On the estimation of the order of Euler - Zagier multiple zeta functions

Hideaki Ishikawa, Kohji Matsumoto

1 Introduction

Let r be a positive integer, and define

$$\zeta_r(s_1,\ldots,s_r) = \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\cdots\sum_{n_r=1}^{\infty} n_1^{-s_1}(n_1+n_2)^{-s_2}\ldots(n_1+n_2+\cdots+n_r)^{-s_r}, \quad (1)$$

where $s_1,\ldots,s_r$ are complex variables. This series is called the Euler-Zagier r-ple sum, and its values at positive integer arguments have been studied extensively by many mathematicians.

The series (1) may be regarded as an analytic function of several complex variables. From this viewpoint, we should consider first of all the problem of analytic continuation. In the case $r = 2$, this problem was already discussed by Atkinson [4]. However, the investigation of the problem of analytic continuation for $r \geq 3$ has begun recently. First, Arakawa and Kaneko [3] proved the analytic continuation of (1) as a function of one variable s_r. The continuation to the whole $\mathbb{C}^r$ spase as a function of r variables was established by Zhao [15], and by Akiyama, Egami and Tanigawa [1], independently. The method of continuation given in those three papers are all different from each other.

Still another proof of the analytic continuation was given by Matsumoto [10]. His method is based on the Mellin-Barnes integral formula ((2)below), which had been used successfully by Katsurada [8],[9] who discovered a new elegant proof of the analytic continuation of the case $r = 2$ of (1). The analytic continuation of various generalizations of (1) has been shown by [2], [11], [12], [13],[14].

A natural next problem is the estimation of the order of $|\zeta_r(s_1,\ldots,s_r)|$. Some upper bounds with respect to $t_r = \Im s_r$ were given in [6], [10] [12]. It is desirable, however, to obtain upper bounds with respect to all $t_j = \Im s_j$, $1 \leq j \leq r$.

For this purpose, we apply the method of using the Mellin-Barnes formula in the present paper. After reviewing the argument in [10] briefly in

Section 2, we will give an upper bound of $|\zeta_r(s_1,\ldots,s_r)|$ for general r (Theorem 1) in Section 3. This theorem is a direct consequence of the formula (4), which is shown by a "right-shift" of the path of integration, and the estimate of the theorem is by no means best-possible. In Section 4 we will prove a refinement (Theorem 2) in the case $r=2$, which is shown by a suitable "left-shift" of (4). The method presented in Section 4 can be applied to more general $r\geq 3$ in principle, but the procedure will become much more complicated in practice. Therefore, to explain the essence of the idea clearly, we will discuss a typical example of the case $r=3$ in the last section.

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2 A review of the proof of analytic continuation

In this section we sketch the argument in [10] how to prove the analytic continuation of (1) by using the Mellin-Barnes integral formula

$$\Gamma(s)(1+\lambda)^{-s}=\frac{1}{2\pi i}\int_{(c)}\Gamma(s+z)\Gamma(-z)\lambda^z dz, \tag{2}$$

where s and λ are complex numbers with $\Re s>0$, $\lambda\neq 0$, $|\arg\lambda|<\pi$, and c is real with $-\Re s<c<0$. The path of integration is the vertical line from $c-i\infty$ to $c+i\infty$.

Let $r\geq 2$, $\xi>0$, and first assume $\sigma_j=\Re s_j\geq 1+\xi$ $(1\leq j\leq r)$. Then (1) is convergent absolutely. Putting $\lambda=n_r/(n_1+n_2+\cdots+n_{r-1})$, $c=-1-\xi/2$ and $s=s_r$ in (2), dividing the both sides by

$$\Gamma(s_r)n_1^{s_1}(n_1+n_2)^{s_2}\ldots(n_1+\cdots+n_{r-2})^{s_{r-2}}(n_1+\cdots+n_{r-1})^{s_{r-1}+s_r},$$

and then summing with respect to $n_1,\ldots,n_r$, we obtain

$$\begin{aligned}
&\zeta(s_1,s_2,\ldots s_r)\\
&= \frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)}\zeta_{r-1}(s_1,\ldots,s_{r-2},s_{r-1}+s_r+z)\zeta(-z)dz\\
&= I_r(c;s_1,s_2,\ldots,s_r),
\end{aligned} \tag{3}$$

say. Here $\zeta(-z)$ is the Riemann zeta-function. In the case $r=2$, Katsurada [8] [9] obtained this formula in a somewhat more generalized form. His aim was to shift the path of integration and deduce asymptotic expansion formulas for certain mean values of Dirichlet $L-$functions and Lerch zeta-functions, but his argument includes a new proof of the analytic continuation of $\zeta_2(s_1,s_2)$ (see Section 4 of [11]). Then Matsumoto [10] extended Katsurada's shifting argument to general r.

Shift the path of integration in (3) to $\Re z = c_{r-1}$, where c_{r-1} is an arbitrary positive number. Counting the residues of relevant poles, we obtain

$$\begin{aligned}
&\zeta_r(s_1,\ldots,s_r)\\
&= \sum_{k_{r-1}=-1}^{[c_{r-1}]} \frac{B_{k_{r-1}+1}}{(k_{r-1}+1)!}\langle s_r\rangle_{k_{r-1}}\zeta_{r-1}(s_1,\ldots,s_{r-2},s_{r-1}+s_r+k_{r-1})\\
&\quad +I_r(c_{r-1};s_1,\ldots,s_r),
\end{aligned} \tag{4}$$

where B_k is the kth Bernoulli number and

$$\langle s\rangle_k = \begin{cases} s(s+1)\ldots(s+k-1) & if \quad k\geq 1,\\ 1 & if \quad k=0,\\ (s-1)^{-1} & if \quad k=-1. \end{cases}$$

The series (1) is convergent absolutely in the region

$$A(r)=\{(s_1,\ldots,s_r)\in\mathbb{C}^r \mid \sigma_j+\sigma_{j+1}+\cdots+\sigma_r>r-j+1\ (1\leq j\leq r)\}$$

(Theorem 3 of [11]). Applying this fact to the factor ζ_{r-1}, we find easily that $I_r(c_{r-1};s_1,\ldots s_r)$ is holomorphic in the region

$$D_r(c_{r-1})=\{(s_1,\ldots,s_r)\in\mathbb{C}^r \mid \sigma_j+\sigma_{j+1}+\cdots+\sigma_r>r-j-c_{r-1}\,(1\leq j\leq r)\}.$$

If we already know that ζ_{r-1} can be continued to the whole $\mathbb{C}^{r-1}$ space, then (4) implies that ζ_r can be continued to $D_r(c_{r-1})$. Since c_{r-1} is arbitrary, by induction on r, we can conclude that $\zeta_r(s_1,\ldots,s_r)$ can be continued meromorphically to the whole $\mathbb{C}^r$ space, and the possible singularities are located only on one of the following hyperplanes:

$$\begin{aligned}
s_r &= 1,\\
s_{r-1}+s_r &= 2,1,0,-2,-4,-6,\ldots,\\
s_{r-j+1}+\cdots+s_r &= j-n\ \ (3\leq j\leq r,\ \ n=0,1,2,3,\ldots).
\end{aligned}$$

We denote the union of these hyperplanes by $S(r)$. It is known that these are indeed singularities (Theorem 1 of Akiyama, Egami and Tanigawa [1]).

