The joint universality of twisted automorphic L-functions

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Abstract

The simultaneous universality of twisted automorphic L-functions, associated with a new form with respect to a congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$ and twisted by Dirichlet characters, is proved. Applications to the functional independence and the zero density of linear combinations of those L-functions are given.

1 Introduction and statement of results

In 1975, Voronin [29] proved the universality theorem for the Riemann zeta-function $\zeta(s)$. Let K be a compact subset in the strip $D_0 = \{s = \sigma + it \in \mathbb{C} \mid \frac{1}{2} < \sigma < 1\}$ with connected complement. Let $\mathcal{F}(K)$ be the family of functions which are non-vanishing, continuous on K and holomorphic in the interior of K. We use the notation

$$\nu_T(\cdots) = T^{-1}\mathrm{meas}\{\tau \in [0, \mathrm{T}] \mid \ldots\}$$

for $T > 0$, where $\mathrm{meas}\{\mathrm{A}\}$ denotes the Lebesgue measure of the set A, and in place of dots we write some condition satisfied by τ. Then Voronin's theorem asserts

$$\liminf_{T\to\infty} \nu_T(\sup_{s\in K} |\zeta(s + i\tau) - f(s)| < \varepsilon) > 0 \tag{1.1}$$

for any $f \in \mathcal{F}(K)$ and any $\varepsilon > 0$ (see Chapter 6 of [13]).

After Voronin, the universality theorem was generalized by many mathematicians. The following "joint universality" theorem for Dirichlet L-functions was shown by Voronin [30], Gonek [6] and Bagchi [1, 2]. Let $\chi_1, \ldots, \chi_m$ be pairwise non-equivalent Dirichlet characters and $L(s, \chi_j)$ the Dirichlet L-function attached to χ_j $(1 \leqslant j \leqslant m)$. Let K_j be a compact subset of D_0 with connected complement, and $f_j \in \mathcal{F}(K_j)$ $(1 \leqslant j \leqslant m)$. Then, for any $\varepsilon > 0$,

$$\liminf_{T\to\infty} \nu_T\Big(\sup_{1\leqslant j\leqslant m} \sup_{s\in K_j} |L(s+i\tau\chi_j) - f_j(s)| < \varepsilon\Big) > 0. \tag{1.2}$$

It is desirable to prove universality theorems for more general zeta and L-functions. Recently, the universality of automorphic L-functions has been studied. Let $F(z)$ be a holomorphic normalized Hecke-eigen cusp form of weight κ for the full modular group $\mathrm{SL}(2,\mathbb{Z})$. Then $F(z)$ has the Fourier expansion of the form

$$F(z) = \sum_{n=1}^{\infty} c(n) \mathrm{e}^{2\pi i n z}, \quad c(1) = 1. \tag{1.3}$$

The associated Dirichlet series

$$L(s,F) = \sum_{n=1}^{\infty} c(n) n^{-s} \tag{1.4}$$

is convergent absolutely for $\sigma > (\kappa+1)/2$, and can be continued to an entire function. The universality of $L(s,F)$ was first discussed by Kačėnas-Laurinčikas [9], who proved a certain conditional result. The general result of Laurinčikas [14] can also be applied to $L(s,F)$, which gives another conditional universality. Then in [18], the universality of $L(s,F)$ was proved unconditionally. Let K be a compact subset in the strip $D = \{s \in \mathbb{C} \mid \kappa/2 < \sigma < (\kappa+1)/2\}$ with connected complement, and $f \in \mathcal{F}(K)$. Then it is proved in [18] that

$$\liminf_{T\to\infty} \nu_T\Big(\sup_{s\in K_j} |L(s+i\tau, F) - f(s)| < \varepsilon\Big) > 0 \tag{1.5}$$

for any $\varepsilon > 0$. In [19], this result has been generalized to the case when F is a new form with respect to the congruence subgroup

$$\Gamma_0(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \mid c \equiv 0 \pmod{M} \right\},$$

where M is a positive integer.

