Asymptotic expansions of double gamma-functions and related remarks

Kohji Matsumoto

Abstract

Let $\Gamma_2(\beta,(w_1,w_2))$ be the double gamma-function. We prove asymptotic expansions of $\log\Gamma_2(\beta,(1,w))$ with respect to w, both when $|w|\to+\infty$ and when $|w|\to 0$. Our proof is based on the results on Barnes' double zeta-functions given in the author's former article [11]. We also prove asymptotic expansions of $\log\Gamma_2(2\varepsilon_n-1,(\varepsilon_n-1,\varepsilon_n))$, $\rho_2(\varepsilon_n-1,\varepsilon_n)$ and $\rho_2(\varepsilon_n,\varepsilon_n^2-\varepsilon_n)$, where ε_n is the fundamental unit of $K=\mathbf{Q}(\sqrt{4n^2+8n+3})$. Combining those results with Fujii's formula [6][7], we obtain an expansion formula for $\zeta'(1;v_1)$, where $\zeta(s;v_1)$ is Hecke's zeta-function associated with K.

1 Introduction

This is a continuation of the author's article [11]. We first recall Theorem 1 and its corollaries in [11].

Let $\beta>0$, and w is a non-zero complex number with $|\arg w|<\pi$. The Barnes double zeta-function is defined by

$$\zeta_2(v;\beta,(1,w))=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\beta+m+nw)^{-v}. \tag{1.1}$$

This series is convergent absolutely for $\Re v>2$, and can be continued meromorphically to the whole v-plane, holomorphic except for the poles at $v=1$ and $v=2$.

Let $\zeta(v)$, $\zeta(v,\beta)$ be the Riemann zeta and the Hurwitz zeta-function, respectively, $\mathbf{C}$ the complex number field, θ_0 a fixed number satisfying $0<\theta_0<\pi$, and put

$$\mathcal{W}_\infty=\{w\in\mathbf{C}\mid |w|\geq 1,\ |\arg w|\leq\theta_0\}$$

and

$$\mathcal{W}_0=\{w\in\mathbf{C}\mid |w|\leq 1,\ w\neq 0, |\arg w|\leq\theta_0\}.$$

Define

$$\binom{v}{n} = \begin{cases} v(v-1)\cdots(v-n+1)/n! & \text{if } n \text{ is a positive integer,} \\ 1 & \text{if } n=0. \end{cases}$$

Theorem 1 in [11] asserts that for any positive integer N we have

$$\zeta_2(v;\beta,(1,w)) = \zeta(v,\beta) + \frac{\zeta(v-1)}{v-1}w^{1-v} + \sum_{k=0}^{N-1}\binom{-v}{k}\zeta(-k,\beta)\zeta(v+k)w^{-v-k} + O(|w|^{-\Re v-N}) \tag{1.2}$$

in the region $\Re v > -N+1$ and $w \in \mathcal{W}_\infty$, and also

$$\zeta_2(v;\beta,(1,w)) = \zeta(v,\beta) + \frac{\zeta(v-1,\beta)}{v-1}w^{-1} + \sum_{k=0}^{N-1}\binom{-v}{k}\zeta(v+k,\beta)\zeta(-k)w^{k} + O(|w|^{N}) \tag{1.3}$$

in the region $\Re v > -N+1$ and $w \in \mathcal{W}_0$ The implied constants in (1.2) and (1.3) depend only on N, v, β and θ_0.

There are two corollaries of these results stated in [11]. Corollary 1 gives the asymptotic expansions of Eisenstein series, which we omit here. The detailed proof of Corollary 1 are described in [11]. On the other hand, Corollary 2 in [11] is only stated without proof. Here we state it as the following Theorem 1.

Let $\Gamma_2(\beta,(1,w))$ be the double gamma-function defined by

$$\log\left(\frac{\Gamma_2(\beta,(1,w))}{\rho_2(1,w)}\right) = \zeta_2'(0;\beta,(1,w)), \tag{1.4}$$

where 'prime' denotes the differentiation with respect to v and

$$-\log\rho_2(1,w) = \lim_{\beta\to 0}\{\zeta_2'(0;\beta,(1,w)) + \log\beta\}. \tag{1.5}$$

Let $\psi(v) = (\Gamma'/\Gamma)(v)$ and γ the Euler constant. Then we have

Theorem 1 *For any positive integer $N \geq 2$, we have*

$$\begin{aligned}\log\Gamma_2(\beta,(1,w)) = &-\frac{1}{2}\beta\log w + \log\Gamma(\beta) + \frac{1}{2}\beta\log 2\pi \\ &+(\zeta(-1,\beta)-\zeta(-1))w^{-1}\log w - (\zeta(-1,\beta)-\zeta(-1))\gamma w^{-1} \\ &+\sum_{k=2}^{N-1}\frac{(-1)^k}{k}(\zeta(-k,\beta)-\zeta(-k))\zeta(k)w^{-k} \\ &+O(|w|^{-N})\end{aligned} \tag{1.6}$$

for $w \in \mathcal{W}_\infty$, where the implied constant depends only on N, β and θ_0. Also we have

$$\begin{aligned}
&\log \Gamma_2(\beta,(1,w)) \\
&\quad = \log \Gamma(\beta w^{-1}+1) + \frac{1}{2}\log \Gamma(\beta+1) - \log\beta + \beta w^{-1}\log w \\
&\quad + \left\{ \zeta(-1) + \zeta'(-1) - \frac{\gamma}{12} - \zeta_1(-1,\beta) - \zeta_1'(-1,\beta) - \frac{\psi(\beta+1)}{12} \right\} w^{-1} \\
&\quad - \sum_{k=2}^{N-1} \frac{(-1)^k}{k} (\zeta(k) - \zeta_1(k,\beta))\zeta(-k) w^k + O(|w|^N)
\end{aligned} \tag{1.7}$$

for $w \in \mathcal{W}_0$, where $\zeta_1(v,\beta) = \zeta(v,\beta) - \beta^{-v}$ and the implied constant depends only on N and θ_0.

Note that when $w > 0$, the formula (1.6) was already obtained in [10] by a different method.

We will show the proof of Theorem 1 in Section 2, and will give additional remarks in Section 3. In Section 4 we will prove Theorem 2, which will give a uniform error estimate with respect to β. In Section 5 we will state Theorem 3, our second main result in the present paper, that is an asymptotic expansion formula for $\zeta'(1; v_1)$, where $\zeta(s; v_1)$ is Hecke's zeta-function associated with the real quadratic field $\mathbf{Q}(\sqrt{4n^2+8n+3})$. This can be proved by combining Fujii's result [6][7] with certain expansions of double gamma-functions, and the proof of the latter will be described in Sections 6 to 8. Throughout this paper, the empty sum is to be considered as zero.

2 Proof of Theorem 1

Double gamma-functions were first introduced and studied by Barnes [3] [4] and others about one hundred years ago. In 1970s, Shintani [13][14] discovered the importance of double gamma-functions in connection with Kronecker limit formulas for real quadratic fields. Now the usefulness of double gama-functions in number theory is a well-known fact. For instance, see Vignéras [17], Arakawa [1] [2], Fujii [6][7]. Therefore it is desirable to study the asymptotic behaviour of double gamma-functions. Various asymptotic formulas of double gamma-functions are obtained by Billingham and King [5]. Also, when $w > 0$, the formula (1.6) was already proved in the author's article [10], by using a certain contour integral. It should be noted that in [10], it is claimed that the error term on the right-hand side of (1.6) is uniform in β (or α in the notation of [10]), but this is not true. See [12], and also Section 4 of the present paper.

Here we prove Theorem 1 by using the results given in [11]. The special case $u = 0$, $\alpha = \beta$ of the formula (3.8) of [11] implies

$$\begin{aligned}\zeta_2(v; \beta, (1, w)) &= \zeta(v, \beta) + \frac{\zeta(v-1)}{v-1} w^{1-v} \\ &+ \sum_{k=0}^{N-1} \binom{-v}{k} \zeta(-k, \beta) \zeta(v+k) w^{-v-k} \\ &+ R_N(v; \beta, w) \end{aligned} \tag{2.1}$$

for any positive integer N, where

$$R_N(v; \beta, w) = \frac{1}{2\pi i} \int_{(c_N)} \frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)} \zeta(v+z, \beta) \zeta(-z) w^z dz, \tag{2.2}$$

$c_N = -\Re v - N + \varepsilon$ with an arbitrarily small positive ε, and the path of the above integral is the vertical line $\Re z = c_N$. In Section 5 of [11] it is shown that (2.1) holds for $\Re v > -N + 1 + \varepsilon$, $w \in \mathcal{W}_\infty$. The result (1.2) is an immediate consequence of the above facts.

