. \end{aligned}$$

Therefore the proof of Lemma 2 is now reduced to the validity of Lemma 6.

§4. Proof of Lemma 6.

An essential ingredient of the proof is the following

Lemma 7. Let $f(s)$ be an entire function of exponential type, and let $\{\lambda_m\}$ be a sequence of complex numbers. Let α, β and δ be positive real numbers such that

$$\begin{aligned} &(a) \qquad \limsup_{y\to\infty} \frac{\log|f(\pm iy)|}{y} \leqslant \alpha, \\ &(b) \qquad |\lambda_m - \lambda_n| \geqslant \delta|m-n|, \\ &(c) \qquad \lim_{m\to\infty} \frac{\lambda_m}{m} = \beta, \\ &(d) \qquad \alpha\beta < \pi. \end{aligned}$$

Then

$$\limsup_{m\to\infty} \frac{\log|f(\lambda_m)|}{|\lambda_m|} = \limsup_{r\to\infty} \frac{\log|f(r)|}{r}.$$

This is a variant of the Bernstein theorem, and is given as Theorem 6.4.12 of [12] with a proof.

To prove Lemma 6, we apply Lemma 7 with $f = \varrho$. Since the support of the measure μh^{-1} is included in $D_{1,N}$, we see that

$$\left|\varrho(\pm iy)\right| \leqslant e^{Ny} \int_{\mathbb{C}} \left|d\mu h^{-1}(s)\right|$$

for $y > 0$, hence we can take $\alpha = N$ in the condition (a) of Lemma 7. Let us take a fixed positive number β satisfying

$$\beta < \frac{\pi}{N}\,. \tag{4.1}$$

Consider the set A of all positive integers m, such that there exists a real number $r \in \big((m-1/4)\beta, (m+1/4)\beta\big]$ with $|\varrho(r)| \leqslant e^{-r}$.

We fix a number μ, satisfying $0 < \mu < 1$, and put

$$\mathcal{P}_\mu = \big\{p;\ \text{primes},\ |c_p| > \mu\big\}.$$

Then from (3.9) it follows that

$$\sum_{p\in\mathcal{P}_\mu} \big|\varrho(\log p)\big| < \infty. \tag{4.2}$$

On the other hand, we have

$$\begin{aligned}\sum_{p\in\mathcal{P}_\mu} \big|\varrho(\log p)\big| &\geqslant \sum_{m\notin A} \sum\nolimits_m' \big|\varrho(\log p)\big| \\ &\geqslant \sum_{m\notin A} \sum\nolimits_m' p^{-1},\end{aligned} \tag{4.3}$$

where $\sum_m'$ denotes the sum running over all primes $p \in \mathcal{P}_\mu$ satisfying $(m-1/4)\beta < \log p \leqslant (m+1/4)\beta$. Therefore, putting

$$a = \exp\big((m-1/4)\beta\big), \qquad b = \exp\big((m+1/4)\beta\big),$$

from (4.2) and (4.3) we obtain

$$\sum_{m\notin A} \sum_{\substack{p\in\mathcal{P}_\mu\\ a<p\leqslant b}} p^{-1} < \infty. \tag{4.4}$$

Let $\pi_\mu(x)$ be the number of primes $p \in \mathcal{P}_\mu$ up to x. Then, using (2.3), we have, for $a \leqslant u \leqslant b$,

$$\begin{aligned}\sum_{a<p\leqslant u} c_p^2 &\leqslant 4 \sum_{\substack{p\in\mathcal{P}_\mu\\ a<p\leqslant u}} 1 + \mu^2 \sum_{\substack{p\notin\mathcal{P}_\mu\\ a<p\leqslant u}} 1 \\ &= 4\big(\pi_\mu(u) - \pi_\mu(a)\big) + \mu^2\Big(\big(\pi(u) - \pi_\mu(u)\big) - \big(\pi(a) - \pi_\mu(a)\big)\Big) \\ &= (4-\mu^2)\big(\pi_\mu(u) - \pi_\mu(a)\big) + \mu^2\big(\pi(u) - \pi(a)\big).\end{aligned} \tag{4.5}$$

On the other hand, by Rankin's formula (1.3), we have

$$\sum_{a<p\leqslant u} c_p^2 = \pi(u)\big(1+o(1)\big) - \pi(a)\big(1+o(1)\big) \tag{4.6}$$

as $m \to \infty$.