To prove (1.5) unconditionally, a new idea was introduced in [18]; this idea is called "the positive density method" in [21]. A similar idea was applied to prove the joint universality of Lerch zeta-functions in [17].

It is the purpose of the present paper to combine this positive density method with Bagchi's idea [2] for the proof of (1.2), and establish the following joint universality theorem for twisted automorphic L-functions.

Theorem 1 *Let F(z) be a holomorphic normalized Hecke-eigen new form of weight κ with respect to $\Gamma_0(M)$, whose Fourier expansion is given by (1.3). Let q_j be positive integers prime to M $(1 \leqslant j \leqslant m)$, χ_j be pairwise non-equivalent Dirichlet characters* mod q_j *$(1 \leqslant j \leqslant m)$, and define the twisted L-functions*

$$L_j(s,F) = \sum_{n=1}^{\infty} c(n)\chi_j(n) n^{-s}, \tag{1.6}$$

which can be continued to the whole complex plane. For each j, let K_j be a compact subset of D with connected complement, and $f_j \in \mathcal{F}(K_j)$ $(1 \leqslant j \leqslant m)$. Then

$$\liminf_{T\to\infty} \nu_T(\sup_{1\leqslant j\leqslant m} \sup_{s\in K_j} |L_j(s+i\tau,F) - f_j(s)| < \varepsilon) > 0 \tag{1.7}$$

for any $\varepsilon > 0$.

The following three results are simple consequences of Theorem 1. In the following theorems, $L_j(s,F)$'s are the same as (1.6).

Theorem 2 *Let $\kappa/2 < \sigma < (\kappa+1)/2$, N be a positive integer, and ψ be the mapping from $\mathbb{R}$ to $\mathbb{C}^{Nm}$ defined by*

$$\begin{aligned}\psi(t) &= (L_1(\sigma+it,F),\ldots,L_m(\sigma+it,F),L_1'(\sigma+it,F),\ldots,L_m'(\sigma+it,F),\\ &\quad \ldots,L_1^{(N-1)}(\sigma+it,F),\ldots,L_m^{(N-1)}(\sigma+it,F)).\end{aligned}$$

Then $\psi(\mathbb{R})$ is dense in $\mathbb{C}^{Nm}$.

This is an analogue of Voronin's result [28] that

$$\{(\zeta(\sigma+it),\zeta'(\sigma+it),\ldots\zeta^{(N-1)}(\sigma+it)) \mid t\in\mathbb{R}\}$$

is dense in $\mathbb{C}^N$ for $1/2 < \sigma < 1$. Voronin proved this result earlier than his discovery of the universality theorem, but now it is known that this result as well as the above Theorem 2 is easily deduced from the universality.

Theorem 3 *If continuous functions $f_l : \mathbb{C}^{Nm} \to \mathbb{C}$ $(0 \leqslant l \leqslant L)$ satisfy*

$$\begin{aligned}\sum_{l=0}^{L} s^l f_l(L_1(s,F),\ldots,L_m(s,F),L_1'(s,F),\ldots,L_m'(s,F),\\ \ldots,L_1^{(N-1)}(s,F),\ldots,L_m^{(N-1)}(s,F)) \equiv 0\end{aligned} \tag{1.8}$$

for all $s \in \mathbb{C}$, then $f_l \equiv 0$ $(0 \leqslant l \leqslant L)$.

This result of functional independence is related with a problem mentioned in Hilbert's famous "Mathematische Probleme". Some history of this problem is written in [5].

Theorem 4 *Let $m \geqslant 2$, $u_j \in \mathbb{C}$ $(1 \leqslant j \leqslant m)$, and assume at least two of u_js are not zero. Let $\kappa/2 < \sigma_1 < \sigma_2 < (\kappa+1)/2$, $T \geqslant 2$, and $N(\sigma_1,\sigma_2,T)$ be the numbers of zeros (counted with multiplicity) of the function*

$$V(s) = \sum_{j=1}^{m} u_j L_j(s,F)$$

in the rectangle $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, $0 \leqslant t \leqslant T$. Then we have, for T sufficiently large,

$$N(\sigma_1,\sigma_2,T) \geqslant BT$$

with a constant $B > 0$.