3 A general estimate

We first quote the following lemma.

Lemma 1 (Matsumoto-Tanigawa [14], Lemma 2) *Let u, v, p, q, r be real numbers. Then*

$$\begin{aligned}
&\int_{-\infty}^{\infty}(1+|u+y|)^p(1+|v+y|)^q(1+|y|)^r\exp\Big(-\frac{\pi}{2}|u+y|-\frac{\pi}{2}|y|\Big)dy\\
&\quad = O\Big((1+U^p)(1+U^q+V^q)(1+U^{r+1})\exp(-\frac{\pi}{2}|u|)\Big),
\end{aligned}$$

where $U = 1 + |u|$, $V = 1 + |v|$, and the implied constant depends only on p, q and r.

In this section we estimate the right-hand side of (4) to obtain an upper bound of $|\zeta_r(s_1, \ldots, s_r)|$.

Assume $(s_1, \ldots, s_r) \in D_r(c_{r-1}) \backslash S(r)$. By the functional equation of the Riemann zeta-function, we have

$$\begin{aligned} I_r(c_{r-1}; s_1, \ldots s_r) &= \frac{1}{2\pi i} \int_{(c_{r-1})} \frac{\Gamma(s_r + z)}{\Gamma(s_r)} \frac{\zeta(1+z)}{2(2\pi)^z \cos(\pi z/2)} \\ &\quad \times \zeta_{r-1}(s_1, s_2, \ldots, s_{r-2}, s_{r-1} + s_r + z) dz. \end{aligned} \tag{5}$$

If $(s_1, \ldots, s_r) \in D_r(c_{r-1})$ and $\Re z = c_{r-1}$, then $(s_1, \ldots, s_{r-1} + s_r + z) \in A(r-1)$, so

$$\zeta_{r-1}(s_1, \ldots, s_{r-2}, s_{r-1} + s_r + z) = O(1)$$

on the right-hand side of (5). Hence by using Stirling's formula we have

$$\begin{aligned} &I_r(c_{r-1}; s_1, \ldots s_r) \\ &\ll \frac{1}{|\Gamma(s_r)|} \int_{-\infty}^{\infty} (1 + |t_r + y|)^{\sigma_r + c_{r-1} - 1/2} \exp\Big(-\frac{\pi}{2}|t_r + y| - \frac{\pi}{2}|y|\Big) dy. \end{aligned}$$

Applying Lemma 1, we obtain

$$I_r(c_{r-1}; s_1, \ldots s_r) \ll (1 + (1 + |t_r|)^{\sigma_r + c_{r-1} - 1/2})(1 + |t_r|)^{3/2 - \sigma_r},$$

hence

$$I_r(c_{r-1}; s_1, \ldots s_r) \ll (1 + |t_r|)^{c_{r-1} + 1} \tag{6}$$

if $c_{r-1} > -\sigma_r + 1/2$.

Next, to the factor $\zeta_{r-1}(s_1, \ldots, s_{r-2}, s_{r-1} + s_r + k_{r-1})$ on the right-hand of (4), we apply the same "right-shift" argument as in Section 2 to obtain

$$\begin{aligned} &\zeta_{r-1}(s_1, \ldots, s_{r-2}, s_{r-1} + s_r + k_{r-1}) \\ &= \sum_{k_{r-2}=-1}^{[c_{r-2}]} \frac{B_{k_{r-2}+1}}{(k_{r-2}+1)!} \langle s_{r-1} + s_r + k_{r-1} \rangle_{k_{r-2}} \\ &\quad \times \zeta_{r-2}(s_1, \ldots, s_{r-3}, s_{r-2} + s_{r-1} + s_r + k_{r-1} + k_{r-2}) \\ &\quad + I_{r-1}(c_{r-2}; s_1, \ldots, s_{r-2}, s_{r-1} + s_r + k_{r-1}), \end{aligned} \tag{7}$$

where c_{r-2} is an arbitrary positive number. We see that $I_{r-1}(c_{r-2}; s_1, \ldots, s_{r-2}, s_{r-1} + s_r + k_{r-1})$ is holomorphic in the region

$$\{(s_1, \ldots, s_r) \in \mathbb{C}^r \,|\, \sigma_j + \cdots + \sigma_r + k_{r-1} > (r-1) - j - c_{r-2} \quad (1 \le j \le r-1)\}. \tag{8}$$

If we choose

$$c_{r-2} = c_{r-2}(k_{r-1}) \geq c_{r-1} - k_{r-1} - 1, \tag{9}$$

then (8) contains $D_r(c_{r-1})$. Hence $I_{r-1}(c_{r-2}; s_1, \ldots, s_{r-2}, s_{r-1} + s_r + k_{r-1})$ is holomorphic for $(s_1, \ldots, s_r) \in D_r(c_{r-1})$, and similarly to (6) we obtain

$$I_{r-1}(c_{r-2}; s_1, \ldots, s_{r-2}, s_{r-1} + s_r + k_{r-1}) \ll (1 + |t_{r-1} + t_r|)^{c_{r-2}+1} \tag{10}$$

if $c_{r-2} > -(\sigma_{r-1} + \sigma_r + k_{r-1}) + 1/2$.

