On analytic continuation of various multiple zeta-functions

Kohji Matsumoto

Abstract

In this article we describe the development of the problem of analytic continuation of multiple zeta-functions. We begin with the work of E. W. Barnes and H. Mellin, and then discuss the Euler sum and its multi-variable generalization. Recently, M. Katsurada discovered that the classical Mellin-Barnes integral formula is useful to the study of analytic continuation of the Euler sum. We will explain Katsurada's idea in Section 4. Then in the last two sections we will present new results of the author, which are obtained by using the Mellin-Barnes formula to more general multiple zeta-functions.

1 Barnes multiple zeta-functions

The problem of analytic continuation of multiple zeta-functions was first considered by Barnes [7][8] and Mellin [48][49]. Barnes [7] introduced the double zeta-function of the form

$$\zeta_2(s;\alpha,(w_1,w_2))=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}(\alpha+m_1w_1+m_2w_2)^{-s}, \tag{1.1}$$

where α, w_1, w_2 are complex numbers with positive real parts, and s is the complex variable. The series (1.1) is convergent absolutely in the half-plane $\Re s>2$. Actually Barnes first defined his function as the contour integral

$$\zeta_2(s;\alpha,(w_1,w_2))=-\frac{\Gamma(1-s)}{2\pi i}\int_{\mathcal{C}}\frac{e^{-\alpha z}(-z)^{s-1}}{(1-e^{-w_1z})(1-e^{-w_2z})}dz, \tag{1.2}$$

where $\mathcal{C}$ is the contour which consists of the half-line on the positive real axis from infinity to a small positive constant δ, a circle of radius δ counterclockwise round the origin, and the other half-line on the positive real axis from δ to infinity. It is easy to see that (1.2) coincides with (1.1) when $\Re s>2$. The expression (1.2) gives the meromorphic continuation of $\zeta_2(s;\alpha,(w_1,w_2))$ to the whole s-plane. Moreover, Barnes [7] studied very carefully how to extend the definition of $\zeta_2(s;\alpha,(w_1,w_2))$ to the situation when the real parts of α, w_1, w_2 are not necessarily positive.

Barnes introduced his double zeta-function for the purpose of constructing the theory of double gamma-functions. As for the theory of double gamma-functions, there were several predecessors such as Kinkelin, Hölder, Méray, Pincherle, and Alexeiewsky, but Barnes developed the theory most systematically. Then Barnes [8] proceeded to the theory of more general multiple gamma-functions, and in this research he introduced the multiple zeta-function defined by

$$\zeta_r(s;\alpha,(w_1,\ldots,w_r))=\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}(\alpha+m_1w_1+\cdots+m_rw_r)^{-s}, \tag{1.3}$$

where r is a positive integer, and α, $w_1,\ldots,w_r$ are complex numbers. Barnes assumed the following condition to ensure the convergence of the series. Let ℓ be any line on the complex s-plane crossing the origin. Then ℓ divides the plane into two half-planes. Let $H(\ell)$ be one of those half-planes, not including ℓ itself. The assumption of Barnes is that

$$w_j\in H(\ell)\qquad(1\le j\le r). \tag{1.4}$$

Then excluding the finitely many possible $(m_1,\ldots,m_r)$ satisfying $m_1w_1+\cdots+m_rw_r=-\alpha$ from the sum, we see easily that (1.3) is convergent absolutely when $\Re s>r$. Barnes [8] proved an integral expression similar to (1.2) for $\zeta_r(s;\alpha,(w_1,\ldots,w_r))$, which yields the meromorphic continuation.

On the other hand, Mellin [48][49] studied the meromorphic continuation of the multiple series

$$\sum_{m_1=1}^{\infty}\cdots\sum_{m_k=1}^{\infty}P(m_1,\ldots,m_k)^{-s}, \tag{1.5}$$

where $P(X_1,\ldots,X_k)$ is a polynomial of k indeterminates and of complex coefficients with positive real parts. Mellin's papers include a prototype of the method in the present paper, though he treated the one variable case only. For example, the formula (4.1) appears in p.21 of [48]. After Mellin's works, many subsequent researches on (1.5) and its generalizations were done; main contributors include K. Mahler, P. Cassou-Nogués, P. Sargos, B. Lichtin, M. Eie and M. Peter. Most of them concentrated on the one variable case, hence we do not discuss the details of their works. However, Lichtin's series of papers [36][37][38][39] and [40] should be mentioned here. In [36] Lichtin proposed the problem of studying the analytic continuation of Dirichlet series in several variables

$$\sum_{m_1=1}^{\infty}\cdots\sum_{m_k=1}^{\infty}P_0(m_1,\ldots,m_k)\times P_1(m_1,\ldots,m_k)^{-s_1}\cdots P_r(m_1,\ldots,m_k)^{-s_r}, \tag{1.6}$$

where $P_0, P_1, \dots, P_r$ are polynomials of k indeterminates, and he carried out such investigations in [37][38][39][40]. In particular Lichtin proved that the series (1.6) can be continued meromorphically to the whole space when the associated polynomials are hypoelliptic (and also satisfy some other conditions).

2 The Euler sum

The two-variable double sum

$$\zeta_2(s_1, s_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{-s_1}(m+n)^{-s_2} \tag{2.1}$$

is convergent absolutely if $\Re s_2 > 1$ and $\Re(s_1 + s_2) > 2$. The investigation of this sum goes back to Euler. He was interested in the values of (2.1) when s_1 and s_2 are positive integers. Various properties of the values of (2.1) at positive integers were given in Nielsen's book [54]. Ramanujan also had an interest in such kind of problems, and some of their formulas were rediscovered by later authors (see the comments in pp.252-253 of Berndt [9]). Even in very recent years, the Euler sum is an object of active researches; see, for instance, [10][17].

As far as the author knows, the first study on the analytic continuation of $\zeta_2(s_1, s_2)$ was done by Atkinson [6], in his research on the mean square of the Riemann zeta-function $\zeta(s)$. When $\Re s_1 > 1$ and $\Re s_2 > 1$, it holds that

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1 + s_2) + \zeta_2(s_1, s_2) + \zeta_2(s_2, s_1). \tag{2.2}$$

Atkinson's aim was to integrate the left-hand side with respect to t, when $s_1 = \frac{1}{2} + it$ and $s_2 = \frac{1}{2} - it$. Hence he was forced to show the analytic continuation of the right-hand side. Atkinson [6] used the Poisson summation formula to deduce a certain integral expression, and by which he succeeded in showing the analytic continuation.