This type of application of the universality theorem was first noticed again by Voronin in his papers [31, 32], and studied further by Laurinčikas [11, 12, 15].

In Section 2 we will prove an analogue of the prime number theorem in arithmetic progressions for Fourier coefficients of $F(z)$, which will be used in Section 5. In Section 3 we will prepare the probabilistic setting and will give a limit theorem, which is one of the keys of the proof of Theorem 1. Another key is Lemma 4 (the "denseness" lemma), which will be proved in Sections 4 and 5. Then in Section 6 we will complete the proof of Theorem 1. Proofs of other theorems will be shown in the final section.

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2 A prime number theorem for the coefficients of cusp forms

As was mentioned in Section 1, the universality of automorphic L-functions was first fully proved in [18]. A key ingredient used in [18] is the prime number formula

$$\sum_{p\leqslant x}\tilde{c}(p)^2=\frac{x}{\log x}(1+\mathrm{o}(1)) \tag{2.1}$$

due to Rankin [27], where p runs over prime numbers up to x and $\tilde{c}(p) = c(p)p^{-(\kappa-1)/2}$. Rankin proved (2.1) for $M = 1$. For our present purpose it is necessary to generalize (2.1) to the case of arbitrary M with adding the condition $p \equiv h \pmod q$, $(h, q) = 1$, where q is a positive integer prime to M. In this paper we will prove

Lemma 1 *The formula*

$$\sum_{\substack{p\leqslant x\\ p\equiv h(\mathrm{mod}\, q)}}\tilde{c}(p)^2=\frac{1}{\varphi(q)}\frac{x}{\log x}(1+o(1)) \tag{2.2}$$

holds when $(h, q) = 1$, where $\varphi(q)$ is Euler's totient function and the implied constant depends on q.

Let $F(z)$ be a holomorphic normalized Hecke-eigen new form of weight κ with respect to $\Gamma_0(M)$. The associated Dirichlet series (1.4) has the Euler product expansion

$$L(s,F)=\prod_p(1-c(p)p^{-s}+\psi_0(p)p^{\kappa-1-2s})^{-1},$$

where $\psi_0(p) = 1$ if $(p, M) = 1$ and $\psi_0(p) = 0$ if $p \mid M$. We write each Euler factor as

$$1-c(p)p^{-s}+\psi_0(p)p^{\kappa-1-2s}=(1-\alpha(p)p^{-s})(1-\beta(p)p^{-s}),$$

$\beta(p) = \overline{\alpha(p)}$ if $(p, M) = 1$ and $\beta(p) = 0$ if $p \mid M$. Hence $\alpha(p) + \beta(p) = c(p)$, and if $(p, M) = 1$, then $\alpha(p)\beta(p) = p^{\kappa-1}$.

Let χ be a Dirichlet character mod q. The twisted Rankin-Selberg L-function attached to F and χ is defined by

$$L(s, F\otimes F, \chi) = L(2s, \psi_0\chi^2)\sum_{n=1}^{\infty} c^2(n)\chi(n)n^{1-\kappa-s}, \tag{2.3}$$

where $L(2s, \psi_0\chi^2)$ is the Dirichlet L-function attached to $\psi_0\chi^2$. The Euler product expansion is

$$\begin{aligned} L(s, F\otimes F, \chi) &= \prod_{p\nmid M}(1-\chi(p)\alpha(p)^2p^{1-\kappa-s})^{-1}(1-\chi(p)p^{-s})^{-2} \\ &\quad \times (1-\chi(p)\beta(p)^2p^{1-\kappa-s})^{-1} \\ &\quad \times \prod_{p\mid M}(1-\chi(p)c(p)^2p^{1-\kappa-s})^{-1}. \end{aligned} \tag{2.4}$$

The expressions (2.3) and (2.4) are convergent absolutely for $\sigma > 1$, but $L(s, F\otimes F, \chi)$ can be continued to the whole plane, and is entire if χ is non-principal. If $\chi = \chi_0$ is principal, then it has a simple pole at $s = 1$.