Repeating the above process, we obtain

$$\begin{aligned}
&\zeta_r(s_1, \ldots, s_r) = \\
&\quad \Biggl\{ \sum_{k_{r-1}=-1}^{[c_{r-1}]} \frac{B_{k_{r-1}+1}}{(k_{r-1}+1)!} \langle s_r \rangle_{k_{r-1}} \Biggl\{ \sum_{k_{r-2}=-1}^{[c_{r-2}]} \frac{B_{k_{r-2}+1}}{(k_{r-2}+1)!} \langle s_{r-1} + s_r + k_{r-1} \rangle_{k_{r-2}} \cdots \\
&\quad \cdots \Biggl\{ \sum_{k_2=-1}^{[c_2]} \frac{B_{k_2+1}}{(k_2+1)!} \langle s_3 + s_4 + \cdots + s_r + k_3 + \cdots + k_{r-1} \rangle_{k_2} \Biggl\{ \sum_{k_1=-1}^{[c_1]} \frac{B_{k_1+1}}{(k_1+1)!} \\
&\quad \times \langle s_2 + \cdots + s_r + k_2 + \cdots + k_{r-1} \rangle_{k_1} \zeta(s_1 + s_2 + \cdots + s_r + k_1 + \cdots + k_{r-1}) \\
&\quad + I_2(c_1) \Biggr\} + I_3(c_2) \Biggr\} \cdots + I_{r-1}(c_{r-2}) \Biggr\} + I_r(c_{r-1}) \Biggr\},
\end{aligned} \tag{11}$$

where $c_1, \ldots, c_{r-1}$ are positive numbers and

$$I_m(c_{m-1}) = I_m(c_{m-1}; s_1, \ldots, s_{m-1}, (s_m + \cdots + s_r) + (k_m + \cdots + k_{r-1}))$$

(for $2 \leq m \leq r$; the empty sum is interpreted as zero), which is holomorphic in the region

$$\Bigl\{ (s_1, \ldots, s_r) \in \mathbb{C}^r \,|\, (\sigma_j + \cdots + \sigma_r) + (k_m + \cdots + k_{r-1}) > m - j - c_{m-1} (1 \leq j \leq m) \Bigr\}.$$

This region contains $D_r(c_{r-1})$ if

$$c_{m-1} = c_{m-1}(k_m, \ldots, k_{r-1}) \geq c_{r-1} - (k_m + \cdots + k_{r-1}) - (r - m). \tag{12}$$

Under this condition, we obtain

$$I_m(c_{m-1}) \ll (1 + |t_m + \cdots + t_r|)^{c_{m-1}+1} \tag{13}$$

if

$$c_{m-1} > -\Bigl((\sigma_m + \cdots + \sigma_r) + (k_m + \cdots + k_{r-1}) \Bigr) + 1/2. \tag{14}$$

Let $\theta(\sigma)$ be the infimum of the numbers α satisfying $\zeta(\sigma + it) = O\Bigl((1 + |t|)^\alpha\Bigr)$. It is known that $\theta(\sigma) = \frac{1}{2} - \sigma$ when $\sigma \leq 0$. As for the best known result on $\theta(\sigma)$ for $0 < \sigma < 1$, see Huxley [5]. From (11) and (13) we obtain

$$
\begin{aligned}
&\zeta_r(s_1,\dots,s_r) \ll \\
&\quad \left\{ \sum_{k_{r-1}=-1}^{[c_{r-1}]} (1+|t_r|)^{k_{r-1}} \left\{ \sum_{k_{r-2}=-1}^{[c_{r-2}]} (1+|t_{r-1}+t_r|)^{k_{r-2}} \dots \left\{ \sum_{k_2=-1}^{[c_2]} (1+|t_3+t_4\cdots+t_r|)^{k_2} \right.\right.\right. \\
&\quad \times \left\{ \sum_{k_1=-1}^{[c_1]} (1+|t_2+\cdots+t_r|)^{k_1} (1+|t_1+t_2+\cdots+t_r|)^{\theta(\sigma_1+\cdots+\sigma_r+k_1+\cdots+k_{r-1})} \right. \\
&\quad \left.\left.\left. +(1+|t_2+\cdots+t_r|)^{c_1+1} \right\} + (1+|t_3+\cdots+t_r|)^{c_2+1} \right\} \cdots + (1+|t_{r-1}+t_r|)^{c_{r-2}+1} \right\} \\
&\quad \left. +(1+|t_r|)^{c_{r-1}+1} \right\} \qquad (15)
\end{aligned}
$$

for $(s_1,\dots,s_r) \in D_r(c_{r-1})\backslash S(r)$, if it further satisfies the conditions (12) and (14) for $2 \le m \le r$. Therefore we now arrive at the following

Theorem 1 *Let $r \ge 2$ and $c_{r-1} > 0$. Choose positive numbers $c_{r-2} = c_{r-2}(k_{r-1})$, $c_{r-3} = c_{r-3}(k_{r-2},k_{r-1})$, $\dots$, $c_1 = c_1(k_2,\dots,k_{r-1})$ satisfying (12), where $k_2,\dots,k_{r-1}$ are integers with $-1 \le k_m \le [c_m]$ $(2 \le m \le r-1)$. Then we have*

$$
\begin{aligned}
&\zeta_r(s_1,s_2,\dots,s_r) \ll (1+|t_r|)^{c_{r-1}+1} \\
&\quad + \sum_{j=2}^{r-1} \max_{\substack{-1\le k_{r-1}\le [c_{r-1}] \\ -1\le k_{r-2}\le [c_{r-2}] \\ \cdots \\ -1\le k_j\le [c_j]}} (1+|t_r|)^{k_{r-1}} (1+|t_{r-1}+t_r|)^{k_{r-2}} \times \dots \\
&\quad \cdots \times (1+|t_{j+1}+\cdots+t_r|)^{k_j} (1+|t_j+\cdots+t_r|)^{c_{j-1}+1} \\
&\quad + \max_{\substack{-1\le k_{r-1}\le [c_{r-1}] \\ -1\le k_{r-2}\le [c_{r-2}] \\ \cdots \\ -1\le k_1\le [c_1]}} (1+|t_r|)^{k_{r-1}} (1+|t_{r-1}+t_r|)^{k_{r-2}} \times \dots \\
&\quad \cdots \times (1+|t_2+\cdots+t_r|)^{k_1} (1+|t_1+\cdots+t_r|)^{\theta(\sigma_1+\cdots+\sigma_r+k_1+\cdots+k_{r-1})}
\end{aligned}
$$

for any $(s_1,\dots,s_r) \in D_r(c_{r-1})\backslash S(r)$ which further satisfies (14) for $2 \le m \le r$.