From (2.1) we have

$$\begin{aligned}\zeta_2'(v; \beta, (1, w)) &= \zeta'(v, \beta) - \frac{\zeta(v-1)}{(v-1)^2} w^{1-v} + \frac{\zeta'(v-1)}{v-1} w^{1-v} \\ &- \frac{\zeta(v-1)}{v-1} w^{1-v} \log w + \sum_{k=0}^{N-1} A_k(v; \beta, w) \\ &+ R_N'(v; \beta, w), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} A_k(v; \beta, w) = \Bigg\{ & \binom{-v}{k}' \zeta(v+k) + \binom{-v}{k} \zeta'(v+k) \\ & - \binom{-v}{k} \zeta(v+k) \log w \Bigg\} \zeta(-k, \beta) w^{-v-k}. \end{aligned}$$

Noting the facts

$$\begin{aligned} &\zeta(0) = -\frac{1}{2}, \quad \zeta(-1) = -\frac{1}{12}, \quad \zeta'(0) = -\frac{1}{2} \log 2\pi, \\ &\zeta(0, \beta) = \frac{1}{2} - \beta, \quad \zeta'(0, \beta) = \log \Gamma(\beta) - \frac{1}{2} \log 2\pi, \end{aligned} \tag{2.4}$$

and

$$\left. \binom{-v}{k}' \right|_{v=0} = \begin{cases} 0 & \text{if } k = 0, \\ (-1)^k k^{-1} & \text{if } k \geq 1, \end{cases}$$

we find

$$A_0(0;\beta,w)=\frac{1}{2}\left(\beta-\frac{1}{2}\right)(\log 2\pi-\log w),$$
$$\lim_{v\to 0}A_1(v;\beta,w)=\zeta(-1,\beta)w^{-1}(\log w-\gamma),$$
$$A_k(0;\beta,w)=\frac{(-1)^k}{k}\zeta(-k,\beta)\zeta(k)w^{-k}\quad (k\geq 2),$$

and so

$$\begin{aligned}&\zeta_2'(0;\beta,(1,w))\\ &\quad=-\frac{1}{12}w\log w+\left(\frac{1}{12}-\zeta'(-1)\right)w+\frac{1}{2}\left(\frac{1}{2}-\beta\right)\log w\\ &\qquad+\log\Gamma(\beta)+\left(\frac{1}{2}\beta-\frac{3}{4}\right)\log 2\pi\\ &\qquad+\zeta(-1,\beta)w^{-1}\log w-\zeta(-1,\beta)\gamma w^{-1}\\ &\qquad+\sum_{k=2}^{N-1}\frac{(-1)^k}{k}\zeta(-k,\beta)\zeta(k)w^{-k}+R_N'(0;\beta,w)\end{aligned}\tag{2.5}$$

for $w\in\mathcal{W}_\infty$ and $N\geq 2$. (The above calculations are actually the same as in pp.395-396 of [10].)

Put $z=-v-N+\varepsilon+iy$ in (2.2), and differentiate with respect to v. We obtain

$$\begin{aligned}R_N'(v;\beta,w) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma(-N+\varepsilon+iy)\zeta(-N+\varepsilon+iy,\beta)w^{-v-N+\varepsilon+iy}\\ &\times\left\{\frac{\Gamma'(v+N-\varepsilon-iy)}{\Gamma(v)}\zeta(v+N-\varepsilon-iy)\right.\\ &\quad-\Gamma(v+N-\varepsilon-iy)\frac{\Gamma'(v)}{\Gamma(v)^2}\zeta(v+N-\varepsilon-iy)\\ &\quad+\frac{\Gamma(v+N-\varepsilon-iy)}{\Gamma(v)}\zeta'(v+N-\varepsilon-iy)\\ &\quad\left.-\frac{\Gamma(v+N-\varepsilon-iy)}{\Gamma(v)}\zeta(v+N-\varepsilon-iy)\log w\right\}dy.\end{aligned}$$

Noting

$$\frac{\Gamma'(v)}{\Gamma(v)^2}=\frac{1}{\Gamma(v)}\left(\psi(v+1)-\frac{1}{v}\right)=\frac{1}{\Gamma(v+1)}(v\psi(v+1)-1)\to -1$$

(as $v\to 0$), we have

$$\begin{aligned}R_N'(0;\beta,w) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma(-N+\varepsilon+iy)\zeta(-N+\varepsilon+iy,\beta)w^{-N+\varepsilon+iy}\\ &\quad\times\Gamma(N-\varepsilon-iy)\zeta(N-\varepsilon-iy)dy.\end{aligned}\tag{2.6}$$

Hence, using Stirling's formula and Lemma 2 of [11] we can show that

$$R'_N(0;\beta,w)=O(|w|^{-N+\varepsilon}), \tag{2.7}$$

and this estimate is uniform in β if $0<\beta\leq 1$.

Consider (2.5) with $N+1$ instead of N, and compare it with the original (2.5). Then we find

$$\begin{aligned} R'_N(0;\beta,w) &= \frac{(-1)^N}{N}\zeta(-N,\beta)\zeta(N)w^{-N}+R'_{N+1}(0;\beta,w)\\ &= \frac{(-1)^N}{N}\zeta(-N,\beta)\zeta(N)w^{-N}+O(|w|^{-N-1+\varepsilon}), \end{aligned}$$

hence

$$R'_N(0;\beta,w)=O(|w|^{-N}), \tag{2.8}$$

which is again uniform in β if $0<\beta\leq 1$. Noting this uniformity and the fact

$$\log\Gamma(\beta)=\log\Gamma(\beta+1)-\log\beta,$$

from (2.5) we obtain

$$\begin{aligned} -\log\rho_2(1,w) &= -\frac{1}{12}w\log w+\left(\frac{1}{12}-\zeta'(-1)\right)w+\frac{1}{4}\log w\\ &- \frac{3}{4}\log 2\pi+\zeta(-1)w^{-1}\log w-\zeta(-1)\gamma w^{-1}\\ &+ \sum_{k=2}^{N-1}\frac{(-1)^k}{k}\zeta(-k)\zeta(k)w^{-k}+O(|w|^{-N}) \end{aligned} \tag{2.9}$$

for $w\in\mathcal{W}_\infty$. The first assertion (1.6) of Theorem 1 follows from (2.5), (2.8) and (2.9).

Next we prove (1.7). Our starting point is the special case $u=0$, $\alpha=\beta$ of (6.7) of [11], that is

$$\begin{aligned} &\zeta_2(v;\beta,(1,w))\\ &\quad=\zeta(v,\beta)+\zeta_1(v,\beta/w)w^{-v}+\frac{1}{v-1}\zeta_1(v-1,\beta)w^{-1}\\ &\quad+\sum_{k=0}^{N-1}\binom{-v}{k}\zeta_1(v+k,\beta)\zeta(-k)w^k+S_{1,N}(v;\beta,w), \end{aligned} \tag{2.10}$$

where

$$S_{1,N}(v;\beta,w)=\frac{1}{2\pi i}\int_{(N-\varepsilon)}\frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)}\zeta_1(v+z,\beta)\zeta(-z)w^z dz. \tag{2.11}$$

In Section 6 of [11] it is shown that (2.10) holds for $\Re v > 1 - N + \varepsilon$, $w \in \mathcal{W}_0$. From (2.10) it follows that

$$\begin{aligned}
&\zeta_2'(v;\beta,(1,w)) \\
&\quad = \zeta'(v,\beta) + \zeta_1'(v,\beta/w)w^{-v} - \zeta_1(v,\beta/w)w^{-v}\log w \\
&\quad -\frac{1}{(v-1)^2}\zeta_1(v-1,\beta)w^{-1} + \frac{1}{v-1}\zeta_1'(v-1,\beta)w^{-1} \\
&\quad + \sum_{k=0}^{N-1} B_k(v;\beta)\zeta(-k)w^k + S_{1,N}'(v;\beta,w), \qquad (2.12)
\end{aligned}$$

where

$$B_k(v;\beta) = \binom{-v}{k}'\zeta_1(v+k,\beta) + \binom{-v}{k}\zeta_1'(v+k,\beta).$$