On the other hand, Matsuoka [47] obtained the analytic continuation of

$$\sum_{m=2}^{\infty} m^{-s} \sum_{n<m} n^{-1},$$

which is actually equal to $\zeta_2(1, s)$. Apostol and Vu [4], independently of Matsuoka [47], proved that $\zeta_2(s_1, s_2)$ may be continued meromorphically with respect to s_1 for each fixed s_2, and also with respect to s_2 for each fixed s_1. Both of the proofs of Matsuoka and Apostol-Vu are based on the Euler-Maclaurin summation formula. The main aim of those papers is the investigation of special values of $\zeta_2(s_1, s_2)$ at (not necessarily positive) integer points, and they deduced various formulas.

Note that Apostol and Vu [4] also considered the series

$$T(s_1, s_2) = \sum_{m=1}^{\infty} \sum_{n<m} \frac{1}{m^{s_1} n^{s_2} (m+n)}, \tag{2.3}$$

and discussed its analytic continuation.

Let q be a positive integer (≥ 2), $\varphi(q)$ the Euler function, χ a Dirichlet character mod q, and $L(s,\chi)$ the corresponding Dirichlet L-function. Inspired by Atkinson's work [6], Meurman [50] and Motohashi [53] independently of each other considered the sum

$$Q((s_1,s_2);q)=\varphi(q)^{-1}\sum_{\chi \bmod q} L(s_1,\chi)L(s_2,\bar{\chi}).$$

Corresponding to (2.2), the decomposition

$$Q((s_1,s_2);q)=L(s_1+s_2,\chi_0)+f((s_1,s_2);q)+f((s_2,s_1);q)$$

holds, where χ_0 is the principal character mod q and

$$f((s_1,s_2);q)=\sum_{\substack{1\leq a\leq q\\(a,q)=1}}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(qm+a)^{-s_1}(q(m+n)+a)^{-s_2}. \tag{2.4}$$

This is a generalization of the Euler sum (2.1). Meurman [50] proved the analytic continuation of (2.4) by generalizing the argument of Atkinson [6]. On the other hand, Motohashi showed a double contour integral expression of (2.4), which yields the analytic continuation. By refining Motohashi's argument, Katsurada and the author [32][33] proved the asymptotic expansions of

$$\sum_{\chi \bmod q}|L(s,\chi)|^2 \quad (s\neq 1) \qquad \text{and} \qquad \sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0}}|L(1,\chi)|^2 \tag{2.5}$$

with respect to q. See also Katsurada [28], where a somewhat different argument using confluent hypergeometric functions is given.

Let $\zeta(s,\alpha)$ be the Hurwitz zeta-function defined by the analytic continuation of the series $\sum_{n=0}^{\infty}(\alpha+n)^{-s}$, where $\alpha>0$. Katsurada and the author [34] proved the asymptotic expansion of the mean value

$$\int_0^1|\zeta(s,\alpha)-\alpha^{-s}|^2d\alpha \tag{2.6}$$

with respect to $\Im s$. The starting point of the argument in [34] is the following generalization of (2.2):

$$\zeta(s_1,\alpha)\zeta(s_2,\alpha)=\zeta(s_1+s_2,\alpha)+\zeta_2((s_1,s_2);\alpha)+\zeta_2((s_2,s_1);\alpha), \tag{2.7}$$

where

$$\zeta_2((s_1,s_2);\alpha)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(\alpha+m)^{-s_1}(\alpha+m+n)^{-s_2}. \tag{2.8}$$

This is again a generalization of (2.1). In [34], the meromorphic continuation of $\zeta_2((s_1,s_2);\alpha)$ was achieved by using the formula

$$\begin{aligned}\zeta_2((s_1,s_2);\alpha) &= \frac{\Gamma(s_1+s_2-1)\Gamma(1-s_1)}{\Gamma(s_2)}\zeta(s_1+s_2-1)\\ &+\frac{1}{\Gamma(s_1)\Gamma(s_2)(e^{2\pi i s_1}-1)(e^{2\pi i s_2}-1)}\int_{\mathcal{C}}\frac{y^{s_2}-1}{e^y-1}\\ &\times\int_{\mathcal{C}}h(x+y;\alpha)x^{s_1-1}dxdy, \end{aligned}\tag{2.9}$$

where

$$h(z;\alpha)=\frac{e^{(1-\alpha)z}}{e^z-1}-\frac{1}{z}.$$

This formula is an analogue of Motohashi's integral expression for (2.4).

The author [42] considered the more general series

$$\zeta_2((s_1,s_2);\alpha,w)=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}(\alpha+m_1)^{-s_1}(\alpha+m_1+m_2w)^{-s_2},\tag{2.10}$$

where $w>0$, and proved its analytic continuation in a way similar to the above. This method also gives the asymptotic expansion of $\zeta_2((s_1,s_2);\alpha,w)$ with respect to w when $w\to+\infty$. This especially implies the asymptotic expansion of the Barnes double zeta-function $\zeta_2(s;\alpha,(1,w))$ with respect to w, because this function is nothing but $\zeta_2((0,s);\alpha,w)$. These results and also the asymptotic expansion of the double gamma-function are proved in [42]. Note that some claims in [42] on the uniformity of the error terms are not true, which are corrected in [43] (see also [45]).

3 Multi-variable Euler-Zagier sums

The r-variable generalization of the Euler sum (2.1), defined by

$$\begin{aligned}&\zeta_r(s_1,\ldots,s_r)\\ &\quad=\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}m_1^{-s_1}(m_1+m_2)^{-s_2}\cdots(m_1+\cdots+m_r)^{-s_r},\end{aligned}\tag{3.1}$$

is absolutely convergent in the region

$$\mathcal{A}_r=\{(s_1,\ldots,s_r)\in\mathbf{C}^r\mid\Re(s_{r-k+1}+\cdots+s_r)>k\quad(1\le k\le r)\},\tag{3.2}$$

as will be shown in Theorem 3 below (in Section 6). (The condition of absolute convergence given by Proposition 1 of Zhao [66] is not sufficient.) In connection with knot theory, quantum groups and mathematical physics, the properties of

(3.1) has been investigated recently by Zagier [64][65], Goncharov [19] and others, and is called the Euler-Zagier sum or the multiple harmonic series. The case $r = 3$ of (3.1) was actually already studied by Sitaramachandrarao and Subbarao [58]. The Euler-Zagier sum also appears in the works of Butzer, Markett and Schmidt ([15], [16], [41]).