4 The case of the double zeta-function

In the case of the double zeta-function, Theorem 1 implies

$$
\zeta_2(s_1,s_2) \ll (1+|t_2|)^{c_1+1} + \max_{-1\le k_1\le [c_1]} (1+|t_2|)^{k_1} (1+|t_1+t_2|)^{\theta(\sigma_1+\sigma_2+k_1)} \qquad (16)
$$

for $(s_1, s_2) \in D_2(c_1)\backslash S(2)$, under the additional condition $c_1 > -\sigma_2 + 1/2$. But this estimate is by no means best-possible. For instance, consider the case $s_1 = it$, $s_2 = i\alpha t$, where $t > 0$ and α is a real constant. Then $(s_1, s_2) \in D_2(c_1)$ if $c_1 > 1$. Taking $c_1 = 1 + \epsilon$, from (16) we obtain

$$\zeta_2(it, i\alpha t) \ll (1 + t)^{2+\epsilon}. \tag{17}$$

However, in the case $\alpha = 1$, by the obvious relation

$$\zeta(it)\zeta(it) = \zeta(2it) + 2\zeta_2(it, it), \tag{18}$$

we immediately obtain

$$\zeta_2(it, it) \ll (1 + |t|)^{1+\epsilon}, \tag{19}$$

which is much better than (17). This is the consequence of the fortunate relation (18), but actually we can improve (17) without using this relation. The purpose of this section is to prove such an improvement (Theorem 2 below), and from which we can deduce the following

Corollary *For any fixed real α ($\neq -1$), we have*

$$\zeta_2(it, i\alpha t) \ll (1 + |t|)^{3/2+\epsilon}. \tag{20}$$

We begin with the formula (4) with $r = 2$, $0 < c_1 < 1$. Then

$$\zeta_2(s_1, s_2) = \frac{\zeta(s_1 + s_2 - 1)}{s_2 - 1} + \zeta(s_1 + s_2) + I_2(c_1; s_1, s_2) \tag{21}$$

in the region $D_2(c_1)$. Let η be a small positive number satisfying $0 < \eta < c_1$, and

$$D_2^*(\eta) = \left\{ (s_1, s_2) \in \mathbb{C}^2 \middle| \begin{array}{ccc} \sigma_2 & > & -\eta \\ \sigma_1 + \sigma_2 & < & 1 - \eta \end{array} \right\}.$$

Then $D_2(c_1) \cap D_2^*(\eta) \neq \emptyset$. Fix an element (s_1, s_2) of this intersection, and shift the path of integration of $I_2(c_1; s_1, s_2)$ to $\Re z = \eta$. (The "left-shift" argument.) The only relevant pole is $z = 1 - s_1 - s_2$, hence we have

$$\begin{aligned} \zeta_2(s_1, s_2) \quad = \quad & \frac{\zeta(s_1 + s_2 - 1)}{s_2 - 1} + \zeta(s_1 + s_2) \\ & + \frac{\Gamma(1 - s_1)}{\Gamma(s_2)} \Gamma(s_1 + s_2 - 1)\zeta(s_1 + s_2 - 1) + I_2(\eta; s_1, s_2), \end{aligned} \tag{22}$$

and $I_2(\eta; s_1, s_2)$ is holomorphic in $D_2^*(\eta)$. This gives the meromorphic continuation of $\zeta_2(s_1, s_2)$ to $D_2^*(\eta)$.

Now we assume that (s_1, s_2) is an arbitrary element of $D_2^*(\eta)$, and estimate $\zeta_2(s_1, s_2)$ by using (22). By Stirling's formula we have

$$\frac{\Gamma(1-s_1)}{\Gamma(s_2)}\Gamma(s_1+s_2-1)\zeta(s_1+s_2-1) \ll (1+|t_1|)^{1/2-\sigma_1}(1+|t_2|)^{1/2-\sigma_2} \tag{23}$$

and

$$\begin{aligned} I_2(\eta; s_1, s_2) \quad \ll \quad & e^{\frac{\pi}{2}|t_2|}(1+|t_2|)^{1/2-\sigma_2}\int_{-\infty}^{\infty} e^{-\frac{\pi}{2}|t_2+y|-\frac{\pi}{2}|y|} \\ & \times(1+|t_2+y|)^{\sigma_2+\eta-1/2}(1+|t_1+t_2+y|)^{\theta(\sigma_1+\sigma_2+\eta)}dy. \end{aligned} \tag{24}$$

Therefore our problem is reduced to the evaluation of the integral

$$J = \int_{-\infty}^{\infty}(1+|y+u|)^p(1+|y+v|)^q \exp\Big(-\frac{\pi}{2}|y+u|-\frac{\pi}{2}|y|\Big)dy, \tag{25}$$

where u, v, p, q are real numbers and $p>-1$, $q\geq 0$. We can obtain some estimate if we apply Lemma 1; but here we show more refined estimates.

Lemma 2 *Assume $p>-1$ and $q\geq 0$. Then the integral J can be estimated as follows.*

(i) When $uv\leq 0$, then

$$J \ll e^{-\frac{\pi}{2}|u|}(1+|u|)^{p+1}(1+|u-v|)^q. \tag{26}$$

(ii) When $uv>0$ and $|u|<|v|$, then

$$J \ll e^{-\frac{\pi}{2}|u|}(1+|u|)^{p+1}(1+|v|)^q. \tag{27}$$

(iii) When $uv>0$ and $|u|\geq|v|$, then

$$\begin{aligned} J \quad \ll \quad & e^{-\frac{\pi}{2}|u|}\Big\{\max\{(1+|u|)^p,(1+|u-v|)^p\}(1+|v|)^{q+1} \\ & +(1+|u-v|)^{p+q+1}\Big\}. \end{aligned} \tag{28}$$

Remark 1 The following proof can also be applied to the case $p\leq -1$. In this case, the conclusion is as follows. In (26) and (27), the factor $(1+|u|)^{p+1}$ is to be replaced by $\log(1+|u|)$ (if $p=-1$) or 1 (if $p<-1$). In (28), the factor $(1+|u-v|)^{p+q+1}$ is to be replaced by $(1+|u-v|)^q\log(1+|u-v|)$ (if $p=-1$) or $(1+|u-v|)^q$ (if $p<-1$).

To prove Lemma 2, we may assume $u\geq 0$ without loss of generality, because the results in the case $u<0$ can be deduced from the case $u\geq 0$ by replacing u, v, y by $-u$, $-v$, $-y$ respectively in (25).