It is clear that $B_0(0;\beta) = \zeta_1'(0,\beta)$ and

$$B_k(0;\beta) = \frac{(-1)^k}{k}\zeta_1(k,\beta) \qquad (k \geq 2).$$

Also, since

$$\lim_{v\to 0}\left(\zeta_1(v+1,\beta) - \frac{1}{v}\right) = -\psi(\beta) - \beta^{-1},$$

we see that

$$\lim_{v\to 0} B_1(v;\beta) = \psi(\beta) + \beta^{-1}.$$

Hence from (2.12) (with (2.4)) we get

$$\begin{aligned}
\zeta_2'(0;\beta,(1,w)) &= \frac{1}{2}\log w + \frac{1}{2}\log\Gamma(\beta+1) - \log\beta \\
&- \frac{3}{4}\log 2\pi + \log\Gamma\left(\frac{\beta}{w}+1\right) + \beta w^{-1}\log w \\
&- \left\{\zeta_1(-1,\beta) + \zeta_1'(-1,\beta) + \frac{1}{12}\psi(\beta+1)\right\} w^{-1} \\
&+ \sum_{k=2}^{N-1} \frac{(-1)^k}{k}\zeta_1(k,\beta)\zeta(-k)w^k + S_{1,N}'(0;\beta,w) \qquad (2.13)
\end{aligned}$$

for $N \geq 2$.

The estimate

$$S_{1,N}'(0;\beta,w) = O(|w|^N) \qquad (2.14)$$

can be shown similarly to (2.8); this time, instead of Lemma 2 of [11], we use the fact that $\zeta_1(v,\beta)$ and $\zeta_1'(v,\beta)$ are uniformly bounded with respect to β in the domain of absolute convergence. Hence (2.14) is uniform for any $\beta > 0$. From (2.13), (2.14) and this uniformity, we obtain

$$\begin{aligned} -\log\rho_2(1,w) &= \frac{1}{2}\log w - \frac{3}{4}\log 2\pi + \left(\frac{1}{12}\gamma - \zeta(-1) - \zeta'(-1)\right)w^{-1} \\ &+ \sum_{k=2}^{N-1}\frac{(-1)^k}{k}\zeta(k)\zeta(-k)w^k + O(|w|^N) \end{aligned} \tag{2.15}$$

for $N \geq 2$, $w \in \mathcal{W}_0$. From (2.13), (2.14) and (2.15), the assertion (1.7) follows.

3 Additional remarks on Theorem 1

In this supplementary section we give two additional remarks.

First we mention an alternative proof of (1.6). Shintani [15] proved

$$\begin{aligned} \Gamma_2(\beta,(1,w)) = (2\pi)^{\beta/2}\exp\left\{\left(\frac{\beta-\beta^2}{2w} - \frac{\beta}{2}\right)\log w + \frac{(\beta^2-\beta)\gamma}{2w}\right\} \\ \times\Gamma(\beta)\prod_{n=1}^{\infty}\frac{\Gamma(\beta+nw)}{\Gamma(1+nw)}\exp\left\{\frac{\beta-\beta^2}{2nw} + (1-\beta)\log(nw)\right\} \end{aligned} \tag{3.1}$$

(see also Katayama-Ohtsuki [9], p.179). Shintani assumed that $w > 0$, but (3.1) holds for any complex w with $|\arg w| < \pi$ by analytic continuation. We recall Stirling's formula of the form

$$\begin{aligned} \log\Gamma(w+a) &= \left(w + a - \frac{1}{2}\right)\log w - w + \frac{1}{2}\log 2\pi \\ &+ \sum_{m=1}^{M}\frac{(-1)^{m-1}B'_{m+2}(a)}{m(m+1)(m+2)w^m} + O(|w|^{-M-1/2}), \end{aligned} \tag{3.2}$$

given in p.278, Section 13.6 of Whittaker-Watson [18], where $B'_{m+2}(a)$ is the derivative of the $(m+2)$th Bernoulli polynomial and M is any positive integer. Noting

$$\zeta(-m,a) = -\frac{B'_{m+2}(a)}{(m+1)(m+2)} \tag{3.3}$$

(p.267, Section 13.14 of [18]), we obtain

$$\begin{aligned} \log\left(\prod_{n=1}^{\infty}\frac{\Gamma(\beta+nw)}{\Gamma(1+nw)}\exp\left\{\frac{\beta-\beta^2}{2nw} + (1-\beta)\log(nw)\right\}\right) = \sum_{n=1}^{\infty}\left\{\frac{\beta-\beta^2}{2nw}\right. \\ \left. - \sum_{m=1}^{M}\frac{(-1)^{m-1}}{mn^m}(\zeta(-m,\beta) - \zeta(-m))w^{-m} + O\left((n|w|)^{-M-1/2}\right)\right\}. \end{aligned} \tag{3.4}$$

From (3.3) and the fact $B_3(a) = a^3 - (3/2)a^2 + (1/2)a$ it follows that

$$\zeta(-1, a) = \frac{1}{2}(a - a^2) - \frac{1}{12}. \tag{3.5}$$

Hence the coefficient of the term of order w^{-1} on the right-hand side of (3.4) vanishes, and so the right-hand side of (3.4) is equal to

$$-\sum_{m=2}^{M} \frac{(-1)^{m-1}}{m}(\zeta(-m,\beta) - \zeta(-m))\zeta(m)w^{-m} + O\left(|w|^{-M-1/2}\right).$$

Substituting this into the right-hand side of (3.1), and noting (3.5), we arrive at the formula (1.6).

Next we discuss a connection with the Dedekind eta-function

$$\eta(w) = e^{\pi i w/12} \prod_{n=1}^{\infty}(1 - e^{2\pi i n w}). \tag{3.6}$$

In the rest of this section we assume $\pi/2 \le \theta_0 \le \pi$, and define

$$\mathcal{W}(\theta_0) = \{w \in \mathbf{C} \mid \pi - \theta_0 \le \arg w \le \theta_0\}.$$

For $w \in \mathcal{W}(\theta_0)$ we have $\log(-w) = -\pi i + \log w$. Hence from (2.9) and the facts $\zeta(-1) = -1/12$ and $\zeta(-k) = 0$ for even k it follows that

$$\begin{aligned} &\log \rho_2(1, w) + \log \rho_2(1, -w) \\ &\quad = \frac{1}{12}\pi i w - \frac{1}{2}\log w + \frac{1}{4}\pi i + \frac{3}{2}\log 2\pi + \frac{1}{12}\pi i w^{-1} + O(|w|^{-N}) \end{aligned} \tag{3.7}$$

for $w \in \mathcal{W}_\infty \cap \mathcal{W}(\theta_0)$ and $N \ge 2$. Similarly, from (2.15) we get

$$\begin{aligned} &\log \rho_2(1, w) + \log \rho_2(1, -w) \\ &\quad = -\log w + \frac{1}{2}\pi i + \frac{3}{2}\log 2\pi + O(|w|^{N}) \end{aligned} \tag{3.8}$$

for $w \in \mathcal{W}_0 \cap \mathcal{W}(\theta_0)$ and $N \ge 2$. The reason why the terms of order $w^{\pm k}$ $(2 \le k \le N-1)$ vanish in (3.7) and (3.8) can be explained by the modularity of $\eta(w)$, by using the formula

$$\rho_2(1, w)\rho_2(1, -w) = (2\pi)^{3/2} w^{-1/2} \eta(w) \exp\left(\pi i \left(\frac{1}{4} + \frac{1}{12w}\right)\right) \tag{3.9}$$

due to Shintani [15]. In fact, in view of (3.9), we see that (3.7) is a direct consequence of the definition (3.6) of $\eta(w)$. Also (3.8) follows easily from (3.9) and the modular relation of $\eta(w)$.

4 The uniformity of the error terms

A difference between (1.6) and (1.7) is that the error estimate in (1.7) is uniform in β, while that in (1.6) is not. From the proof it can be seen that the implied constant in (1.6) does not depend on β if $0 < \beta \le 1$. For general β, it is possible to separate the parts depending on β from the error term on the right-hand side of (1.6). An application can be found in [12].