If χ is primitive, then the twisted form

$$F_\chi(z) = \sum_{n=1}^{\infty} c(n)\chi(n)\mathrm{e}^{2\pi inz} \tag{2.5}$$

is a new form of level Mq^2. Hence $L(s, F\otimes F, \chi)$ satisfies a standard form of functional equation (Li [20]). Also $L(1+it, F\otimes F, \chi)\neq 0$ for any real t (cf. Ogg [24], Theorem 4 and Rankin [27], Theorem 1). Hence, applying the lemma in p.295 of Perelli [25], we can see that

$$-\frac{L'}{L}(s, F\otimes F, \chi) - \frac{\varepsilon_\chi}{s-1} \tag{2.6}$$

is holomorphic in the closed half-plane $\sigma \geqslant 1$, where $\varepsilon_\chi = 1$ if χ is principal and $\varepsilon_\chi = 0$ otherwise. Note that (2.6) is valid for any (not necessarily primitive) χ, as we can see easily by an argument similar to that in p. 89 of Davenport [4].

Define $\Lambda_F(n) = (\alpha(p)^m + \beta(p)^m)^2\log p$ if $n = p^m$ is a prime power and $\Lambda_F(n) = 0$ otherwise. Then we can write

$$-\frac{L'}{L}(s, F\otimes F, \chi) = \sum_{n=1}^{\infty}\Lambda_F(n)\chi(n)n^{1-\kappa-s}$$

when $\sigma > 1$.

Now we quote the following Tauberian theorem.

Lemma 2 ([7], Chapter 5, Corollary 3) *Let $a(n)$, $b(n)$ be arithmetical functions satisfying $a(n)\geqslant 0$, $b(n) = O(a(n))$ and*

$$\sum_{n\leqslant x} a(n) = O(x). \tag{2.7}$$

If the Dirichlet series

$$A(s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s},\qquad B(s)=\sum_{n=1}^{\infty}\frac{b(n)}{n^s}$$

are holomorphic for $\sigma>1$ *and*

$$A(s)-\frac{A_0}{s-1},\qquad B(s)-\frac{B_0}{s-1}$$

(where A_0*,* B_0 *are constants) are holomorphic for* $\sigma\geqslant 1$*, then*

$$\sum_{n\leqslant x}b(n)=B_0x(1+o(1)). \tag{2.8}$$

We apply this lemma with $a(n)=\Lambda_F(n)n^{1-\kappa}$ and $b(n)=\Lambda_F(n)\chi(n)n^{1-\kappa}$. Then

$$A(s)=-\frac{L'}{L}(s,F\otimes F),\qquad B(s)=-\frac{L'}{L}(s,F\otimes F,\chi).$$

hence

$$A(s)-\frac{1}{s-1},\qquad B(s)-\frac{\varepsilon_\chi}{s-1}$$

are holomorphic for $\sigma\geqslant 1$. Then conditions $a(n)\geqslant 0$, $b(n)=O(a(n))$ are clearly valid. Therefore, if we can show (2.7) for our present $a(n)$, we can deduce

$$\sum_{n\leqslant x}\Lambda_F(n)\chi(n)n^{1-\kappa}=\varepsilon_\chi x(1+o(1)) \tag{2.9}$$

by Lemma 2.