There are various interesting relations among values of (3.1) at positive integers. Some of them (for small r) can be found in earlier references, but systematic studies were begun by Hoffman [22] (see also [23]). He proved a class of relations, including some previous results and conjectures, and stated the sum conjecture and the duality conjecture. The sum conjecture, originally due to M. Schmidt (see Markett [41]) and also to C. Moen, was proved by Granville [21] and Zagier (unpublished). On the other hand, the duality conjecture has turned out to be an immediate consequence of iterated integral representations of Drinfel'd and Kontsevich (cf. Zagier [65]). Further generalizations were done by Ohno [55] and Hoffman-Ohno [24]. Other families of relations, coming from the theory of knot invariants, were discovered by Le-Murakami [35] and Takamuki [60]. Various relations were also discussed by Borwein et al. [11][12], Flajolet-Salvy [18] and Minh-Petitot [51]. For instance, a conjecture mentioned in Zagier [65] was proved in [12][13]. See also [14] and [56] for the latest developments. (Recent developments in this direction are really enormous; it is impossible to mention all of them here.)

The works mentioned above were mainly devoted to the study of the values of $\zeta_r(s_1, \ldots, s_r)$ at positive integers. On the other hand, except for the case $r = 2$ explained in the preceding section, the study of analytic continuation of $\zeta_r(s_1, \ldots, s_r)$ has begun very recently. First, Arakawa and Kaneko [5] proved that if $s_1, \ldots, s_{r-1}$ are fixed, then (3.1) can be continued meromorphically with respect to s_r to the whole complex plane. The analytic continuation of (3.1) to the whole $\mathbf{C}^r$-space as an r-variable function was established by Zhao [66], and independently by Akiyama, Egami and Tanigawa [1]. Zhao's proof is based on properties of generalized functions in the sense of Gel'fand and Shilov. The method in [1] is more elementary; an application of the Euler-Maclaurin summation formula. Akiyama, Egami and Tanigawa [1] further studied the values of $\zeta_r(s_1, \ldots, s_r)$ at non-positive integers (see also Akiyama and Tanigawa [3]). Note that the statements about the trivial zeros of ζ_2 in Zhao [66] are incorrect. T. Arakawa pointed out that the method of Arakawa and Kaneko [5] can also be refined to give an alternative proof of analytic continuation of $\zeta_r(s_1, \ldots, s_r)$ as an r-variable function.

Akiyama and Ishikawa [2] considered the multiple L-function

$$\begin{aligned} &L_r((s_1, \ldots, s_r); (\chi_1, \ldots, \chi_r)) \\ &\quad = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{\chi_1(m_1)}{m_1^{s_1}} \frac{\chi_2(m_1+m_2)}{(m_1+m_2)^{s_2}} \cdots \frac{\chi_r(m_1+\cdots+m_r)}{(m_1+\cdots+m_r)^{s_r}}, \end{aligned} \tag{3.3}$$

where $\chi_1, \ldots, \chi_r$ are Dirichlet characters. This series itself was introduced earlier by Goncharov [20], but the purpose of Akiyama and Ishikawa [2] was to prove the analytic continuation of (3.3). For this purpose, they first wrote (3.3) as a linear combination of

$$\zeta_r((s_1,\ldots,s_r);(\alpha_1,\ldots,\alpha_r)) = \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}(\alpha_1+m_1)^{-s_1} \times(\alpha_2+m_1+m_2)^{-s_2}\cdots(\alpha_r+m_1+\cdots+m_r)^{-s_r}, \tag{3.4}$$

where $\alpha_1,\ldots,\alpha_r$ are positive, and considered the analytic continuation of the latter. They established this continuation by generalizing the argument in Akiyama, Egami and Tanigawa [1]. Ishikawa [26] proved more refined properties of the special case $s_1=\cdots=s_r$ in (3.3), and he applied those results to the study of certain multiple character sums (Ishikawa [27]).

4 Katsurada's idea

In Section 2 we mentioned the works of Katsurada and the author on asymptotic expansions of (2.5) and (2.6). The essence of those works are the treatment of the functions (2.4) and (2.8), and in [32][34] these functions are expressed by certain double contour integrals.

Katsurada [29][30] reconsidered this problem, and discovered a simple elegant alternative way of proving the expansions of (2.5) and (2.6). The key tool of Katsurada's method is the Mellin-Barnes integral formula

$$\Gamma(s)(1+\lambda)^{-s} = \frac{1}{2\pi i}\int_{(c)}\Gamma(s+z)\Gamma(-z)\lambda^z dz, \tag{4.1}$$

where s and η are complex with $\Re s>0$, $|\arg\lambda|<\pi$, $\lambda\neq 0$, and c is real with $-\Re s<c<0$. The path of integration is the vertical line from $c-i\infty$ to $c+i\infty$. This formula is classically known (e.g. Whittaker and Watson [62], Section 14.51, p.289, Corollary), or can be easily proved as follows. First assume $|\lambda|<1$, and shift the path to the right. The relevant poles of the integrand are located at $z=n$ $(n=0,1,2,\ldots)$ with the residue $(-1)^{n+1}\Gamma(s+n)\lambda^n/n!$. Hence the right-hand side of (4.1) is equal to

$$\Gamma(s)\sum_{n=0}^{\infty}\binom{-s}{n}\lambda^n = \Gamma(s)(1+\lambda)^{-s},$$

which is the left-hand side. The case of larger $|\lambda|$ now follows by analytic continuation.

Katsurada [30] used (4.1) to obtain a simple argument of deducing the analytic continuation and the asymptotic expansion of the function (2.4). Then, Katsurada [29] (this article was published earlier, but written later than [30]) proved that the same idea can be applied to the function (2.8) to obtain its analytic continuation. In [29], this idea is combined with some properties of hypergeometric functions, hence the technical details are not so simple. Therefore, to illustrate the essence of Katsurada's idea clearly, we present here a simple proof of the analytic continuation of the Euler sum (2.1) by his method.