We write $\beta = A + \tilde{\beta}$, where A is a non-negative integer and $0 < \tilde{\beta} \le 1$. Then we have

Theorem 2 *For any positive integer N and $\Re v > -N+1$, we have*

$$\begin{aligned} R_N(v;\beta,w) &= -\sum_{k=N}^{\infty}\binom{-v}{k}\zeta(v+k)\sum_{j=0}^{A-1}(\tilde{\beta}+j)^k w^{-v-k} \\ &\quad + O(|w|^{-\Re v-N}) \end{aligned} \tag{4.1}$$

if $w \in \mathcal{W}_\infty$ and $|w| > \beta - 1$, where the implied constant depends only on v, N and θ_0.

In the case $w > 0$, this result has been proved in [12], but the proof presented below is simpler.

Corollary *Let $N \ge 2$, $w \in \mathcal{W}_\infty$ and $|w| > \beta - 1$. Then the error term on the right-hand side of* (1.6) *can be replaced by*

$$-\sum_{k=N}^{\infty}\frac{(-1)^k}{k}\zeta(k)\sum_{j=0}^{A-1}(\tilde{\beta}+j)^k w^{-k} + O(|w|^{-N}),$$

and the implied constant depends only on N and θ_0.

Now we prove the theorem. Since

$$\zeta(v+z,\beta) = \zeta(v+z,\tilde{\beta}) - \sum_{j=0}^{A-1}(\tilde{\beta}+j)^{-v-z},$$

from (2.2) we have

$$\begin{aligned} R_N(v;\beta,w) &= -\sum_{j=0}^{A-1}\frac{1}{2\pi i}\int_{(c_N)}\frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)}(\tilde{\beta}+j)^{-v-z}\zeta(-z)w^z dz \\ &\quad + R_N(v;\tilde{\beta},w) \\ &= -\sum_{j=0}^{A-1} T_N(j) + R_N(v;\tilde{\beta},w), \end{aligned} \tag{4.2}$$

say. Let L be a large positive integer ($L > N$), and shift the path of integration of $T_N(j)$ to $\Re z = c_L$. Counting the residues of the poles $z = -v-k$ ($N \le k \le L-1$), we obtain

$$T_N(j) = \sum_{k=N}^{L-1} \binom{-v}{k} \zeta(v+k)(\tilde{\beta}+j)^k w^{-v-k} + T_L(j).$$

Using Stirling's formula we can see that $T_L(j) \to 0$ as $L \to +\infty$ if $|w| > \tilde{\beta} + j$. The resulting infinite series expression of $-\sum_{j=0}^{A-1} T_N(j)$ coincides with the explicit term on the right-hand side of (4.1). The remainder term $R_N(v; \tilde{\beta}, w)$ can be estimated by (5.4) of [11]. Since $0 < \tilde{\beta} \le 1$, the estimate is uniform in $\tilde{\beta}$. Hence the proof of Theorem 2 is complete.

5 An Asymptotic expansion of the derivative of Hecke's zeta-function at $s = 1$

Let D be a square-free positive integer, $D \equiv 2$ or $3 \pmod 4$. Hecke [8] introduced and studied the zeta-function (following the notation of Hecke)

$$\zeta(s; v_1) = \sum_{(\mu)} \frac{\mathrm{sgn}(\mu\mu')}{|N(\mu)|^s} \tag{5.1}$$

associated with the real quadratic field $\mathbf{Q}(\sqrt{D})$, where (μ) runs over all non-zero principal integral ideals of $\mathbf{Q}(\sqrt{D})$, $N(\mu)$ is the norm of (μ), μ' is the conjugate of μ, and $\mathrm{sgn}(\mu\mu')$ is the sign of $\mu\mu'$. Hecke's motivation is to study the Dirichlet series

$$Z_{\sqrt{D}}(s) = \sum_{n=1}^{\infty} \frac{\{n\sqrt{D}\} - 1/2}{n^s},$$

where $\{x\}$ is the fractional part of x. The coefficients $G_1(\sqrt{D})$ and $G_2(\sqrt{D})$ in the Laurent expansion

$$Z_{\sqrt{D}}(s) = G_1(\sqrt{D})s^{-1} + G_2(\sqrt{D}) + G_3(\sqrt{D})s + \cdots$$

are important in the study of the distribution of $\{n\sqrt{D}\} - 1/2$, a famous classical problem in number theory. Hecke's paper [8] implicitly includes the evaluation of $G_1(\sqrt{D})$ and $G_2(\sqrt{D})$ in terms of $\zeta(1; v_1)$ and $\zeta'(1; v_1)$. In particular,

$$G_2(\sqrt{D}) = \frac{\zeta(1; v_1)\sqrt{D}}{\pi^2 \log \varepsilon_D}(\gamma + \log 2\pi) - \frac{\zeta'(1; v_1)\sqrt{D}}{2\pi^2 \log \varepsilon_D} - \frac{1}{12}\sqrt{D} + \frac{1}{8} \tag{5.2}$$

if $N(\varepsilon_D) = 1$, where ε_D is the fundamental unit of $\mathbf{Q}(\sqrt{D})$.

Fujii [6] proved different expressions for $G_1(\sqrt{D})$ and $G_2(\sqrt{D})$. Combining them with Hecke's results, Fujii obtained new expressions for $\zeta(1;v_1)$ and $\zeta'(1;v_1)$. An interesting feature of Fujii's results is that double gamma-functions appear in his expressions. This is similar to Shintani's theorem [14]. Shintani [14] proved a formula which expresses the value $L_F(1,\chi)$ of a certain Hecke L-function (associated with a real quadratic field F) in terms of double gamma-functions. Combining Shintani's result with our expansion formula for double gamma-functions, we have shown an expansion for $L_F(1,\chi)$ in [12]. In a similar way, in this paper we prove an expansion formula for $\zeta'(1;v_1)$.

Fujii [6] proved his results for any $\mathbf{Q}(\sqrt{D})$, D is square-free, positive, $\equiv 2$ or 3 (mod 4). However, his general statement is very complicated. Therefore in this paper we content ourselves with considering a typical example, given as Example 2 in Fujii [7].

To state Fujii's results, we introduce more general form of double zeta and double gamma-functions. Let α, w_1, w_2 be positive numbers, and define

$$\zeta_2(v;\alpha,(w_1,w_2)) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha+mw_1+nw_2)^{-v}, \tag{5.3}$$

$$-\log\rho_2(w_1,w_2) = \lim_{\alpha\to 0}\{\zeta_2'(0;\alpha,(w_1,w_2))+\log\alpha\}, \tag{5.4}$$

and

$$\log\frac{\Gamma_2(\alpha,(w_1,w_2))}{\rho_2(w_1,w_2)} = \zeta_2'(0;\alpha,(w_1,w_2)). \tag{5.5}$$

Let n be a non-negative integer such that $4n^2+8n+3$ is square-free. We consider the case $D=4n^2+8n+3$. Then $\varepsilon_n=\sqrt{D}+2n+2$ is the fundamental unit of $\mathbf{Q}(\sqrt{D})$. Example 2 of Fujii [7] asserts that

$$\zeta(1;v_1) = \frac{\pi^2}{12}\frac{4n+1}{\sqrt{4n^2+8n+3}} \tag{5.6}$$

and

$$\begin{aligned}\zeta'(1;v_1) \;=\; \frac{2\pi^2}{\sqrt{4n^2+8n+3}}\Bigg\{&-\log\frac{\Gamma_2(\varepsilon_n^2,(\varepsilon_n,\varepsilon_n^2-\varepsilon_n))\rho_2(\varepsilon_n-1,\varepsilon_n)}{\rho_2(\varepsilon_n,\varepsilon_n^2-\varepsilon_n)\Gamma_2(2\varepsilon_n-1,(\varepsilon_n-1,\varepsilon_n))}\\ &+\frac{4n+1}{12}\left(\gamma+\log 2\pi-\log(\varepsilon_n-\varepsilon_n^{-1})\right)\\ &-\frac{\log\varepsilon_n}{24}\left(\frac{\sqrt{4n^2+8n+3}}{2n+1}+8n+5\right)\Bigg\}.\end{aligned} \tag{5.7}$$