To prove (2.7), we divide the sum as

$$\begin{aligned}\sum_{n\leqslant x}\Lambda_F(n)n^{1-\kappa} &= \sum_{p\leqslant x}\Lambda_F(p)p^{1-\kappa}+\sum_{m\geqslant 2}\sum_{p^m\leqslant x}\Lambda_F(p^m)p^{m(1-\kappa)}\\ &= S_1+S_2,\end{aligned}$$

say. If $(p,M)=1$, then $|\alpha(p)|=|\beta(p)|=p^{(\kappa-1)/2}$. If $p\mid M$, then $\beta(p)=0$ and $|\alpha(p)|=|c(p)|\leqslant p^{(\kappa-2)/2}$ (see [22], Theorem 4.6.17). Hence

$$|\Lambda_F(p^m)p^{m(1-\kappa)}|\leqslant 4\log p$$

for any prime p and any $m\geqslant 1$. Therefore

$$S_2\ll\sum_{1\leqslant m\leqslant\log x/\log 2}\ \sum_{p\leqslant x^{1/2}}\log p\ll x^{1/2}(\log x)^2 \tag{2.10}$$

and

$$S_1\ll\sum_{p\leqslant x}\log p\ll x$$

by the classical prime number theorem. Hence (2.7) holds, and therefore (2.9) is established.

Then, using (2.9), we have

$$\sum_{\substack{n\leqslant x\\ n\equiv h(\mathrm{mod}\ q)}} \Lambda_F(n)n^{1-\kappa} = \frac{1}{\varphi(q)}\sum_{\chi \mathrm{mod}\ q}\overline{\chi}(h)\sum_{n\leqslant x}\Lambda_F(n)\chi(n)n^{1-\kappa}$$
$$= \frac{x}{\varphi(q)}(1+o(1)). \tag{2.11}$$

The left-hand side of the above is again divided as

$$\sum_{\substack{p\leqslant x\\ p\equiv h(\mathrm{mod}\ q)}} \Lambda_F(p)p^{1-\kappa} + \sum_{m\geqslant 2}\sum_{\substack{p^m\leqslant x\\ p^m\equiv h(\mathrm{mod}\ q)}} \Lambda_F(p^m)p^{m(1-\kappa)},$$

and the second term is $O(x^{1/2}(\log x)^2)$ by (2.10). Hence

$$\sum_{\substack{p\leqslant x\\ p\equiv h(\mathrm{mod}\ q)}} \Lambda_F(p)p^{1-\kappa} = \frac{x}{\varphi(q)}(1+o(1)). \tag{2.12}$$

Since $\Lambda_F(p) = c(p)^2\log p$, the assertion of Lemma 1 follows easily from (2.12) by partial summation.

The above proof of Lemma 1 is an analogue of the proof of Theorem 1, Chapter 6 of [7].

It is possible to refine Lemma 1. In fact, as in Perelli [26] and Ichihara [8], we can argue along the same line as in Davenport's book [4] to obtain

$$\sum_{\substack{p\leqslant x\\ p\equiv h(\mathrm{mod}\ q)}} \Lambda_F(p)p^{1-\kappa} = \frac{x}{\varphi(q)} + O(x\exp(-c_1\sqrt{\log x})) \tag{2.13}$$

with a certain $c_1 > 0$. Since $L(s, F\otimes F, \chi)$ satisfies a functional equation, we can use the results in Perelli [25] to prove (2.13). From (2.13) we can immediately obtain a refinement of Lemma 1; or we may quote Remark 5.2.2 (especially formula (14)) of Moreno's paper [23]. However, Lemma 1 is enough for our present purpose.

3 A limit theorem

Let $\lambda > 0$, and put

$$D_\lambda = \{s\in\mathbb{C} \mid \kappa/2 < \sigma < (\kappa+1)/2,\ |t| < \lambda\}.$$

By $H(D_\lambda)$ we denote the space of functions analytic on D_λ, equipped with the topology of uniform convergence on compacta, and put $H^m = H(D_\lambda)^m$ (direct product). Denote by $\mathcal{B}(S)$ the family of all Borel subsets of the space S. The measure P_T, defined by

$$P_T(A) = \nu_T((L_1(s+i\tau, F), \ldots, L_m(s+i\tau, F)) \in A)$$

for $T \geqslant 2$ and $A \in \mathcal{B}(H^m)$, is a probability measure on $(H^m, \mathcal{B}(H^m))$.