Assume $\Re s_2 > 1$ and $\Re(s_1+s_2) > 2$. Putting $s = s_2$ and $\lambda = n/m$ in (4.1), and dividing the both sides by $\Gamma(s_2)m^{s_1+s_2}$, we obtain

$$m^{-s_1}(m+n)^{-s_2} = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s_2+z)\Gamma(-z)}{\Gamma(s_2)} m^{-s_1-s_2-z} n^z dz. \tag{4.2}$$

We may assume $\min\{-\Re s_2, 1-\Re(s_1+s_2)\} < c < -1$. Then we can sum up the both sides of (4.2) with respect to m and n to obtain

$$\zeta_2(s_1,s_2) = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s_2+z)\Gamma(-z)}{\Gamma(s_2)} \zeta(s_1+s_2+z)\zeta(-z) dz. \tag{4.3}$$

Now we shift the path to $\Re z = M-\varepsilon$, where M is a positive integer and ε is a small positive number. The validity of this shifting is easily shown by using Stirling's formula. The relevant poles of the integrand are at $z = -1, 0, 1, 2, \ldots, M-1$. Counting the residues of those poles, we get

$$\begin{aligned}\zeta_2(s_1,s_2) = &\frac{1}{s_2-1}\zeta(s_1+s_2-1) + \sum_{k=0}^{M-1}\binom{-s_2}{k}\zeta(s_1+s_2+k)\zeta(-k) \\ &+\frac{1}{2\pi i}\int_{(M-\varepsilon)} \frac{\Gamma(s_2+z)\Gamma(-z)}{\Gamma(s_2)} \zeta(s_1+s_2+z)\zeta(-z) dz.\end{aligned} \tag{4.4}$$

The last integral can be continued holomorphically to the region

$$\left\{(s_1,s_2) \in \mathbf{C}^2 \mid \Re s_2 > -M+\varepsilon, \Re(s_1+s_2) > 1-M+\varepsilon\right\},$$

because in this region the poles of the integrand are not on the path of integration. Hence (4.4) gives the analytic continuation of $\zeta_2(s_1,s_2)$ to this region. Since M is arbitrary, the proof of the continuation to the whole $\mathbf{C}^2$-space is complete. Moreover, from (4.4) we can see that the singularities of $\zeta_2(s_1,s_2)$ are located only on the subsets of $\mathbf{C}^2$ defined by one of the equations

$$s_2 = 1, \quad s_1+s_2 = 2-\ell \quad (\ell \in \mathbf{N}_0), \tag{4.5}$$

where $\mathbf{N}_0$ denotes the set of non-negative integers.

Katsurada applied (4.1) to various other types of problems. Here we mention his short note [31], in which he introduced (inspired by [42]) the double zeta-function of the form

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} e^{2\pi i(ms_1+ns_2)}(\alpha+m)^{-s_1}(\alpha+\beta+m+n)^{-s_2},$$

expressed it as an integral similar to (4.3), and obtained some asymptotic results in the domain of absolute convergence.

5 The Mordell-Tornheim zeta-function and the Apostol-Vu zeta-function

Let $\Re s_j > 1$ $(j = 1, 2, 3)$ and define

$$\zeta_{MT}(s_1, s_2, s_3) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{-s_1} n^{-s_2} (m+n)^{-s_3}. \tag{5.1}$$

This series was first considered by Tornheim [61], and the special case $s_1 = s_2 = s_3$ was studied independently by Mordell [52]. We call (5.1) as the Mordell-Tornheim zeta-function. Tornheim himself called it the harmonic double series. Zagier [65] quoted Witten's paper [63] and studied (5.1) under the name of the Witten zeta-function.

The analytic continuation of $\zeta_{MT}(s_1, s_2, s_3)$ was established by S. Akiyama and also by S. Egami in 1999. Akiyama's method is based on the Euler-Maclaurin summation formula, while Egami's proof is a modification of the method of Arakawa and Kaneko [5]. Both of their proofs have been unpublished yet.

Here, by using the method explained in the preceding section, we give a simple proof of

Theorem 1 *The function $\zeta_{MT}(s_1, s_2, s_3)$ can be meromorphically continued to the whole $\mathbf{C}^3$-space, and its singularities are only on the subsets of $\mathbf{C}^3$ defined by one of the equations $s_1 + s_3 = 1 - \ell$, $s_2 + s_3 = 1 - \ell$ $(\ell \in \mathbf{N}_0)$ or $s_1 + s_2 + s_3 = 2$.*

Proof. Assume $\Re s_1 > 1$, $\Re s_2 > 0$ and $\Re s_3 > 1$. Then the series (5.1) is absolutely convergent. Putting $s = s_3$ and $\lambda = n/m$ in (4.1), and dividing the both sides by $\Gamma(s_3) m^{s_1+s_3} n^{s_2}$, we obtain

$$m^{-s_1-s_3} n^{-s_2} (1 + \frac{n}{m})^{-s_3} = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)} m^{-s_1-s_3-z} n^{-s_2+z} dz.$$

We may assume $-\Re s_3 < c < \min\{\Re s_2 - 1, 0\}$. Summing up with respect to m and n we get

$$\zeta_{MT}(s_1, s_2, s_3) = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)} \zeta(s_1+s_3+z)\zeta(s_2-z) dz. \tag{5.2}$$

Let M be a positive integer which is larger than $\Re s_2 - 1 + \varepsilon$, and shift the path to $\Re z = M - \varepsilon$. First assume that s_2 is not a positive integer. Then all the relevant poles are simple, and we obtain

$$\zeta_{MT}(s_1,s_2,s_3)=\frac{\Gamma(s_2+s_3-1)\Gamma(1-s_2)}{\Gamma(s_3)}\zeta(s_1+s_2+s_3-1)$$
$$+\sum_{k=0}^{M-1}\binom{-s_3}{k}\zeta(s_1+s_3+k)\zeta(s_2-k)$$
$$+\frac{1}{2\pi i}\int_{(M-\varepsilon)}\frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)}\zeta(s_1+s_3+z)\zeta(s_2-z)dz. \tag{5.3}$$

When $s_2=1+h$ ($h\in\mathbf{N}_0$, $h\le M-1$), the right-hand side of (5.3) contains two singular factors, but they cancel each other. In fact, we obtain

$$\zeta_{MT}(s_1,1+h,s_3)$$
$$=\binom{-s_3}{h}\left\{\left(1+\frac{1}{2}+\cdots+\frac{1}{h}-\psi(s_3+h)\right)\zeta(s_1+s_3+h)-\zeta'(s_1+s_3+h)\right\}$$
$$+\sum_{\substack{k=0\\k\neq h}}^{M-1}\binom{-s_3}{k}\zeta(s_1+s_3+k)\zeta(1+h-k)$$
$$+\frac{1}{2\pi i}\int_{(M-\varepsilon)}\frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)}\zeta(s_1+s_3+z)\zeta(1+h-z)dz, \tag{5.4}$$

where $\psi=\Gamma'/\Gamma$. The empty sum is to be considered as zero. The desired assertions of Theorem 1 now follow from (5.3) and (5.4), as in the argument described in the preceding section.