Let α, β be positive numbers with $\alpha<\beta$, and define

$$\zeta_2((u,v);(\alpha,\beta)) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha+m)^{-u}(\beta+m+n)^{-v}. \tag{5.8}$$

In [11] we have shown that $\zeta_2((u,v);(\alpha,\beta))$ can be continued meromorphically to the whole $\mathbf{C}^2$-space. In Section 7 we will see that $\zeta_2((0,v);(\alpha,\beta))$ is holomorphic at $v=0$, while $\zeta_2((-k,v+k);(\alpha,\beta))$ has a pole of order 1 at $v=0$ for any positive integer k. Denote the Laurent expansion at $v=0$ by

$$\begin{aligned}&\zeta_2((-k,v+k);(\alpha,\beta))\\&\quad=C_{-1}(k;(\alpha,\beta))\frac{1}{v}+C_0(k;(\alpha,\beta))+C_1(k;(\alpha,\beta))v+\cdots\end{aligned}\tag{5.9}$$

for $k\geq 1$. We shall prove

Theorem 3 *Let $D=4n^2+8n+3$, $\varepsilon_n=\sqrt{D}+2n+2$, and $\xi=\xi_n=\varepsilon_n-1$. Then, for any positive integer $N\geq 2$, we have*

$$\begin{aligned}&\log\frac{\Gamma_2(\varepsilon_n^2,(\varepsilon_n,\varepsilon_n^2-\varepsilon_n))\rho_2(\varepsilon_n-1,\varepsilon_n)}{\rho_2(\varepsilon_n,\varepsilon_n^2-\varepsilon_n)\Gamma_2(2\varepsilon_n-1,(\varepsilon_n-1,\varepsilon_n))}\\
&=\ -\frac{1}{12}\xi\log\xi-\frac{1}{12}\xi\log(1+\xi)+\left(\frac{1}{12}-\zeta'(-1)\right)\xi+\frac{1}{6}\log\xi\\
&\quad-\frac{1}{4}\log(1+\xi)+\frac{1}{4}\log 2\pi-\zeta_2'((0,0);(1,2))\\
&\quad-\frac{1}{12}\xi^{-1}\log\xi-\frac{1}{12}\xi^{-1}\log(1+\xi)+\left(\frac{1}{12}\gamma+C_0(1;(1,2))\right)\xi^{-1}\\
&\quad+\sum_{k=2}^{N-1}\frac{(-1)^k}{k}\Big\{\zeta(-k)\zeta(k)-C_0(k;(1,2))\\
&\qquad-\frac{k}{12}\left(1+\frac{1}{2}+\cdots+\frac{1}{k-1}-\log\xi\right)\Big\}\xi^{-k}\\
&\quad+O\left(\xi^{-N}\log\xi\right).\end{aligned}\tag{5.10}$$

From this theorem and (5.7), we obtain the asymptotic expansion of $\zeta'(1;v_1)$ with respect to $\xi=\xi_n$ (or with respect to ε_n) when $n\to+\infty$. Moreover, combining with (5.2) and (5.6), we can deduce the asmptotic expansion of $G_2(\sqrt{D})=G_2(\sqrt{4n^2+8n+3})$. It should be noted that, by expanding the factor $\log(1+\xi)$ on the right-hand side of (5.10), we can write down the asymptotic expansion with respect to ξ in the most strict sense. This is an advantage of the above theorem; the formula for $L_F(1,\chi)$ proved in [12] is not the asymptotic expansion in the strict sense.

The rest of this paper is devoted to the proof of Theorem 3. It is desirable to extend our consideration to the case of Fujii's general formula (Fujii [6], Theorem 6 and Corollary 2). It is also an interesting problem to evaluate the quantities $\zeta_2'((0,0);(1,2))$ and $C_0(k;(1,2))$ appearing on the right-hand side of (5.10).

6 The behaviour of $\Gamma_2(\varepsilon_n^2, (\varepsilon_n, \varepsilon_n^2 - \varepsilon_n))$ and $\rho_2(\varepsilon_n, \varepsilon_n^2 - \varepsilon_n)$

Let $\beta = \alpha/w_1$ and $w = w_2/w_1$. From (5.3) we have

$$\begin{aligned} \zeta_2(v;\alpha,(w_1,w_2)) &= w_1^{-v}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\beta+m+nw)^{-v} \\ &= w_1^{-v}\zeta_2(v;\beta,(1,w)) \end{aligned} \tag{6.1}$$

for $\Re v > 2$. This formula gives the analytic continuation of $\zeta_2(v;\alpha,(w_1,w_2))$ to the whole complex v-plane, and yields

$$\zeta_2'(0;\alpha,(w_1,w_2)) = -\zeta_2(0;\beta,(1,w))\log w_1 + \zeta_2'(0;\beta,(1,w)). \tag{6.2}$$

From (5.4) and (6.2),we have

$$\begin{aligned} &-\log\rho_2(w_1,w_2) \\ &\quad = \lim_{\alpha\to 0}\{1-\zeta_2(0;\beta,(1,w))\}\log w_1 + \lim_{\alpha\to 0}\{\zeta_2'(0;\beta,(1,w)) + \log\beta\} \\ &\quad = \{1-\zeta_2(0;0,(1,w))\}\log w_1 - \log\rho_2(1,w), \end{aligned} \tag{6.3}$$

where the existence of the limit

$$\zeta_2(0;0,(1,w)) = \lim_{\beta\to 0}\zeta_2(0;\beta,(1,w))$$

can be seen from the expression

$$\begin{aligned} \zeta_2(0;\beta,(1,w)) &= -B_1(\beta) + \sum_{\ell=0}^{2}\frac{B_\ell(0)B_{2-\ell}(1-\beta)}{\ell!(2-\ell)!}w^{\ell-1} \\ &= \frac{1}{12}w + \frac{1}{2}\left(\frac{1}{2}-\beta\right) + \frac{1}{2}\left(\beta^2-\beta+\frac{1}{6}\right)w^{-1}, \end{aligned} \tag{6.4}$$

which is the special case $m = 0$ of Theorem 5 in [10]. From (6.4) it follows that

$$\zeta_2(0;0,(1,w)) = \frac{1}{12}w + \frac{1}{4} + \frac{1}{12}w^{-1}, \tag{6.5}$$

$$\zeta_2(0;1,(1,w)) = \frac{1}{12}w - \frac{1}{4} + \frac{1}{12}w^{-1}. \tag{6.6}$$

Now we consider the case $(w_1, w_2) = (\varepsilon_n, \varepsilon_n^2 - \varepsilon_n)$. Our aim in this section is to prove the following

Proposition 1 *We have*

$$\log\rho_2(\varepsilon_n,\varepsilon_n^2-\varepsilon_n)=\Big(\frac{1}{12}\xi-\frac{3}{4}+\frac{1}{12}\xi^{-1}\Big)\log(1+\xi)+\frac{1}{12}\xi\log\xi$$
$$-\Big(\frac{1}{12}-\zeta'(-1)\Big)\xi-\frac{1}{4}\log\xi+\frac{3}{4}\log 2\pi+\frac{1}{12}\xi^{-1}\log\xi-\frac{1}{12}\gamma\xi^{-1}$$
$$-\sum_{k=2}^{N-1}\frac{(-1)^k}{k}\zeta(-k)\zeta(k)\xi^{-k}+O(\xi^{-N}) \tag{6.7}$$

for any $N\geq 2$*, and*

$$\log\Gamma_2(\varepsilon_n^2,(\varepsilon_n,\varepsilon_n^2-\varepsilon_n))=-\log(1+\xi)-\frac{1}{2}\log\xi+\log 2\pi. \tag{6.8}$$

Proof. Putting $(w_1,w_2)=(\varepsilon_n,\varepsilon_n^2-\varepsilon_n)$, the formula (6.3) gives

$$\log\rho_2(\varepsilon_n,\varepsilon_n^2-\varepsilon_n)=\{\zeta_2(0;0,(1,\xi))-1\}\log(1+\xi)+\log\rho_2(1,\xi) \tag{6.9}$$

because $w=(\varepsilon_n^2-\varepsilon_n)/\varepsilon_n=\varepsilon_n-1=\xi$. Substituting (2.9) and (6.5) into the right-hand side of (6.9), we obtain (6.7).