First consider the case (i), that is $u \geq 0$ and $v \leq 0$. We divide the integral J as

$$J = \int_{-\infty}^{-u} + \int_{-u}^{0} + \int_{0}^{-v} + \int_{-v}^{\infty} = J_1 + J_2 + J_3 + J_4,$$

say. We put $-y - u = \tau$ in J_1 to obtain

$$\begin{aligned}
J_1 &= e^{-\frac{\pi}{2}u} \int_0^\infty e^{-\pi\tau}(1+\tau)^p(1+\tau+u-v)^q d\tau \\
&\ll e^{-\frac{\pi}{2}u}\left\{\int_0^{u-v} e^{-\pi\tau}(1+\tau)^p(1+u-v)^q d\tau + \int_{u-v}^\infty e^{-\pi\tau}(1+\tau)^{p+q}d\tau\right\} \\
&\ll e^{-\frac{\pi}{2}u}(1+u-v)^q.
\end{aligned}$$

As for J_2, we put $y + u = \tau$ to obtain

$$\begin{aligned}
J_2 &= e^{-\frac{\pi}{2}u} \int_0^u (1+\tau)^p(1-\tau+u-v)^q d\tau \\
&\leq e^{-\frac{\pi}{2}u}(1+u-v)^q \int_0^u (1+\tau)^p d\tau \\
&\ll e^{-\frac{\pi}{2}u}(1+u)^{p+1}(1+u-v)^q.
\end{aligned}$$

The integral J_3 can be treated similarly to J_2 and we have

$$J_3 = O(e^{-\frac{\pi}{2}u}(1+u)^p(1+u-v)^q).$$

As for J_4, we put $y+v=\tau$ and proceed similarly to the case of J_1 to obtain $J_4 = O(e^{-\frac{\pi}{2}u+\pi v}(1+u-v)^p)$, and from which we can show

$$J_4 \ll e^{-\frac{\pi}{2}u}(1+u)^p. \tag{29}$$

In fact, if $0 \leq -v < u$ we simply use $e^{\pi v} \leq 1$ and $(1+u-v)^p \ll (1+u)^p$. If $-v \geq u$, then

$$e^{-\frac{\pi}{2}u+\pi v} \leq e^{-\pi u - \frac{\pi}{4}(u-v)}$$

and $e^{-\frac{\pi}{4}(u-v)}(1+u-v)^p = O(1)$, hence (29) follows. Collecting the above results, we obtain (26).

The proof of (27) in the case $v > u > 0$ is similar; we divide

$$J = \int_{-\infty}^{-v} + \int_{-v}^{-u} + \int_{-u}^{0} + \int_{0}^{\infty} = J_1' + J_2' + J_3' + J_4',$$

say. We omit the details, only noting that J_4' is treated by splitting the integral further at $y = u$ and $y = v$ and estimating each part separately.

In the case $u \geq v > 0$, we divide

$$J = \int_{-\infty}^{-u} + \int_{-u}^{-v} + \int_{-v}^{0} + \int_{0}^{\infty} \quad = J_1^{''} + J_2^{''} + J_3^{''} + J_4^{''},$$

say. The treatment of $J_1^{''}$ is completely the same as J_1. Next we put $-y-v = \tau$ in $J_2^{''}$ to obtain

$$\begin{aligned} J_2^{''} &= e^{-\frac{\pi}{2}u} \int_0^{u-v} (1-\tau+u-v)^p (1+\tau)^q d\tau \\ &\leq e^{-\frac{\pi}{2}u} \Big\{ \int_0^{(u-v)/2} (1+u-v)^p (1+\tau)^q d\tau \\ &\quad + \int_{(u-v)/2}^{u-v} (1-\tau+u-v)^p (1+u-v)^q d\tau \Big\} \\ &\ll e^{-\frac{\pi}{2}u} (1+u-v)^{p+q+1}. \end{aligned}$$

As for $J_3^{''}$, we put $y+u=\tau$ to obtain

$$\begin{aligned} J_3^{''} &\ll e^{-\frac{\pi}{2}u} \int_{u-v}^{u} (1+\tau)^p (1+\tau-u+v)^q d\tau \\ &\ll e^{-\frac{\pi}{2}u} \max\Big\{ (1+u)^p,\ (1+u-v)^p \Big\} (1+v)^{q+1}. \end{aligned}$$

The integral $J_4^{''}$ can be treated similarly to $J_4^{'}$ and we obtain $J_4^{''} = O(e^{-\frac{\pi}{2}u}(1+u)^p(1+v)^q)$. The estimate (28) now follows, and the proof of Lemma 2 is complete.

We can estimate $I_2(\eta; s_1, s_2)$ by applying Lemma 2 to the right-hand side of (24). Then, combining with (22) and (23), we obtain the following

Theorem 2 *Let $0 < \eta < 1$. If $(s_1, s_2) \in D_2^*(\eta)$, then we have*

$$\begin{aligned} \zeta_2(s_1, s_2) &\ll (1+|t_2|)^{-1}(1+|t_1+t_2|)^{\theta(\sigma_1+\sigma_2-1)} + (1+|t_1+t_2|)^{\theta(\sigma_1+\sigma_2)} \\ &\quad + (1+|t_1|)^{1/2-\sigma_1}(1+|t_2|)^{1/2-\sigma_2} + |I_2(\eta; s_1, s_2)|, \end{aligned} \tag{30}$$

and $I_2(\eta; s_1, s_2)$ is

$$\ll (1+|t_2|)^{\eta+1}(1+|t_1|)^{\theta(\sigma_1+\sigma_2+\eta)} \tag{31}$$

if $t_2(t_1+t_2) \leq 0$,

$$\ll (1+|t_2|)^{\eta+1}(1+|t_1+t_2|)^{\theta(\sigma_1+\sigma_2+\eta)} \tag{32}$$

if $t_2(t_1+t_2) > 0$ and $|t_2| < |t_1+t_2|$, and

$$\begin{aligned} \ll \quad & (1+|t_2|)^{1/2-\sigma_2} \Bigg(\max\Big\{ (1+|t_1|)^{\sigma_2+\eta-1/2}, (1+|t_2|)^{\sigma_2+\eta-1/2} \Big\} \\ & \times (1+|t_1+t_2|)^{\theta(\sigma_1+\sigma_2+\eta)+1} + (1+|t_1|)^{\sigma_2+\eta+1/2+\theta(\sigma_1+\sigma_2+\eta)} \Bigg) \end{aligned} \tag{33}$$

if $t_2(t_1+t_2) > 0$ and $|t_2| \geq |t_1+t_2|$.