After the papers of Tornheim [61] and Mordell [52], the values of $\zeta_{MT}(s_1,s_2,s_3)$ at positive integers have been investigated by many authors (Subbarao and Sitaramachandrarao [59], Huard, Williams and Zhang [25], and Zagier [65]). It is now an interesting problem to study the properties of the values of $\zeta_{MT}(s_1,s_2,s_3)$ at non-positive integers.

Next, recall the series (2.3) considered by Apostol and Vu [4]. They were inspired by the work of Sitaramachandrarao and Sivaramasarma [57], and various formulas on the special values of (2.3) were obtained in these papers.

Here we introduce the following three-variable Apostol-Vu zeta-function:

$$\zeta_{AV}(s_1,s_2,s_3)=\sum_{m=1}^{\infty}\sum_{n<m}m^{-s_1}n^{-s_2}(m+n)^{-s_3}\qquad(\Re s_j>1). \tag{5.5}$$

Note that there is the following simple relation between ζ_{AV} and ζ_{MT}:

$$\zeta_{MT}(s_1,s_2,s_3)=2^{-s_3}\zeta(s_1+s_2+s_3)+\zeta_{AV}(s_1,s_2,s_3)+\zeta_{AV}(s_2,s_1,s_3). \tag{5.6}$$

Also, there is a simple relation between $\zeta_{AV}(s_1,s_2,1)$ and $\zeta_2(s_1,s_2)$ (see (17) of Apostol and Vu [4]).

Now we prove the analytic continuation of the (three-variable) Apostol-Vu zeta-function $\zeta_{AV}(s_1,s_2,s_3)$. The principle of the proof is the same as in Theorem 1, but the details are somewhat more complicated.

Theorem 2 *The function $\zeta_{AV}(s_1,s_2,s_3)$ can be continued meromorphically to the whole $\mathbf{C}^3$-space, and its singularities are only on the subsets of $\mathbf{C}^3$ defined by one of the equations $s_1+s_3=1-\ell$, or $s_1+s_2+s_3=2-\ell$ $(\ell\in\mathbf{N}_0)$.*

Proof. Assume $\Re s_j>1$ $(j=1,2,3)$. Quite similarly to (5.2), this time we obtain

$$\begin{aligned}\zeta_{AV}(s_1,s_2,s_3) &= \frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)}\sum_{m=1}^{\infty}\sum_{n<m}m^{-s_1-s_3-z}n^{-s_2+z}dz\\ &= \frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)}\zeta_2(s_2-z,s_1+s_3+z)dz, \qquad (5.7)\end{aligned}$$

where $-\Re s_3<c<0$. Now we shift the path of integration to $\Re z=M-\varepsilon$. It is not difficult to show from (4.4) that $\zeta_2(s_1,s_2)$ is of polynomial order with respect to $\Im s_1$ and $\Im s_2$. Hence this shifting is possible. From (4.5) we see that the only pole of $\zeta_2(s_2-z,s_1+s_3+z)$ (as a function in z), under the assumption $\Re s_j>1$ $(j=1,2,3)$, is $z=1-s_1-s_3$. This is located on the left-hand side of $\Re z=c$, hence irrelevant now. Counting the residues of the poles at $z=0,1,\ldots,M-1$, we get

$$\begin{aligned}\zeta_{AV}(s_1,s_2,s_3) &= \sum_{k=0}^{M-1}\binom{-s_3}{k}\zeta_2(s_2-k,s_1+s_3+k)\\ &+\frac{1}{2\pi i}\int_{(M-\varepsilon)}\frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)}\zeta_2(s_2-z,s_1+s_3+z)dz. \qquad (5.8)\end{aligned}$$

This formula already implies the meromorphic continuation except for the case $s_1+s_2+s_3=2-\ell$ $(\ell\in\mathbf{N}_0)$. However, $\zeta_2(s_2-z,s_1+s_3+z)$ is singular when $s_1+s_2+s_3=2-\ell$. To clarify the behaviour of the above integral on this polar set, we substitute the formula (4.4) into the integrand on the right-hand side of (5.8). We obtain

$$\begin{aligned}\zeta_{AV}(s_1,s_2,s_3) &= \sum_{k=0}^{M-1}\binom{-s_3}{k}\zeta_2(s_2-k,s_1+s_3+k)\\ &+\zeta(s_1+s_2+s_3-1)P(s_1,s_3)+\sum_{j=0}^{M-1}\zeta(s_1+s_2+s_3+j)\zeta(-j)Q_j(s_1,s_3)\\ &+R(s_1,s_2,s_3), \qquad (5.9)\end{aligned}$$

where

$$P(s_1,s_3)=\frac{1}{2\pi i}\int_{(M-\varepsilon)}\frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)}\frac{dz}{s_1+s_3+z-1},$$

$$Q_j(s_1,s_3)=\frac{1}{2\pi i}\int_{(M-\varepsilon)}\frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)}\binom{-s_1-s_3-z}{j}dz,$$

and

$$\begin{aligned}R(s_1,s_2,s_3)=&\frac{1}{(2\pi i)^2}\int_{(M-\varepsilon)}\frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)}\\ &\times\int_{(M-\varepsilon)}\frac{\Gamma(s_1+s_3+z+z')\Gamma(-z')}{\Gamma(s_1+s_3+z)}\zeta(s_1+s_2+s_3+z')\zeta(-z')dz'dz.\end{aligned}$$

It is easy to see that

(i) $P(s_1,s_3)$ is holomorphic if $\Re s_3>-M+\varepsilon$ and $\Re(s_1+s_3)>1-M+\varepsilon$, and

(ii) $Q_j(s_1,s_3)$ is holomorphic for $0\le j\le M-1$ if $\Re s_3>-M+\varepsilon$.

Also, since the inner integral of $R(s_1,s_2,s_3)$ is holomorphic if $\Re(s_1+s_3+z)>-M+\varepsilon$ and $\Re(s_1+s_2+s_3)>1-M+\varepsilon$ as a function of the four variables (s_1,s_2,s_3,z), we see that

(iii) $R(s_1,s_2,s_3)$ is holomorphic if $\Re s_3>-M+\varepsilon$, $\Re(s_1+s_3)>-2M+2\varepsilon$ and $\Re(s_1+s_2+s_3)>1-M+\varepsilon$.