Next, for $\Re v>2$, we have

$$\begin{aligned}
&\zeta_2(v;\varepsilon_n^2,(\varepsilon_n,\varepsilon_n^2-\varepsilon_n))\\
&\quad=\varepsilon_n^{-v}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\varepsilon_n+m+n(\varepsilon_n-1))^{-v}\\
&\quad=(1+\xi)^{-v}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(1+m+(1+n)\xi)^{-v}\\
&\quad=(1+\xi)^{-v}\left\{\sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty}(1+m+\ell\xi)^{-v}-\sum_{m=0}^{\infty}(1+m)^{-v}\right\}\\
&\quad=(1+\xi)^{-v}\{\zeta_2(v;1,(1,\xi))-\zeta(v)\}.
\end{aligned} \tag{6.10}$$

This formula is valid for any v by analytic continuation. Hence from this formula we obtain

$$\begin{aligned}
\zeta_2'(0;\varepsilon_n^2,(\varepsilon_n,\varepsilon_n^2-\varepsilon_n)) &= -\left\{\zeta_2(0;1,(1,\xi))+\frac{1}{2}\right\}\log(1+\xi)\\
&\quad+\zeta_2'(0;1,(1,\xi))+\frac{1}{2}\log 2\pi,
\end{aligned} \tag{6.11}$$

which with (6.9) and (1.4) yields

$$\begin{aligned}
&\log\Gamma_2(\varepsilon_n^2,(\varepsilon_n,\varepsilon_n^2-\varepsilon_n))\\
&\quad=\left\{\zeta_2(0;0,(1,\xi))-\zeta_2(0;1,(1,\xi))-\frac{3}{2}\right\}\log(1+\xi)\\
&\qquad+\log\Gamma_2(1,(1,\xi))+\frac{1}{2}\log 2\pi.
\end{aligned} \tag{6.12}$$

From (3.1) we find that

$$\Gamma_2(1,(1,\xi)) = \left(\frac{2\pi}{\xi}\right)^{1/2}. \tag{6.13}$$

Substituting (6.5), (6.6) and (6.13) into the right-hand side of (6.12), we obtain (6.8). This completes the proof of Proposition 1.

7 An auxiliary integral

In this section we prove several properties of the integral

$$I(v;(\alpha,\beta)) = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)}\zeta_2((v+z,-z);(\alpha,\beta))\xi^z dz \tag{7.1}$$

which are necessary in the next section. Here $\Re v > 2$, $1-\Re v < c < -1$, and $0<\alpha<\beta$.

Let ε be an arbitrarily small positive number. From (9.2) of [11] we have

$$\begin{aligned}\zeta_2((v+z,-z);(\alpha,\beta)) &= -\frac{1}{1+z}\zeta(v-1,\alpha)\\ &+\sum_{j=0}^{J-1}\binom{z}{j}\zeta(v+j,\alpha)\zeta(-j,\beta-\alpha)+S_{0,J}((v+z,-z);(\alpha,\beta)),\end{aligned} \tag{7.2}$$

where J is any positive integer and $S_{0,J}((v+z,-z);(\alpha,\beta))$ is holomorphic in the region $\Re v > 1-J+\varepsilon$ and $\Re z < J-\varepsilon$. Since J is arbitrary, (7.2) implies that $z=-1$ is the only pole of $\zeta_2((v+z,-z);(\alpha,\beta))$ as a function in z. This pole is irrelevant when we shift the path of integration on the right-hand side of (7.1) to $\Re z = c_N = -\Re v - N + \varepsilon$, where N is a positive integer ≥ 2. It is not difficult to see that $\zeta_2((v+z,-z);(\alpha,\beta))$ is of polynomial order with respect to $\Im z$ (for example, by using (7.2)), hence this shifting is possible. Counting the residues of the poles $z=-v-k$ $(0\leq k\leq N-1)$, we obtain

$$\begin{aligned}I(v;(\alpha,\beta)) &= \sum_{k=0}^{N-1}\binom{-v}{k}\zeta_2((-k,v+k);(\alpha,\beta))\xi^{-v-k}\\ &+ I_N(v;(\alpha,\beta)),\end{aligned} \tag{7.3}$$

where

$$\begin{aligned}I_N(v;(\alpha,\beta)) &= \frac{1}{2\pi i}\int_{(c_N)}\frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)}\zeta_2((v+z,-z);(\alpha,\beta))\xi^z dz\\ &= \frac{1}{2\pi i}\xi^{-v}\int_{(-N+\varepsilon)}\frac{\Gamma(z)\Gamma(v-z)}{\Gamma(v)}\zeta_2((z,v-z);(\alpha,\beta))\xi^z dz.\end{aligned} \tag{7.4}$$

Next, from (9.9) of [11] we have

$$\zeta_2((u,v);(\alpha,\beta)) = \frac{\Gamma(1-u)\Gamma(u+v-1)}{\Gamma(v)}\zeta(u+v-1,\beta-\alpha)$$
$$+\sum_{j=0}^{J-1}\binom{-v}{j}\zeta(u-j,\alpha)\zeta(v+j,\beta-\alpha)+R_{0,J}((u,v);(\alpha,\beta)), \qquad (7.5)$$

where

$$R_{0,J}((u,v);(\alpha,\beta))$$
$$=\frac{1}{2\pi i}\int_{(-J+\varepsilon)}\frac{\Gamma(z')\Gamma(v-z')}{\Gamma(v)}\zeta(u+z',\alpha)\zeta(v-z',\beta-\alpha)dz'. \qquad (7.6)$$

The formula (7.5) is valid in the region

$$\{(u,v)\in\mathbf{C}^2 \mid \Re u < J+1-\varepsilon, \Re v > -J+1+\varepsilon\},$$

and in this region $R_{0,J}((u,v);(\alpha,\beta))$ are holomorphic. In particular, choosing $J=2$ and $(u,v)=(z,v-z)$, we have

$$\zeta_2((z,v-z);(\alpha,\beta)) = \frac{\Gamma(1-z)\Gamma(v-1)}{\Gamma(v-z)}\zeta(v-1,\beta-\alpha)$$
$$+\zeta(z,\alpha)\zeta(v-z,\beta-\alpha)-(v-z)\zeta(z-1,\alpha)\zeta(v-z+1,\beta-\alpha)$$
$$+R_{0,2}((z,v-z);(\alpha,\beta)) \qquad (7.7)$$

for $\Re z < 3-\varepsilon$ and $\Re(v-z) > -1+\varepsilon$. If $\Re z = -N+\varepsilon$, then the right-hand side of (7.7) can be singular only if $v=2,1,0,-1,-2,\ldots$ or $v=z+1$. Hence the integrand of the right-hand side of (7.4) is not singular on the path $\Re z = -N+\varepsilon$ if $\Re v > 1-N+\varepsilon$, which implies that the integral (7.4) can be continued meromorphically to $\Re v > 1-N+\varepsilon$. Moreover, the (possible) pole of $\zeta_2((z,v-z);(\alpha,\beta))$ at $v=0$ cancels with the zero coming from the factor $\Gamma(v)^{-1}$, hence (7.4) is holomorphic at $v=0$.