This theorem with $s_1 = it$, $s_2 = i\alpha t$ and $\eta = \epsilon$ inplies the corollary mentioned before. Therefore this theorem is clearly a refinement of Theorem 1 for $r = 2$.

5 The case of the triple zeta-function

In this section we discuss how to refine Theorem 1 in the case $r = 3$. However the argument is much more complicated than the case $r = 2$ presented in the preceding section. Therefore we restrict our consideration to just one typical example $(s_1, s_2, s_3) = (-it, it, it)$, where t is a non-zero real number.

If we put $c_2 = 2 + \epsilon$ and $c_1 = c_1(k_2) = 1 - k_2 + \epsilon$, then $(-it, it, it) \in D_3(c_2)\backslash S_3$ and we can apply Theorem 1. The result is

$$\zeta_3(-it, it, it) = O\Big((1 + |t|)^{3+\epsilon}\Big). \tag{34}$$

The purpose of this section is to prove the following improvement of (34):

Theorem 3 *We have*

$$\zeta_3(-it, it, it) = O\Big((1 + |t|)^{5/2+\epsilon}\Big)$$

for any $t \neq 0$.

In order to prove this theorem, we again use the "left-shift" argument, but this time we should shift the path to the left twice.

Our starting point is the formula (4) with $r = 3$ and $0 < c_2 < 1$, that is

$$\begin{aligned}\zeta_3(s_1, s_2, s_3) = &\\ &\frac{\zeta_2(s_1, s_2 + s_3 - 1)}{s_3 - 1} - \frac{1}{2}\zeta_2(s_1, s_2 + s_3) + I_3(c_2; s_1, s_2, s_3)\end{aligned}$$

which is valid in $D_3(c_2)$. Let $0 < \mu < c_2$ and

$$D_3^*(\mu) = \left\{ (s_1, s_2, s_3) \in \mathbb{C}^3 \;\middle|\; \begin{array}{ccccc} -\mu & < & \sigma_3 & & \\ & & \sigma_2 + \sigma_3 & < & 1 - \mu \\ 1 - \mu & < & \sigma_1 + \sigma_2 + \sigma_3 & < & 2 - \mu \end{array} \right\}.$$

We fix a point $(s_1, s_2, s_3) \in D_3(c_2) \cap D_3^*(\mu)$, and shift the path of $I_3(c_2; s_1, s_2, s_3)$ to $\Re z = \mu$. The definition of $S(2)$ implies that the poles of $\zeta_2(s_1, s_2+s_3+z)$ as a function in z are $z = 1-s_2-s_3$ and $z = -s_1-s_2-s_3+n$ $(n = 2, 1, 0, -2, -4, -6, \dots)$. Two of them $(z = 1 - s_1 - s_2 - s_3,\ z = 2 - s_1 - s_2 - s_3)$ are located in the strip $\mu < \Re z < c_2$. We may assume these two poles are not at the same point, because we may choose our fixed point with the condition $s_1 \neq 1$. The residues of $\zeta_2(s_1, s_2+s_3+z)$ at $z = 1-s_2-s_3$ and $z = 2-s_1-s_2-s_3$ are $\zeta(s_1)$ and $(1-s_1)^{-1}$, respectively. (These can be calculated by using the expression (21).) These two poles are the only poles of the integrand of $I_3(c_2; s_1, s_2, s_3)$ whose residues we should count, and we obtain

$$\begin{aligned}
&\zeta_3(s_1, s_2, s_3) \\
&= \frac{\zeta_2(s_1, s_2+s_3-1)}{s_3-1} - \frac{1}{2}\zeta_2(s_1, s_2+s_3) \\
&\quad + \frac{\Gamma(1-s_2)}{\Gamma(s_3)}\Gamma(s_2+s_3-1)\zeta(s_2+s_3-1)\zeta(s_1) \\
&\quad + \frac{\Gamma(2-s_1-s_2)}{\Gamma(s_3)}\Gamma(s_1+s_2+s_3-2)\zeta(s_1+s_2+s_3-2)\frac{1}{1-s_1} \\
&\quad + I_3(\mu; s_1, s_2, s_3), \qquad (35)
\end{aligned}$$

which gives the continuation of $\zeta_3(s_1, s_2, s_3)$ to $D_3^*(\mu)$.

Next, let λ be a number satisfying $0 < \lambda < \mu$, and define

$$D_3^{**}(\lambda) = \left\{ (s_1, s_2, s_3) \in \mathbb{C}^3 \;\middle|\; \begin{array}{ccccc} -\lambda & < & \sigma_3 & & \\ & & \sigma_2+\sigma_3 & < & 1-\lambda \\ -\lambda & < & \sigma_1+\sigma_2+\sigma_3 & < & 1-\lambda \end{array} \right\}.$$

Now we fix a point $(s_1, s_2, s_3) \in D_3^*(\mu) \cap D_3^{**}(\lambda)$, and shift the path of $I_3(\mu; s_1, s_2, s_3)$ to $\Re z = \lambda$. This time the only relevant pole is $z = 1 - s_1 - s_2 - s_3$, hence we obtain

$$\begin{aligned}
&\zeta_3(s_1, s_2, s_3) \\
&= \frac{\zeta_2(s_1, s_2+s_3-1)}{s_3-1} - \frac{1}{2}\zeta_2(s_1, s_2+s_3) \\
&\quad + \frac{\Gamma(1-s_2)}{\Gamma(s_3)}\Gamma(s_2+s_3-1)\zeta(s_2+s_3-1)\zeta(s_1) \\
&\quad + \frac{\Gamma(2-s_1-s_2)}{\Gamma(s_3)}\Gamma(s_1+s_2+s_3-2)\zeta(s_1+s_2+s_3-2)\frac{1}{1-s_1} \\
&\quad - \frac{\Gamma(1-s_1-s_2)}{2\Gamma(s_3)}\Gamma(s_1+s_2+s_3-1)\zeta(s_1+s_2+s_3-1) \\
&\quad + I_3(\lambda; s_1, s_2, s_3) \\
&= A_1 - A_2 + A_3 + A_4 - A_5 + I_3(\lambda; s_1, s_2, s_3), \qquad (36)
\end{aligned}$$

say. Since $I_3(\lambda; s_1, s_2, s_3)$ is holomorophic in $D_3^{**}(\lambda)$, the formula (36) gives the continuation of $\zeta_3(s_1, s_2, s_3)$, to $D_3^{**}(\lambda)$.

Since $(-it, it, it) \in D_3^{**}(\lambda)$, we can evaluate the order of $\zeta_3(-it, it, it)$ by using (36). We may assume $t > 0$.