From (i), (ii), (iii) and (5.9), we find that $\zeta_{AV}(s_1,s_2,s_3)$ can be continued meromorphically to the region

$$\left\{(s_1,s_2,s_3)\in\mathbf{C}^3\;\middle|\;\begin{array}{l}\Re s_3>-M+\varepsilon,\\ \Re(s_1+s_3)>1-M+\varepsilon,\\ \Re(s_1+s_2+s_3)>1-M+\varepsilon\end{array}\right\}.$$

Since M is arbitrary, we obtain the analytic continuation of $\zeta_{AV}(s_1,s_2,s_3)$ to the whole $\mathbf{C}^3$-space. The information on singularities can be deduced from the expression (5.9). The proof of Theorem 2 is complete.

6 Generalized multiple zeta-functions

Let $s_1,\ldots,s_r$ be complex variables, $\alpha_1,\ldots,\alpha_r,\ w_1,\ldots,w_r$ be complex parameters, and define the multiple series

$$\begin{aligned}&\zeta_r((s_1,\ldots,s_r);(\alpha_1,\ldots,\alpha_r),(w_1,\ldots,w_r))\\ &=\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}(\alpha_1+m_1w_1)^{-s_1}(\alpha_2+m_1w_1+m_2w_2)^{-s_2}\\ &\qquad\times\cdots\times(\alpha_r+m_1w_1+\cdots+m_rw_r)^{-s_r}.\end{aligned}\tag{6.1}$$

We will explain later (in the proof of Theorem 3) how to choose the branch of logarithms.

When $s_1 = \cdots = s_{r-1} = 0$, then the above series (6.1) reduces to the Barnes multiple zeta-function (1.3). The Euler-Zagier sum (3.1) and its generalization (3.4) are also special cases of (6.1). The multiple series of the form (6.1) was first introduced in the author's article [44], and the meromorphic continuation of the special case $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_r$ and $w_j = 1$ $(1 \leq j \leq r)$ of (6.1) to the whole $\mathbf{C}^r$-space was proved in [44].

To ensure the convergence of (6.1), we assume the condition (1.4) on w_js, which was first introduced by Barnes for his multiple series (1.3). But we do not require any condition on α_js. If $\alpha_j \notin H(\ell)$ for some j, then there might exist finitely many $(m_1, \ldots, m_j)$'s for which

$$\alpha_j + m_1 w_1 + \cdots + m_j w_j = 0 \tag{6.2}$$

holds. We adopt the convention that the terms corresponding to such $(m_1, \ldots, m_j)$'s are removed from the sum (6.1). Under this convention, we now prove

Theorem 3 *If the condition (1.4) holds, then the series (6.1) is absolutely convergent in the region $\mathcal{A}_r$, defined by (3.2), uniformly in any compact subset of $\mathcal{A}_r$.*

Proof. We prove the theorem by induction. When $r = 1$, the assertion is obvious. Assume that the theorem is true for ζ_{r-1}. In what follows, the empty sum is to be considered as zero.

Let $\theta \in (-\pi, \pi]$ be the argument of the vector contained in $H(\ell)$ and orthogonal to ℓ. Then the line ℓ consists of the points whose arguments are $\theta \pm \pi/2$ (and the origin), and

$$H(\ell) = \left\{ w \in \mathbf{C} \setminus \{0\} \;\middle|\; \theta - \frac{\pi}{2} < \arg w < \theta + \frac{\pi}{2} \right\}.$$

We can write $w_j = w_j^{(1)} + w_j^{(2)}$, with $\arg w_j^{(1)} = \theta - \pi/2$ or $\theta + \pi/2$ (or $w_j^{(1)} = 0$) and $\arg w_j^{(2)} = \theta$. Similarly we write $\alpha_j = \alpha_j^{(1)} + \alpha_j^{(2)}$ with $\arg \alpha_j^{(1)} = \theta - \pi/2$ or $\theta + \pi/2$ (or $\alpha_j^{(1)} = 0$) and $\arg \alpha_j^{(2)} = \theta$ or $-\theta$ (or $\alpha_j^{(2)} = 0$). If the set

$$\mathcal{E} = \left\{ \alpha_j^{(2)} \;\middle|\; \arg \alpha_j^{(2)} = -\theta \quad \text{or} \quad \alpha_j^{(2)} = 0 \right\}$$

is not empty, we denote by $\tilde{\alpha}$ (one of) the element(s) of this set whose absolute value is largest. Let μ be the smallest positive integer such that $\tilde{\alpha} + m_1 w_1^{(2)} \in H(\ell)$ for any $m_1 \geq \mu$, and divide (6.1) as

$$\begin{aligned} &\zeta_r((s_1, \ldots, s_r); (\alpha_1, \ldots, \alpha_r), (w_1, \ldots, w_r)) \\ &\quad = \sum_{m_1=0}^{\mu-1} \sum_{m_2=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} + \sum_{m_1=\mu}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} = T_1 + T_2, \end{aligned} \tag{6.3}$$

say. (If $\mathcal{E} = \emptyset$, then we put $\mu = 0$.) For any $m_1 \le \mu - 1$, we put $\alpha'_j(m_1) = \alpha_j + m_1 w_1$. Then

$$\begin{aligned} T_1 &= \sum_{m_1=0}^{\mu-1} \alpha'_1(m_1)^{-s_1} \sum_{m_2=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} (\alpha'_2(m_1) + m_2 w_2)^{-s_2} \\ &\qquad \times \cdots \times (\alpha'_r(m_1) + m_2 w_2 + \cdots + m_r w_r)^{-s_r} \\ &= \sum_{m_1=0}^{\mu-1} \alpha'_1(m_1)^{-s_1} \\ &\qquad \times \zeta_{r-1}((s_2, \ldots, s_r); (\alpha'_2(m_1), \ldots, \alpha'_r(m_1)), (w_2, \ldots, w_r)). \end{aligned} \tag{6.4}$$

As for T_2, we put $\alpha'_j(\mu) = \alpha_j + \mu w_1$ and $m'_1 = m_1 - \mu$. Then

$$\begin{aligned} T_2 = \sum_{m'_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} (\alpha'_1(\mu) + m'_1 w_1)^{-s_1} (\alpha'_2(\mu) + m'_1 w_1 + m_2 w_2)^{-s_2} \\ \times \cdots \times (\alpha'_r(\mu) + m'_1 w_1 + m_2 w_2 + \cdots + m_r w_r)^{-s_r}. \end{aligned} \tag{6.5}$$