Let k be a non-negative integer. Putting $z=-k$ in (7.7), we have

$$\zeta_2((-k,v+k);(\alpha,\beta)) = \frac{\Gamma(1+k)\Gamma(v-1)}{\Gamma(v+k)}\zeta(v-1,\beta-\alpha)$$
$$+\zeta(-k,\alpha)\zeta(v+k,\beta-\alpha)-(v+k)\zeta(-k-1,\alpha)\zeta(v+k+1,\beta-\alpha)$$
$$+R_{0,2}((-k,v+k);(\alpha,\beta)) \qquad (7.8)$$

for $\Re v > -k-1+\varepsilon$. From (7.8) it is easy to see that $\zeta_2((0,v);(\alpha,\beta))$ is holomorphic at $v=0$, while $\zeta_2((-k,v+k);(\alpha,\beta))$ has a pole of order 1 at $v=0$ for $k\geq 1$. We may write the Laurent expansion at $v=0$ as (5.9) for $k\geq 1$. Then

$$\binom{-v}{k}\zeta_2((-k,v+k);(\alpha,\beta))\xi^{-v-k}$$

is holomorphic at $v = 0$, and its Taylor expansion is

$$= \frac{(-1)^k}{k} C_{-1}(k;(\alpha,\beta))\xi^{-k} + \frac{(-1)^k}{k}\Big\{C_0(k;(\alpha,\beta)) \\ + C_{-1}(k;(\alpha,\beta))\left(1+\frac{1}{2}+\cdots+\frac{1}{k-1}-\log\xi\right)\Big\}\xi^{-k}v+\cdots \tag{7.9}$$

for $k \geq 1$; recall that the empty sum is to be considered as zero. Now by (7.3), $I(v;(\alpha,\beta))$ can be continued to the region $\Re v > 1 - N + \varepsilon$, and

$$I'(0;(\alpha,\beta)) = \zeta_2'((0,0);(\alpha,\beta)) - \zeta_2((0,0);(\alpha,\beta))\log\xi \\ + \sum_{k=1}^{N-1}\frac{(-1)^k}{k}\Big\{C_0(k;(\alpha,\beta)) \\ + C_{-1}(k;(\alpha,\beta))\left(1+\frac{1}{2}+\cdots+\frac{1}{k-1}-\log\xi\right)\Big\}\xi^{-k} \\ + I_N'(0;(\alpha,\beta)). \tag{7.10}$$

We claim that the limit values of $\zeta_2'((0,0);(\alpha,\beta))$, $\zeta_2((0,0);(\alpha,\beta))$, $C_0(k;(\alpha,\beta))$, $C_{-1}(k;(\alpha,\beta))$ and $I_N'(0;(\alpha,\beta))$, when $\alpha \to +0$, all exist. We denote them by $\zeta_2'((0,0);(0,\beta))$, $\zeta_2((0,0);(0,\beta))$, $C_0(k;(0,\beta))$, $C_{-1}(k;(0,\beta))$ and $I_N'(0;(0,\beta))$, respectively.

To prove this claim, first recall that if $\Re v < 0$, then $\zeta(v,\alpha)$ is continuous with respect to α when $\alpha \to +0$. This fact can be seen from (2.17.3) of Titchmarsh [16]. Hence the existence of $\zeta_2'((0,0);(0,\beta))$ and $\zeta_2((0,0);(0,\beta))$ follows easily from the case $k = 0$ of (7.8). Similarly we can show the existence of $I_N'(0;0,\beta)$ by using (7.4) and (7.7), and at the same time we find

$$I_N'(0;(\alpha,\beta)) = O(\xi^{-N+\varepsilon}) \tag{7.11}$$

and

$$I_N'(0;(0,\beta)) = O(\xi^{-N+\varepsilon}). \tag{7.12}$$

Next consider the case $k \geq 1$ of (7.8). The Laurent expansion at $v = 0$ of the first term on the right-hand side of (7.8) is

$$= -k\zeta(-1,\beta-\alpha)v^{-1} - P(k;(\alpha,\beta)) + O(|v|),$$

where

$$P(k;(\alpha,\beta)) = k\Big\{\zeta'(-1,\beta-\alpha) + \zeta(-1,\beta-\alpha) \\ - \zeta(-1,\beta-\alpha)\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k-1}\right)\Big\}. \tag{7.13}$$

The other terms on the right-hand side of (7.8) are holomorphic at $v=0$ if $k \geq 2$. If $k=1$, one more pole is coming from the second term on the right-hand side of (7.8), whose Laurent expansion is

$$= \zeta(-1,\alpha)v^{-1} - \zeta(-1,\alpha)\psi(\beta-\alpha) + O(|v|).$$

Collecting the above facts, we obtain

$$C_{-1}(k;(\alpha,\beta)) = \begin{cases} -\zeta(-1,\beta-\alpha)+\zeta(-1,\alpha) & \text{if } k=1, \\ -k\zeta(-1,\beta-\alpha) & \text{if } k \geq 2, \end{cases} \tag{7.14}$$

and

$$\begin{aligned} C_0(k;(\alpha,\beta)) &= -P(k;(\alpha,\beta)) + Q(k;(\alpha,\beta)) \\ &\quad -k\zeta(-k-1,\alpha)\zeta(k+1,\beta-\alpha) + R_{0,2}((-k,k);(\alpha,\beta)), \end{aligned} \tag{7.15}$$

where $P(k;(\alpha,\beta))$ is defined by (7.13) and

$$Q(k;(\alpha,\beta)) = \begin{cases} -\zeta(-1,\alpha)\psi(\beta-\alpha) & \text{if } k=1, \\ \zeta(-k,\alpha)\zeta(k,\beta-\alpha) & \text{if } k \geq 2. \end{cases} \tag{7.16}$$

From the above expressions it is now clear that $C_0(k;(0,\beta))$ and $C_{-1}(k;(0,\beta))$ exist for any $k \geq 1$. We complete the proof of our claim, and therefore from (7.10) we obtain

$$\begin{aligned} I'(0;(0,\beta)) &= \lim_{\alpha\to 0} I'(0;(\alpha,\beta)) \\ &= \zeta_2'((0,0);(0,\beta)) - \zeta_2((0,0);(0,\beta))\log\xi \\ &\quad + \sum_{k=1}^{N-1} \frac{(-1)^k}{k} \left\{ C_0(k;(0,\beta)) + C_{-1}(k;(0,\beta)) \left(1+\frac{1}{2}+\cdots+\frac{1}{k-1}-\log\xi\right)\right\}\xi^{-k} \\ &\quad + I_N'(0;(0,\beta)). \end{aligned} \tag{7.17}$$

8 The behaviour of $\Gamma_2(2\varepsilon_n-1,(\varepsilon_n-1,\varepsilon_n))$ and $\rho_2(\varepsilon_n-1,\varepsilon_n)$; Completion of the proof of Theorem 3

Let $\Re v > 2$ and $0<\alpha<1$. We have

$$\begin{aligned} \zeta_2(v;\alpha,(\varepsilon_n-1,\varepsilon_n)) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha+m\xi+n(1+\xi))^{-v} \\ &= (1+\xi)^{-v}\sum_{n=0}^{\infty}\left(\frac{\alpha}{1+\xi}+n\right)^{-v} \\ &\quad +(\alpha+n)^{-v}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left(1+\frac{(m+n)\xi}{\alpha+n}\right)^{-v}. \end{aligned} \tag{8.1}$$

Using the Mellin-Barnes integral formula ((2.2) of [11]) we get

$$\Gamma(v)\left(1+\frac{(m+n)\xi}{\alpha+n}\right)^{-v}=\frac{1}{2\pi i}\int_{(c)}\Gamma(v+z)\Gamma(-z)\left(\frac{(m+n)\xi}{\alpha+n}\right)^{z}dz, \tag{8.2}$$

where $-\Re v < c < 0$. We may assume $1-\Re v < c < -1$. Then, summing up the both sides of (8.2) with respect to m and n, we obtain

$$\begin{aligned}(\alpha+n)^{-v}&\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left(1+\frac{(m+n)\xi}{\alpha+n}\right)^{-v}\\ &=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(\alpha+n)^{-v-z}(m+n)^{z}\xi^{z}dz\\ &=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)}\zeta_2((v+z,-z);(\alpha,1))\xi^{z}dz.\end{aligned}$$

Therefore, combining with (8.1), we have

$$\zeta_2(v;\alpha,(\varepsilon_n-1,\varepsilon_n))=(1+\xi)^{-v}\zeta\left(v,\frac{\alpha}{1+\xi}\right)+I(v;(\alpha,1)) \tag{8.3}$$

for $\Re v > 2$, and by analytic continuation for $\Re v > 1-N+\varepsilon$. Hence

$$\begin{aligned}&\zeta_2'(0;\alpha,(\varepsilon_n-1,\varepsilon_n))\\ &\quad=-\zeta\left(0,\frac{\alpha}{1+\xi}\right)\log(1+\xi)+\zeta'\left(0,\frac{\alpha}{1+\xi}\right)+I'(0;(\alpha,1)).\end{aligned} \tag{8.4}$$

Applying (2.4) to the right-hand side, we have

$$\begin{aligned}&\zeta_2'(0;\alpha,(\varepsilon_n-1,\varepsilon_n))+\log\alpha\\ &\quad=-\left(\frac{1}{2}-\frac{\alpha}{1+\xi}\right)\log(1+\xi)+\log\Gamma\left(1+\frac{\alpha}{1+\xi}\right)-\frac{1}{2}\log 2\pi\\ &\qquad+\log(1+\xi)+I'(0;(\alpha,1)).\end{aligned}$$