First estimate A_j $(1 \le j \le 5)$ at the point $(s_1, s_2, s_3) = (-it, it, it)$. The corollary in Section 4 implies $A_2 = O((1+t)^{3/2+\epsilon})$. To estimate A_1, we use (4) with $r = 2$, $c_1 = 2 + \epsilon$. We have

$$\zeta_2(-it, 2it-1) = \frac{\zeta(it-2)}{2it-2} - \frac{1}{2}\zeta(it-1) + \frac{1}{12}(2it-1)\zeta(it) + I_2(c_1; -it, 2it-1),$$

because $B_3 = 0$. From Lemma 2 (iii) (or from (6)) we have

$$I_2(c_1; -it, 2it-1) \ll (1+t)^{3+\epsilon}.$$

Hence we obtain $A_1 = O((1+t)^{2+\epsilon})$. By using Stirling's formula it is easy to see that $A_4 \ll (1+t)^{-1/2}$, $A_5 \ll (1+t)^{1/2}$, and A_3 is of exponential decay. Therefore we obtain

$$\zeta_3(-it, it, it) \ll (1+t)^{2+\epsilon} + |I_3(\lambda; -it, it, it)|. \tag{37}$$

Now our remaining task is to estimate the integral $I_3(\lambda; -it, it, it)$. Again by Stirling's formula we have

$$\begin{aligned} I_3(\lambda; -it, it, it) \quad &\ll \quad e^{\frac{1}{2}\pi t}(1+t)^{1/2} \int_{-\infty}^{\infty} \exp\Big(-\frac{\pi}{2}|y| - \frac{\pi}{2}|y-t|\Big) \\ &\quad \times (1+|y-t|)^{\lambda-1/2} |\zeta_2(-it, \lambda + i(2t-y))| dy. \end{aligned} \tag{38}$$

From (22) with $\eta = \epsilon$, with the aid of (23) and the fact $\theta(\sigma) = 1/2 - (2/3)\sigma$ (for $0 \le \sigma \le 1/2$), we obtain

$$\begin{aligned} &\zeta_2(-it, \lambda + i(2t-y)) \\ &\quad \ll \quad (1+|t-y|)^{3/2-\lambda}(1+|2t-y|)^{-1} + (1+|t-y|)^{1/2-(2/3)\lambda} \\ &\qquad\quad + (1+t)^{1/2}(1+|2t-y|)^{1/2-\lambda} + |I_2(\epsilon; -it, \lambda + i(2t-y))| \\ &\quad = \quad h_1(t,y) + h_2(t,y) + h_3(t,y) + h_4(t,y), \end{aligned}$$

say. Substituting this estimate into the right-hand side of (38), we obtain

$$I_3(\lambda; -it, it, it) \ll \sum_{j=1}^{4} H_j, \tag{39}$$

where

$$\begin{aligned} H_j \quad &= \quad e^{\frac{\pi}{2}t}(1+t)^{1/2} \int_{-\infty}^{\infty} \exp\Big(-\frac{\pi}{2}|y| - \frac{\pi}{2}|y-t|\Big) \\ &\quad \times (1+|y-t|)^{\lambda-1/2} h_j(t,y) dy \end{aligned}$$

$(1 \le j \le 4)$. We apply Lemma 1 to H_1, H_2 and H_3 to obtain

$$H_1 \ll (1+t)^{5/2}, \quad H_2 \ll (1+t)^{3/2+(1/3)\lambda}, \quad H_3 \ll (1+t)^{5/2-\lambda}. \tag{40}$$

Concerning H_4, we first estimate $h_4(t,y)$ by (31)-(33) of Theorem 2. The results are

$$h_4(t,y) \ll (1+|2t-y|)^{1+\epsilon}(1+|t-y|)^{1/2-(2/3)(\lambda+\epsilon)}$$

if $y > 2t$,

$$h_4(t,y) \ll (1+|2t-y|)^{1+\epsilon}(1+t)^{1/2-(2/3)(\lambda+\epsilon)}$$

if $t \le y \le 2t$, and

$$\begin{aligned} h_4(t,y) &\ll (1+|2t-y|)^{1/2-\lambda}\Big\{\max\Big((1+t)^{\lambda+\epsilon-1/2}, \quad (1+|2t-y|)^{\lambda+\epsilon-1/2}\Big) \\ &\quad \times(1+|t-y|)^{3/2-(2/3)(\lambda+\epsilon)} + (1+t)^{1+(1/3)(\lambda+\epsilon)}\Big\} \end{aligned}$$

if $y < t$. Therefore

$$H_4 \ll e^{\frac{\pi}{2}t}(1+t)^{1/2}(J_1+J_2+J_3+J_4),$$

where

$$J_1 = \int_{2t}^{\infty} \exp\Big(-\frac{\pi}{2}|y| - \frac{\pi}{2}|y-t|\Big)(1+|y-t|)^{\lambda-(2/3)(\lambda+\epsilon)}(1+|y-2t|)^{1+\epsilon}dy,$$

$$\begin{aligned} J_2 &= (1+t)^{1/2-(2/3)(\lambda+\epsilon)}\int_{t}^{2t} \exp\Big(-\frac{\pi}{2}|y| - \frac{\pi}{2}|y-t|\Big) \\ &\quad \times(1+|y-t|)^{\lambda-1/2}(1+|y-2t|)^{1+\epsilon}dy, \end{aligned}$$

$$\begin{aligned} J_3 &= \int_{-\infty}^{t} \exp\Big(-\frac{\pi}{2}|y| - \frac{\pi}{2}|y-t|\Big)(1+|y-t|)^{\lambda+1-(2/3)(\lambda+\epsilon)}(1+|y-2t|)^{1/2-\lambda} \\ &\quad \times \max\Big\{(1+t)^{\lambda+\epsilon-1/2}, \quad (1+|y-2t|)^{\lambda+\epsilon-1/2}\Big\}dy \end{aligned}$$

and

$$\begin{aligned} J_4 &= (1+t)^{1+(1/3)(\lambda+\epsilon)}\int_{-\infty}^{t} \exp\Big(-\frac{\pi}{2}|y| - \frac{\pi}{2}|y-t|\Big) \\ &\quad \times(1+|y-t|)^{\lambda-1/2}(1+|y-2t|)^{1/2-\lambda}dy. \end{aligned}$$

Now we choose $\lambda = 2\epsilon$. Then $\lambda - (2/3)(\lambda+\epsilon) = 0$, and applying Lemma 1 we obtain

$$J_1 \ll (1+t)^{2+\epsilon}e^{-\frac{\pi}{2}t}. \tag{41}$$

If $y < t$ then $t < |y-2t|$, hence

$$\max\Big\{(1+t)^{\lambda+\epsilon-1/2}, \quad (1+|y-2t|)^{\lambda+\epsilon-1/2}\Big\} = (1+t)^{\lambda+\epsilon-1/2}.$$

Hence, applying Lemma 1 again, we obtain

$$J_3 \ll (1+t)^{2+\epsilon} e^{-\frac{\pi}{2}t}. \tag{42}$$

If we apply Lemma 1 to J_2 and J_4, we only obtain the estimate $O((1+t)^{3+\epsilon})$, hence (34) can not be improved. Therefore we should treat J_2 and J_4 more carefully.