Since $\alpha'_j(\mu) = (\alpha_j^{(1)} + \mu w_1^{(1)}) + (\alpha_j^{(2)} + \mu w_1^{(2)})$, the definitions of $\tilde{\alpha}$ and μ imply that $\alpha'_j(\mu) \in H(\ell)$. The right-hand side of (6.4) is absolutely convergent by induction assumption. Hence we have only to show the absolute convergence of (6.5). In other words, our remaining task is to prove the absolute convergence of (6.1) under the additional assumption that $\alpha_j \in H(\ell)$ $(1 \le j \le r)$. Then always $\alpha_j + m_1 w_1 + \cdots + m_r w_r \in H(\ell)$. Each factor on the right-hand side of (6.1) is to be understood as

$$(\alpha_j + m_1 w_1 + \cdots + m_j w_j)^{-s_j} = \exp(-s_j \log(\alpha_j + m_1 w_1 + \cdots + m_j w_j)),$$

where the branch of the logarithm is chosen by the condition

$$\theta - \frac{\pi}{2} < \arg(\alpha_j + m_1 w_1 + \cdots + m_j w_j) < \theta + \frac{\pi}{2}.$$

Let $\sigma_j = \Re s_j$, $t_j = \Im s_j$, and define $J_+ = \{j \mid \sigma_j \ge 0\}$ and $J_- = \{j \mid \sigma_j < 0\}$. Since

$$\begin{aligned} &|\alpha_j + m_1 w_1 + \cdots + m_j w_j| \\ &\quad \ge |\alpha_j^{(2)} + m_1 w_1^{(2)} + \cdots + m_j w_j^{(2)}| \\ &\quad = |\, |\alpha_j^{(2)}| e^{i\theta} + m_1 |w_1^{(2)}| e^{i\theta} + \cdots + m_j |w_j^{(2)}| e^{i\theta}| \\ &\quad = |\alpha_j^{(2)}| + m_1 |w_1^{(2)}| + \cdots + m_j |w_j^{(2)}|, \end{aligned}$$

we have

$$|\alpha_j + m_1 w_1 + \cdots + m_j w_j|^{-\sigma_j} \le (|\alpha_j^{(2)}| + m_1 |w_1^{(2)}| + \cdots + m_j |w_j^{(2)}|)^{-\sigma_j}$$

for $j \in J_+$. On the other hand, it is clear that

$$|\alpha_j + m_1 w_1 + \cdots + m_j w_j|^{-\sigma_j} \le (|\alpha_j| + m_1|w_1| + \cdots + m_j|w_j|)^{-\sigma_j}$$

for $j \in J_-$. Therefore, denoting

$$\alpha_j^* = \begin{cases} |\alpha_j^{(2)}| & \text{if } j \in J_+ \\ |\alpha_j| & \text{if } j \in J_- \end{cases}$$

and

$$w_j^* = \begin{cases} |w_j^{(2)}| & \text{if } j \in J_+ \\ |w_j| & \text{if } j \in J_-, \end{cases}$$

we find that $\alpha_j^* > 0$, $w_j^* > 0$ for all j and that

$$\begin{aligned} &|(\alpha_j + m_1 w_1 + \cdots + m_j w_j)^{-s_j}| \\ &\quad = |\alpha_j + m_1 w_1 + \cdots + m_j w_j|^{-\sigma_j} \exp(t_j \arg(\alpha_j + m_1 w_1 + \cdots + m_j w_j)) \\ &\quad \le (\alpha_j^* + m_1 w_1^* + \cdots + m_j w_j^*)^{-\sigma_j} \exp(2\pi|t_j|). \end{aligned}$$

Hence

$$\begin{aligned} &|\zeta_r((s_1, \ldots, s_r); (\alpha_1, \ldots, \alpha_r), (w_1, \ldots, w_r))| \\ &\quad \le \exp(2\pi(|t_1| + \cdots + |t_r|) \\ &\qquad \times \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} (\alpha_1^* + m_1 w_1^*)^{-\sigma_1} (\alpha_2^* + m_1 w_1^* + m_2 w_2^*)^{-\sigma_2} \\ &\qquad \times \cdots \times (\alpha_r^* + m_1 w_1^* + \cdots + m_r w_r^*)^{-\sigma_r}. \end{aligned} \tag{6.6}$$

We claim that for any positive integers $k \le r$, the series

$$\begin{aligned} S(k) = &\sum_{m_{r-k+1}=0}^{\infty} \sum_{m_{r-k+2}=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \\ &(\alpha_{r-k+1}^* + m_1 w_1^* + \cdots + m_{r-k+1} w_{r-k+1}^*)^{-\sigma_{r-k+1}} \\ &\times (\alpha_{r-k+2}^* + m_1 w_1^* + \cdots + m_{r-k+2} w_{r-k+2}^*)^{-\sigma_{r-k+2}} \\ &\times \cdots \times (\alpha_r^* + m_1 w_1^* + \cdots + m_r w_r^*)^{-\sigma_r} \end{aligned}$$

is convergent in the region $\sigma_r > 1$, $\sigma_{r-1} + \sigma_r > 2$,..., $\sigma_{r-k+1} + \cdots + \sigma_r > k$, and the estimate

$$\begin{aligned} S(k) \ll &(\beta_1(k) + m_1 w_1^* + \cdots + m_{r-k} w_{r-k}^*) \\ &\times (\beta_2(k) + m_1 w_1^* + \cdots + m_{r-k} w_{r-k}^*)^{c(k)} \end{aligned} \tag{6.7}$$

holds, where $\beta_1(k) > \beta_2(k) > 0$,

$$c(k) = k - 1 - (\sigma_{r-k+1} + \cdots + \sigma_r), \tag{6.8}$$

and the implied constant depends on σ_j, α_j^* and w_j^* $(r-k+1 \leq j \leq r)$. Note that $c(k) < -1$.