We fix a positive parameter δ satisfying $1+\delta < e^{\beta/2}$, and let $0 < \varepsilon < (\frac{1}{100})\delta$. If $m \geqslant m_0(\varepsilon)$, then, for any $u \geqslant a(1+\delta)$, we obtain

$$\begin{aligned} \pi(u)\big(1+o(1)\big) &\geqslant \pi(u)(1-\varepsilon), \\ \pi(a)\big(1+o(1)\big) &\leqslant \pi(a)(1+\varepsilon). \end{aligned}$$

Hence

$$\pi(u)\big(1+o(1)\big) - \pi(a)\big(1+o(1)\big) \geqslant \big(\pi(u)-\pi(a)\big) - \varepsilon\big(\pi(u)+\pi(a)\big). \tag{4.7}$$

Since $u \geqslant a(1+\delta)$, we have, for $m \geqslant m_0(\varepsilon)$,

$$\begin{aligned} \pi(u)-\pi(a) &\geqslant \frac{u}{\log u}(1-\varepsilon) - \frac{a}{\log a}(1+\varepsilon) \\ &\geqslant \frac{a(1+\delta)}{\log a + \log(1+\delta)}(1-\varepsilon) - \frac{a}{\log a}(1+\varepsilon) \\ &\geqslant \frac{a}{\log a}(1+\delta)(1-2\varepsilon) - \frac{a}{\log a}(1+\varepsilon) \\ &\geqslant \frac{a}{\log a}(\delta - 4\varepsilon) \geqslant \frac{a}{\log a}\frac{\delta}{2}. \end{aligned} \tag{4.8}$$

On the other hand, if $u \leqslant b = Ba$ where $B = e^{\beta/2}$, then, for $m \geqslant m_0(\varepsilon)$,

$$\begin{aligned} \pi(u)+\pi(a) \leqslant \pi(b)+\pi(a) &\leqslant \frac{b}{\log b}(1+\varepsilon) + \frac{a}{\log a}(1+\varepsilon) \\ &\leqslant \frac{Ba}{\log a}(1+\varepsilon)^2 + \frac{a}{\log a}(1+\varepsilon) \leqslant \frac{a}{\log a}(2B+2). \end{aligned}$$

Therefore this and (4.8) yield

$$\pi(u)+\pi(a) \leqslant \frac{4B+4}{\delta}\big(\pi(u)-\pi(a)\big).$$

From this and (4.7) we find that for the same u as above and $m \to \infty$

$$\begin{aligned} \pi(u)\big(1+o(1)\big) - \pi(a)\big(1+o(1)\big) &\geqslant \pi(u)-\pi(a) - \varepsilon\frac{4B+4}{\delta}\big(\pi(u)-\pi(a)\big) \\ &= \big(\pi(u)-\pi(a)\big)\big(1+o(1)\big). \end{aligned}$$

Hence, by (4.5) and (4.6), we find

$$\big(\pi(u)-\pi(a)\big)\big(1+o(1)\big) \leqslant (4-\mu^2)\big(\pi_\mu(u)-\pi_\mu(a)\big) + \mu^2\big(\pi_\mu(u)-\pi_\mu(a)\big),$$

so

$$\pi_\mu(u)-\pi_\mu(a) \geqslant \frac{1-\mu^2}{4-\mu^2}\big(\pi(u)-\pi(a)\big)\big(1+o(1)\big)$$

for $u \geqslant a(1+\delta)$, $m \to \infty$. Therefore, using partial summation,

$$\begin{aligned}
\sum_{\substack{p\in\mathcal{P}_\mu \\ a<p\leqslant b}} \frac{1}{p} &= \Big(\sum_{\substack{p\in\mathcal{P}_\mu \\ a<p\leqslant b}} 1\Big) + \int_a^b \Big(\sum_{\substack{p\in\mathcal{P}_\mu \\ a<p\leqslant b}}\Big)\frac{du}{u^2} \\
&= \big(\pi_\mu(b)-\pi_\mu(a)\big)\frac{1}{b} + \int_a^b \big(\pi_\mu(u)-\pi_\mu(a)\big)\frac{du}{u^2} \\
&\geqslant \big(\pi_\mu(b)-\pi_\mu(a)\big)\frac{1}{b} + \int_{a(1+\delta)}^b \big(\pi_\mu(u)-\pi_\mu(a)\big)\frac{du}{u^2} \\
&\geqslant \frac{1-\mu^2}{4-\mu^2}\Bigg(\big(\pi(b)-\pi(a)\big)\frac{1}{b} + \int_{a(1+\delta)}^b \big(\pi(u)-\pi(a)\big)\frac{du}{u^2}\Bigg)\big(1+o(1)\big) \qquad (4.9) \\
&\geqslant \frac{1-\mu^2}{4-\mu^2}\Bigg(\big(\pi(b)-\pi(a(1+\delta))\big)\frac{1}{b} \\
&\quad + \int_{a(1+\delta)}^b \big(\pi(u)-\pi(a(1+\delta))\big)\frac{du}{u^2}\Bigg)\big(1+o(1)\big) \\
&= \frac{1-\mu^2}{4-\mu^2}\Big(\sum_{a(1+\delta)<p\leqslant b}\frac{1}{p}\Big)\big(1+o(1)\big)
\end{aligned}$$

as $m \to \infty$.