Putting $y - t = y'$, we have

$$\begin{aligned} J_2 &= (1+t)^{1/2-2\epsilon} e^{-\frac{\pi}{2}t} \int_0^t e^{-\pi y'} (1+y')^{\lambda-1/2} (1+t-y')^{1+\epsilon} dy' \\ &\leq (1+t)^{1/2-2\epsilon} e^{-\frac{\pi}{2}t} \int_0^t e^{-\pi y'} (1+t)^{1+\epsilon} dy' \\ &\ll (1+t)^{3/2-\epsilon} e^{-\frac{\pi}{2}t}. \end{aligned} \tag{43}$$

Next we divide

$$J_4 = (1+t)^{1+\epsilon} \left(\int_{-\infty}^0 + \int_0^t \right) = (1+t)^{1+\epsilon} (J_{41} + J_{42}),$$

say. Then

$$\begin{aligned} J_{41} &= e^{-\frac{\pi}{2}t} \int_0^\infty e^{-\pi y} (1+t+y)^{\lambda-1/2} (1+2t+y)^{1/2-\lambda} dy \\ &\ll e^{-\frac{\pi}{2}t} \Big\{ \int_0^{2t} e^{-\pi y} (1+t)^{\lambda-1/2} (1+t)^{1/2-\lambda} dy \\ &\quad + \int_{2t}^\infty e^{-\pi y} (1+y)^{\lambda-1/2} (1+y)^{1/2-\lambda} dy \Big\} \\ &\ll e^{-\frac{\pi}{2}t}. \end{aligned}$$

Also putting $t - y = y'$, we have

$$\begin{aligned} J_{42} &= e^{-\frac{\pi}{2}t} \int_0^t (1+y')^{\lambda-1/2} (1+t+y')^{1/2-\lambda} dy' \\ &\ll e^{-\frac{\pi}{2}t} (1+t)^{1/2-\lambda} \int_0^t (1+y')^{\lambda-1/2} dy' \\ &\ll e^{-\frac{\pi}{2}t} (1+t). \end{aligned}$$

Hence we obtain

$$J_4 \ll (1+t)^{2+\epsilon} e^{-\frac{\pi}{2}t}. \tag{44}$$

Collecting (41)-(44) we now obtain

$$H_4 \ll (1+t)^{5/2+\epsilon},$$

therefore with (39) and (40) we now arrive at the assertion of Theorem 3.

Remark 2 The estimate of Lemma 1 is a little rough, and we can improve some of the above estimates, obtained by using Lemma 1, if we consider more carefully. However the most crucial are the estimate of J_3 and J_{42}, which cannot be improved.

References

[1] S. Akiyama, S. Egami, and Y. Tanigawa, Analytic continuation of multiple zeta functions and their values at non-positive integers, *Acta Arith.* **98** (2001), 107-116.

[2] S. Akiyama and H. Ishikawa, On analytic continuation of multiple L-functions and related zeta-functions, in *"Analytic Number Theory."*, C.Jia and K.Matsumoto (eds.), Developments in Math. vol.6, Kluwer Acad. Publ. 2002, pp.1-16.

[3] T. Arakawa and M. Kaneko, Multiple zeta values, poly-Bernoulli numbers, and related zeta functions, *Nagoya Math. J.* **153** (1999), 189-209.

[4] F.V. Atkinson, The mean-value of the Riemann zeta function, *Acta Math.* **81** (1949), 353-376.

[5] M. N. Huxley, Exponential sums and the Riemann zeta function IV, *Proc. London Math. Soc.* (3) **66** (1993), 1-40.

[6] H. Ishikawa, On analytic properties of a multiple L-function, in *"Analytic Extension Formulas and their Applications"*, S.Saitoh et al. (eds.), International Society for Analysis, Applications and Computation Series, Kluwer Acad. Publ. 2001, pp.105-122.

[7] A. Ivić, The Riemann Zeta-Function, Wiley-Interscience Publication, New York, 1985.

[8] M. Katsurada, An application of Mellin-Barnes' type integrals to the mean square of Lerch zeta-functions, *Collect. Math.* **48** (1997), 137-153.

[9] M. Katsurada, An application of Mellin-Barnes type of integrals to the mean square of L-functions, *Liet. Mat. Rink.* **38** (1998), 98-112 = Lithuanian Math. J. **38** (1998), 77-88.

[10] K. Matsumoto, Asymptotic expansions of double zeta-functions of Barnes, of Shintani, and Eisenstein series, Nagoya Math. J., to appear.

[11] K. Matsumoto, On the analytic continuation of various multiple zeta-functions, in *"Number Theory for the Millennium, Proc. Millennial Conf. Number Theory"*, M.A. Bennett et al. (eds.), A K Peters, 2002, to appear.

[12] K. Matsumoto, The analytic continuation and the asymptotic behaviour of certain multiple zeta-functions I, preprint.

[13] K. Matsumoto, The analytic continuation and the asymptotic behaviour of certain multiple zeta-functions II, preprint.

[14] K. Matsumoto and Y. Tanigawa, The analytic continuation and the order estimate of multiple Dirichlet series, J. Théorie des Nombres de Bordeaux, to appear.

[15] J. Zhao, Analytic continuation of multiple zeta functions, *Proc. Amer. Math. Soc.* **128** (2000), 1275-1283.

Graduate School of Science and Technology
Niigata University
Niigata, 950-2181
Japan

E-mail: ishikawa@meso.ge.niigata-u.ac.jp

Graduate school of Mathematics
Nagoya University
Chikusa-ku, Nagoya 464-8602
Japan

E-mail: kohjimat@math.nagoya-u.ac.jp