We prove this claim by induction. For any positive a, b and $\sigma > 1$, we have

$$\sum_{m=0}^{\infty} (a+bm)^{-\sigma} = a^{-\sigma} + \int_0^{\infty} (a+bx)^{-\sigma} dx \ll \left(1 + \frac{a}{b}\right) a^{-\sigma}, \tag{6.9}$$

where the implied constant depends only on σ. Using (6.9) with $m = m_r$, $\sigma = \sigma_r$, $a = \alpha_r^* + m_1 w_1^* + \cdots + m_{r-1} w_{r-1}^*$ and $b = w_r^*$, we can easily show the case $k=1$ of the claim with $\beta_1(1) = \alpha_r^* + w_r^*$ and $\beta_2(1) = \alpha_r^*$. Now we assume that the claim is true for $S(k-1)$. Then we have

$$\begin{aligned} S(k) \ll \sum_{m_{r-k+1}=0}^{\infty} & (\alpha_{r-k+1}^* + m_1 w_1^* + \cdots + m_{r-k+1} w_{r-k+1}^*)^{-\sigma_{r-k+1}} \\ & \times (\beta_1(k-1) + m_1 w_1^* + \cdots + m_{r-k+1} w_{r-k+1}^*) \\ & \times (\beta_2(k-1) + m_1 w_1^* + \cdots + m_{r-k+1} w_{r-k+1}^*)^{c(k-1)}. \end{aligned}$$

If $-\sigma_{r-k+1} \geq 0$, then we replace α_{r-k+1}^* and $\beta_1(k-1)$ by $\max\{\alpha_{r-k+1}^*, \beta_1(k-1)\}$. If $-\sigma_{r-k+1} < 0$, then we replace α_{r-k+1}^* and $\beta_2(k-1)$ by $\min\{\alpha_{r-k+1}^*, \beta_2(k-1)\}$. In any case, we get the estimate of the form

$$\begin{aligned} S(k) \quad \ll \quad & \sum_{m_{r-k+1}=0}^{\infty} (B_1 + m_1 w_1^* + \cdots + m_{r-k+1} w_{r-k+1}^*)^{C_1} \\ & \qquad \times (B_2 + m_1 w_1^* + \cdots + m_{r-k+1} w_{r-k+1}^*)^{C_2} \end{aligned} \tag{6.10}$$

where $B_1 > B_2 > 0$, $C_1 \geq 0$, $C_2 < 0$, and

$$C_1 + C_2 = -\sigma_{r-k+1} + 1 + c(k-1) = c(k). \tag{6.11}$$

Since

$$\begin{aligned} & (B_1 + m_1 w_1^* + \cdots + m_{r-k+1} w_{r-k+1}^*)^{C_1} \\ & \quad = (B_2 + m_1 w_1^* + \cdots + m_{r-k+1} w_{r-k+1}^*)^{C_1} \\ & \qquad \times \left(1 + \frac{B_1 - B_2}{B_2 + m_1 w_1^* + \cdots + m_{r-k+1} w_{r-k+1}^*}\right)^{C_1} \\ & \quad \leq \left(1 + \frac{B_1 - B_2}{B_2}\right)^{C_1} (B_2 + m_1 w_1^* + \cdots + m_{r-k+1} w_{r-k+1}^*)^{C_1}, \end{aligned}$$

from (6.10) and (6.11) it follows that

$$S(k) \quad \ll \quad \sum_{m_{r-k+1}=0}^{\infty} (B_2 + m_1 w_1^* + \cdots + m_{r-k+1} w_{r-k+1}^*)^{c(k)}. \tag{6.12}$$

The claim for $S(k)$ now follows by applying (6.9) to the right-hand side of (6.12), with $\beta_1(k) = B_2 + w^*_{r-k+1}$ and $\beta_2(k) = B_2$. Hence by induction we find that the claim is true for $1 \le k \le r$, and the claim for $k = r$ implies the absolute convergence of the right-hand side of (6.6). This completes the proof of Theorem 3.

Now we apply the method explained in Sections 4 and 5 to the generalized multiple zeta-function (6.1). Besides (1.4), we assume

$$\alpha_j \in H(\ell)\ (1 \le j \le r) \quad \text{and} \quad \alpha_{j+1} - \alpha_j \in H(\ell)\ (1 \le j \le r-1). \tag{6.13}$$

We use (4.1) with $s = s_r$ and

$$\lambda = \frac{\alpha_r - \alpha_{r-1} + m_r w_r}{\alpha_{r-1} + m_1 w_1 + \cdots + m_{r-1} w_{r-1}}.$$

Under the assumption (6.13) both the numerator and the denominator of λ are the elements of $H(\ell)$, hence $|\arg \lambda| < \pi$. Similarly to (4.3), (5.2) or (5.7), we obtain

$$\begin{aligned} &\zeta_r((s_1, \ldots, s_r); (\alpha_1, \ldots, \alpha_r), (w_1, \ldots, w_r)) \\ &\quad = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s_r + z)\Gamma(-z)}{\Gamma(s_r)} \zeta_{r-1}((s_1, \ldots, s_{r-2}, s_{r-1} + s_r + z); \\ &\qquad (\alpha_1, \ldots, \alpha_{r-1}), (w_1, \ldots, w_{r-1}))\zeta\left(-z, \frac{\alpha_r - \alpha_{r-1}}{w_r}\right) w_r^z dz. \end{aligned} \tag{6.14}$$

Hence, shifting the path of integration, we can prove

Theorem 4 *Under the conditions (1.4) and (6.13), the multiple zeta-function (6.1) can be continued meromorphically to the whole $\mathbf{C}^r$-space.*

In the present article we content ourselves with the above very brief outline of the method. The details of the proof, which is induction on r, will be given in [46].

Finally we mention the analytic continuation of Mordell multiple zeta-functions. In Section 5 we quoted Mordell's paper [52], in which he studied the special case $s_1 = s_2 = s_3$ of (5.1). In the same paper, Mordell also considered the multiple series

$$\sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{1}{m_1 m_2 \cdots m_r (m_1 + m_2 + \cdots + m_r + a)} \tag{6.15}$$

where $a > -r$. By using Mordell's result on (6.15), Hoffman [22] evaluated the sum

$$\sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{1}{m_1 m_2 \cdots m_r (m_1 + m_2 + \cdots + m_r)^s} \tag{6.16}$$

when s is a positive integer.

Here we introduce the following multi-variable version of (6.16), which is at the same time a generalization of the Mordell-Tornheim zeta-function (5.1):

$$\begin{aligned}&\zeta_{MOR,r}(s_1,\ldots,s_r,s_{r+1})\\&\quad=\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}m_1^{-s_1}\cdots m_r^{-s_r}(m_1+\cdots+m_r)^{-s_{r+1}}.\end{aligned}\tag{6.17}$$

Theorem 5 *The series (6.17) can be continued meromorphically to the whole* $\mathbf{C}^{r+1}$*-space.*

This and related results will be discussed in a forthcoming paper.

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Graduate Schoolof Mathematics
Nagoya University
Chikusa-ku, Nagoya 464-8602
Japan