Let $\gamma = \{s \in \mathbb{C} \mid |s| = 1\}$, and $\Omega = \prod_p \gamma_p$, where p runs over all primes and $\gamma_p = \gamma$ for any p. We may regard Ω as a compact abelian topological group, hence the probability Haar measure m_H exists. This gives a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. By $\omega(p)$ we denote the projection of $\omega \in \Omega$ to the coordinate space γ_p. Define the H^m-valued random element $\varphi(s, \omega)$ by

$$\varphi(s, \omega) = (\varphi(s, \omega, L_1), \dots, \varphi(s, \omega, L_m)), \tag{3.1}$$

where $\omega \in \Omega$, $s \in D_\lambda$ and

$$\varphi(s, \omega, L_j) = \prod_p (1 - \frac{\alpha(p)\chi_j(p)\omega(p)}{p^s})^{-1}(1 - \frac{\beta(p)\chi_j(p)\omega(p)}{p^s})^{-1} \tag{3.2}$$

for $1 \leqslant j \leqslant m$. Let P_φ be the distribution of $\varphi(s, \omega)$, that is

$$P_\varphi(A) = m_H(\omega \in \Omega \mid \varphi(s, \omega) \in A)$$

for $A \in \mathcal{B}(H^m)$. Then

Lemma 3 *The probability measure P_T converges weakly to P_φ as $T \to \infty$.*

Proof Let $F_1, \dots, F_m$ be normalized eigenforms of weight κ,

$$L(s, F_j) = \sum_{n=1}^{\infty} c_j(n) n^{-s} \quad (1 \leqslant j \leqslant m)$$

the corresponding L-functions, $D_0 = \{s \in \mathbb{C} \mid \Re s > \kappa/2\}$, $H_0^m = H(D_0)^m$, and define

$$P_{T,0}(A) = \nu_T((L(s + i\tau, F_1), \dots, L(s + i\tau, F_m)) \in A)$$

for $A \in \mathcal{B}(H_0^m)$. Also let

$$\varphi_0(s, \omega) = \left(\sum_{n=1}^{\infty} \frac{c_1(n)\omega(n)}{n^s}, \dots, \sum_{n=1}^{\infty} \frac{c_m(n)\omega(n)}{n^s} \right),$$

where

$$\omega(n) = \prod_{p^\alpha || n} \omega(p)^\alpha,$$

and $P_{\varphi,0}$ be the distribution of the H_0^m-valued random element φ_0. Then Laurinčikas [16] proved that $P_{T,0}$ converges weakly to $P_{\varphi,0}$ as $T \to \infty$. Actually Laurinčikas only discussed the case when $F_1, \dots, F_m$ are cusp forms with respect to $\mathrm{SL}(2, \mathbb{Z})$, but we can easily generalize his result to the case of congruence subgroups of higher level.

The function $g : H_0^m \to H^m$ defined by the coordinatewise restriction is continuous. Hence, using a property of weak convergence of probability measures ([3, Theorem 5.1]), we can replace H_0^m in the above statement by H^m. Lemma 3 is a special case of this assertion.

It is also possible to show Lemma 3 as a special case of Theorem 2 of Laurinčikas [15].

4 A denseness lemma

Let

$$f_{jp}(s,a_p) = -\log\left(1 - \frac{\alpha(p)\chi_j(p)a_p}{p^s}\right) - \log\left(1 - \frac{\beta(p)\chi_j(p)a_p}{p^s}\right)$$

for $a_p \in \gamma$ $(1 \leqslant j \leqslant m)$, and

$$\mathbf{f}_p(s,a_p) = (f_{1p}(s,a_p), \ldots, f_{mp}(s,a_p)).$$

Then we have

Lemma 4 *The set of all series $\sum_p \mathbf{f}_p(s,a_p)$, which are convergent in H^m, is dense in H^m.*

This is a generalization of Lemma 2 in [18], and one of the key ingredients of the proof of Theorem 1. This and the next section are devoted to the proof of Lemma 4.