Taking the limit $\alpha\to 0$, and using (7.12) and (7.17) with $\beta=1$, we obtain

Proposition 2 *We have*

$$\begin{aligned}\log\rho_2(\varepsilon_n-1,\varepsilon_n)&=-\frac{1}{2}\log(1+\xi)+\frac{1}{2}\log 2\pi\\ &-\zeta_2'((0,0);(0,1))+\zeta_2((0,0);(0,1))\log\xi\\ &-\sum_{k=1}^{N-1}\frac{(-1)^k}{k}\Big\{C_0(k;(0,1))\\ &\qquad+C_{-1}(k;(0,1))\left(1+\frac{1}{2}+\cdots+\frac{1}{k-1}-\log\xi\right)\Big\}\xi^{-k}\\ &+O(\xi^{-N+\varepsilon}).\end{aligned} \tag{8.5}$$

Remark The error estimate in (8.5) can be strengthened to $O(\xi^{-N}\log\xi)$. (Consider (8.5) with $N+1$ instead of N, and compare it with the original (8.5).)

Our next aim is to prove the following

Proposition 3 *We have*

$$\begin{aligned}\log\Gamma_2(2\varepsilon_n-1,(\varepsilon_n-1,\varepsilon_n)) &= -\frac{1}{2}\log(1+\xi)+\frac{1}{2}\log 2\pi \\ &+\zeta_2'((0,0);(1,2))-\zeta_2'((0,0);(0,1)) \\ &-\{\zeta_2((0,0);(1,2))-\zeta_2((0,0);(0,1))\}\log\xi \\ &+\sum_{k=1}^{N-1}\frac{(-1)^k}{k}\{C_0(k;(1,2))-C_0(k;(0,1))\}\xi^{-k} \\ &+\sum_{k=1}^{N-1}\frac{(-1)^k}{k}\{C_{-1}(k;(1,2))-C_{-1}(k;(0,1))\} \\ &\quad\times\left(1+\frac{1}{2}+\cdots+\frac{1}{k-1}-\log\xi\right)\xi^{-k} \\ &+O(\xi^{-N}\log\xi).\end{aligned}\qquad(8.6)$$

Proof. For $\Re v>2$, we have

$$\begin{aligned}&\zeta_2(v;2\varepsilon_n-1,(\varepsilon_n-1,\varepsilon_n)) \\ &\quad=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(2\xi+1+m\xi+n(1+\xi))^{-v} \\ &\quad=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(n+1+(m+n+2)\xi)^{-v},\end{aligned}$$

which is, again using the Mellin-Barnes integral formula,

$$\begin{aligned}&= \frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(n+1)^{-v}\left(\frac{(m+n+2)\xi}{n+1}\right)^z dz \\ &= \frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(v+z)\Gamma(-z)}{\Gamma(v)}\zeta_2((v+z,-z);(1,2))\xi^z dz.\end{aligned}$$

That is,

$$\zeta_2(v;2\varepsilon_n-1,(\varepsilon_n-1,\varepsilon_n))=I(v;(1,2)),\qquad(8.7)$$

and this identity is valid for $\Re v>1-N+\varepsilon$ by analytic continuation. Hence

$$\zeta_2'(0;2\varepsilon_n-1,(\varepsilon_n-1,\varepsilon_n))=I'(0;(1,2)).\qquad(8.8)$$

Therefore using (7.10), (7.11) (with $\alpha = 1$, $\beta = 2$) and (8.5) we obtain the assertion of Proposition 3. The error estimate $O(\xi^{-N}\log\xi)$ can be shown similarly to the remark just after the statement of Proposition 2.

Now we can easily complete the proof of Theorem 3, by combining Propositions 1, 2 and 3. Since

$$\zeta_2((0,v);(1,2)) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(2+m+n)^{-v} = \zeta_2(v;2,(1,1))$$

(valid at first for $\Re v > 2$ but also valid for any v by analytic continuation), by using (6.4) we find

$$\zeta_2((0,0);(1,2)) = \frac{5}{12}.$$

Also, (7.14) implies that $C_{-1}(1;(1,2)) = 0$ and $C_{-1}(k;(1,2)) = k/12$ for $k \geq 2$. Noting these facts, we can deduce the assertion of Theorem 3 straightforwardly.

It should be remarked finally that if we only want to prove Theorem 3, we can shorten the way; in fact, since the left-hand side of (5.10) is equal to

$$\zeta_2'(0;\varepsilon_n^2,(\varepsilon_n,\varepsilon_n^2-\varepsilon_n)) - \zeta_2'(0;2\varepsilon_n-1,(\varepsilon_n-1,\varepsilon_n)),$$

the formulas (6.11), (6.6), (2.5), (2.8), (8.8), (7.10), (7.11) are sufficient to deduce the conclusion of Theorem 3. However the formulas of Propositions 1, 2 and 3 themselves are of interest, therefore we have chosen the above longer but more informative route.

References

[1] T.Arakawa, Generalized eta-functions and certain ray class invariants of real quadratic fields, Math. Ann. **260** (1982), 475-494.

[2] T.Arakawa, Dirichlet series $\sum_{n=1}^{\infty}\frac{\cot \pi n\alpha}{n^s}$, Dedekind sums, and Hecke L-functions for real quadratic fields, Comment. Math. Univ. St. Pauli **37** (1988), 209-235.

[3] E.W.Barnes, The genesis of the double gamma functions, Proc. London Math. Soc. **31** (1899), 358-381.

[4] E.W.Barnes, The theory of the double gamma function, Philos. Trans. Roy. Soc. (A) **196** (1901), 265-387.

[5] J.Billingham and A.C.King, Uniform asymptotic expansions for the Barnes double gamma function, Proc. Roy. Soc. London Ser. A **453** (1997), 1817-1829.

[6] A.Fujii, Some problems of Diophantine approximation and a Kronecker limit formula, in "Investigations in Number Theory", T.Kubota (ed.), Adv. Stud. Pure Math. **13**, Kinokuniya, 1988, pp.215-236.

[7] A.Fujii, Diophantine approximation, Kronecker's limit formula and the Riemann hypothesis, in "Théorie des Nombres/Number Theory", J.-M.de Koninck and C.Levesque (eds.), Walter de Gruyter, 1989, pp.240-250.

[8] E.Hecke, Über analytische Funktionen und die Verteilung von Zahlen mod. eins, Abh. Math. Sem. Hamburg. Univ. **1** (1921), 54-76 (= Werke, 313-335).

[9] K.Katayama and M.Ohtsuki, On the multiple gamma-functions, Tokyo J. Math. **21** (1998) 159-182.

[10] K.Matsumoto, Asymptotic series for double zeta, double gamma, and Hecke L-functions, Math. Proc. Cambridge Phil. Soc. **123** (1998), 385-405.

[11] K.Matsumoto, Asymptotic expansions of double zeta-functions of Barnes, of Shintani, and Eisenstein series, preprint.

[12] K.Matsumoto, Asymptotic series for double zeta, double gamma, and Hecke L-functions II, preprint.

[13] T.Shintani, On evaluation of zeta functions of totally real algebraic number fields at non-positive integers, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **23** (1976), 393-417.

[14] T.Shintani, On a Kronecker limit formula for real quadratic fields, ibid. **24** (1977), 167-199.

[15] T.Shintani, A proof of the classical Kronecker limit formula, Tokyo J. Math. **3** (1980) 191-199.

[16] E.C.Titchmarsh, "The Theory of the Riemann Zeta-Function", Oxford, 1951.

[17] M.-F.Vignéras, L'equation fonctionelle de la fonction zêta de Selberg du groupe modulaire $PSL(2, \mathbf{Z})$, in "Journées Arithmétiques de Luminy, 1978", Astérisque **61**, Soc. Math. France, 1979, pp.235-249.

[18] E.T.Whittaker and G.N.Watson, "A Course of Modern Analysis", 4th ed., Cambridge Univ. Press, 1927.

Graduate Schoolof Mathematics
Nagoya University
Chikusa-ku, Nagoya 464-8602
Japan