From (1.2) it follows that, for $m \to \infty$,

$$\sum_{a(1+\delta)<p\leqslant b}\frac{1}{p} = \Big(\frac{1}{2} - \frac{\log(1+\delta)}{\beta}\Big)\frac{1}{m} + O\Big(\frac{1}{m^2}\Big),$$

hence and from (4.9)

$$\sum_{\substack{p\in\mathcal{P}_\mu \\ a<p\leqslant b}}\frac{1}{p} \geqslant \frac{1-\mu^2}{4-\mu^2}\Big(\frac{1}{2} - \frac{\log(1+\delta)}{\beta}\Big)\frac{1}{m}\big(1+o(1)\big) + O\Big(\frac{1}{m^2}\Big). \qquad (4.10)$$

Since $0 < \mu < 1$ and $1 + \delta < e^{\beta/2}$, we see that

$$\frac{1-\mu^2}{4-\mu^2}\left(\frac{1}{2} - \frac{\log(1+\delta)}{\beta}\right) > 0.$$

Therefore, from (4.4) and (4.10), we obtain

$$\sum_{m \notin A} \frac{1}{m} < \infty. \tag{4.11}$$

We write

$$A = \{a_m;\, m = 1, 2, \ldots\}, \qquad a_1 < a_2 < \ldots .$$

Then from (4.11) we can easily show that

$$\lim_{m\to\infty} \frac{a_m}{m} = 1.$$

By the definition of the set A, there exists a sequence $\{\lambda_m\}$ such that

$$\left(a_m - \frac{1}{4}\right)\beta < \lambda_m \leqslant \left(a_m + \frac{1}{4}\right)\beta$$

and

$$\left|\varrho(\lambda_m)\right| \leqslant \exp(-\lambda_m).$$

Then

$$\lim_{m\to\infty} \frac{\lambda_m}{m} = \beta,$$

and

$$\limsup_{m\to\infty} \frac{\log|\varrho(\lambda_m)|}{\lambda_m} \leqslant -1\,.$$

Now by Lemma 7 we have

$$\limsup_{r\to\infty} \frac{\log|\varrho(r)|}{r} \leqslant -1\,. \tag{4.12}$$

Assume $\varrho(z) \not\equiv 0$. We can write

$$\varrho(s) = \int_{\mathbb{C}} e^{sz} d\nu(s),$$

where the measure ν is defined by $\nu(A) = \mu h^{-1}(-A), A \in \mathcal{B}(\mathbb{C}),$ so its support is included in $\{s \in \mathbb{C}; -1 < \sigma < -1/2\}$. Hence, by Lemma 5, we get

$$\limsup_{r\to\infty} \frac{\log|\varrho(r)|}{r} > -1,$$

which contradicts (4.12). Therefore we conclude that $\varrho(z) \equiv 0$, which is the assertion of Lemma 6. The proof of Lemma 2 is now completed.

§5. The support of the measure P_φ .

Now we can deduce our theorem from Lemma 2 in much the same way as described in Section 6.5 of [12]. In this section we determine the support of the measure P_φ defined in Section 2. Let

$$S_N = \big\{ f \in H(D_N);\ f(s) \neq 0 \text{ for any } s \in D_N,\ \text{or } f(s) \equiv 0 \big\}.$$

Lemma 8. *The support of the measure P_φ is the set S_N .*

In order to deduce this lemma from Lemma 2, we need two more lemmas:

Lemma 9. *Let $\{f_n(s)\}$ be a sequence of functions analytic on D_N such that $f_n(s) \to f(s)$ (as $n \to \infty$) uniformly on D_N . Suppose $f(s) \not\equiv 0$. Then an interior point s_0 of D_N is a zero of $f(s)$ if and only if there exists a sequence $\{s_n\}$ in D_N such that $s_n \to s_0$ (as $n \to \infty$) and $f_n(s_n) = 0$ for $n > n_0 = n_0(s_0)$.*

This is the Hurwitz theorem (see Section 3.45 of Titchmarsh [20]). The next lemma is Theorem 1.7.10 of [12]. Denote by $S(\xi)$ the support of the random element ξ .

Lemma 10. *Let $\{\xi_m\}$ be a sequence of independent $H(D_N)$ -valued random elements such that the series*

$$\sum_{m=1}^{\infty} \xi_m \tag{5.1}$$

converges almost surely. Then the support of the sum (5.1) is the closure of the set of all $f \in H(D_N)$ which may be written as a convergent series

$$f = \sum_{m=1}^{\infty} f_m, \qquad f_m \in S(\xi_m).$$

Proof of Lemma 8. By the definition $\{\omega(p)\}$ is a sequence of independent random variables defined on $\big(\Omega, \mathcal{B}(\Omega), m_H\big)$, and the support of each $\omega(p)$ is the unit circle γ . Hence

$$\left\{ \log\left(1 - \frac{\alpha(p)\omega(p)}{p^s}\right)^{-1} + \log\left(1 - \frac{\beta(p)\omega(p)}{p^s}\right)^{-1} \right\}$$

is a sequence of independent $H(D_N)$ -valued random elements, and the set

$$\left\{ f \in H(D_N);\ f(s) = -\log\left(1 - \frac{\alpha(p)a}{p^s}\right) + \log\left(1 - \frac{\beta(p)a}{p^s}\right), a \in \gamma \right\}$$

is the support of each element. Consequently, by Lemma 10, the support of the $H(D_N)$ -valued random element

$$\log\varphi(s,\omega,F) = -\sum_p \left\{ \log\left(1 - \frac{\alpha(p)\omega(p)}{p^s}\right) + \log\left(1 - \frac{\beta(p)\omega(p)}{p^s}\right) \right\}$$

is the closure of the set of all convergent series

$$\sum_p f_p(s; a_p).$$

By Lemma 2 the latter set is dense in $H(D_N)$.

The mapping $\exp : H(D_N) \to H(D_N)$ is continuous, sending $\log \varphi(s, \omega, F)$ to $\varphi(s, \omega, F)$, and sending $H(D_N)$ onto $S_N \setminus \{0\}$. Therefore the support of $\varphi(s, \omega, F)$ contains the set $S_N \setminus \{0\}$. By the definition the support is a closed set (see Definition 1.2.13 of [12]), and by Lemma 9 we have $\overline{S_N \setminus \{0\}} = S_N$. Thus

$$S(\varphi) \supseteq S_N. \tag{5.2}$$

On the other hand,

$$\left(1 - \frac{\alpha(p)\omega(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)\omega(p)}{p^s}\right)^{-1}, \qquad s \in D_N, \omega \in \Omega,$$

is non-zero for all primes p. Hence $\varphi(s, \omega, F)$ is an almost surely convergent product of non-vanishing factors. Again by Lemma 9 we see that $\varphi(s, \omega, F) \in S_N$ almost surely. Thus $S(\varphi) \subseteq S_N$. This and (5.2) give the assertion of Lemma 8.

§6. Completion of the proof of the theorem.

Let K be a compact subset of D with connected complement. Then we can find $N > 0$ such that $K \subset D_N$. Let $f(s)$ be a non-vanishing continuous function on K which is analytic in the interior of K.

First we assume that $f(s)$ has a non-vanishing analytic continuation to $H(D_N)$. Denote by G the set of functions $g \in H(D_N)$ for which

$$\sup_{s \in K} \left| g(s) - f(s) \right| < \varepsilon$$

holds. The set G is open, hence by Lemma 1 we have

$$\liminf_{T \to \infty} \nu_T \left(\sup_{s \in K} \left| \varphi(s + i\tau, F) - f(s) \right| < \varepsilon \right) \geqslant P_\varphi(G). \tag{6.1}$$

Obviously $f \in S_N$, hence by Lemma 8 it is contained in the support of the random element $\varphi(s, \omega, F)$. Since G is a neighbourhood of f, we have $P_\varphi(G) > 0$. This and (6.1) implies the assertion of the theorem in this case.

Now consider the general case. First we quote

Lemma 11. *Let K be a compact subset of $\mathbb{C}$ whose complement is connected. Then any continuous function $f(s)$ on K which is analytic in the interior of K is approximable uniformly on K by the polynomials of s.*

This is the Mergelyan theorem, and the proof can be found, for example, in Walsh [22].

Since $f(s) \neq 0$ on K, by Lemma 11 we can find a polynomial $p(s)$ such that $p(s) \neq 0$ on K and

$$\sup_{s \in K} \left| f(s) - p(s) \right| < \frac{\varepsilon}{4}. \tag{6.2}$$

Since $p(s)$ has only finitely many zeros, we can find a region G_1 such that $K \subset G_1$ and $p(s) \neq 0$ on G_1. We choose $\log p(s)$ to be analytic in the interior of G_1. Applying Lemma 11 to $\log p(s)$, we find another polynomial $q(s)$ such that

$$\sup_{s \in K} \left| p(s) - e^{q(s)} \right| < \frac{\varepsilon}{4}.$$

From this and (6.2) it follows that

$$\sup_{s \in K} \left| f(s) - e^{q(s)} \right| < \frac{\varepsilon}{2}. \tag{6.3}$$

Since $e^{q(s)} \neq 0$ for all s, we can use the result of the proved case, which yields

$$\liminf_{T \to \infty} \nu_T \left(\sup_{s \in K} \left| \varphi(s + i\tau, F) - e^{q(s)} \right| < \frac{\varepsilon}{2} \right) > 0.$$

This and (6.3) complete the proof of the theorem.

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