Let $p_0 > 0$, and put

$$\hat{\mathbf{f}}_p(s) = \begin{cases} \mathbf{f}_p(s,1) & \text{if } p > p_0, \\ 0 & \text{if } p \leqslant p_0. \end{cases}$$

Then there exists an $\hat{a}_p \in \gamma$ for which

$$\sum_p \hat{a}_p \hat{\mathbf{f}}_p(s)$$

is convergent in H^m. This can be shown similarly to the first step of the proof of Lemma 2 in [18]. Let

$$\mathbf{g}_p = (g_{1p}, \ldots, g_{mp}) = \hat{a}_p \hat{\mathbf{f}}_p(s).$$

Then we have

Lemma 5 *The set of all series $\sum_p a_p \mathbf{g}_p$ $(a_p \in \gamma)$, which are convergent in H^m, is dense in H^m.*

By using this lemma, we can complete the proof of Lemma 4 similarly to the third step of the proof of Lemma 2 in [18], the details being omitted here.

The proof of Lemma 5 is based on the following

Lemma 6 *Let $\mathbf{f}_l = (f_{1l}, \ldots, f_{ml}) \in H^m$ $(l = 1, 2, 3, \ldots)$. Assume that the sequence $\{\mathbf{f}_l\}$ satisfies*

(a) *if $\mu_1, \ldots, \mu_m$ are complex measures on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, whose supports are compact and contained in D_λ, and*

$$\sum_{l=1}^{\infty} \left| \sum_{j=1}^{m} \int_{\mathbb{C}} f_{jl}\, d\mu_j \right| < \infty,$$

then

$$\int_{\mathbb{C}} s^r \, d\mu_j(s) = 0 \tag{4.1}$$

for $1 \leqslant j \leqslant m$ *and* $r = 0, 1, 2, \ldots$;

(b) *the series* $\sum_{l=1}^{\infty} \mathbf{f}_l$ *is convergent in* H^m;

(c) *for any compact subset* K_j *of* D_λ $(1 \leqslant j \leqslant m)$,

$$\sum_{l=1}^{\infty} \sum_{j=1}^{m} \sup_{s \in K_j} |f_{jl}(s)|^2 < \infty.$$

Then the set of all convergent series $\sum_{l=1}^{\infty} a_l \mathbf{f}_l$ $(a_l \in \gamma)$ *is dense in* H^m.

This is Lemma 5.2.9 of Bagchi [1]. The proof in the case $m = 1$ is given in [13] (see Theorem 6.3.10), and the proof of the general case is similar.

We apply Lemma 6 to $\mathbf{f}_l = \mathbf{g}_p$. The condition (b) is clearly satisfied by the definition of $\mathbf{g}_p$. Since

$$f_{jp}(s, 1) = \frac{c(p)\chi_j(p)}{p^s} + r_{jp}(s) \tag{4.2}$$

with $r_{jp}(s) = \mathrm{O}(p^{\kappa - 2\sigma - 1})$, it can be easily seen that the condition (c) is also satisfied.

Next we check the condition (a). Let μ_j $(1 \leqslant j \leqslant m)$ be complex measures whose supports are compact and contained in D_λ, and

$$\sum_p \left| \sum_{j=1}^{m} \int_{\mathbb{C}} g_{jp} \, d\mu_j \right| < \infty. \tag{4.3}$$

Using (4.2), we find that (4.3) is equivalent to

$$\sum_{p > p_0} \left| \sum_{j=1}^{m} \int_{\mathbb{C}} \hat{a}_p c(p) \chi_j(p) p^{-s} \, d\mu_j(s) \right| < \infty.$$

Since $|\hat{a}_p| = 1$, this is further equivalent to

$$\sum_{p > p_0} \left| \tilde{c}(p) \sum_{j=1}^{m} \chi_j(p) \int_{\mathbb{C}} p^{-s} \, d\mu_j w^{-1}(s) \right| < \infty, \tag{4.4}$$

where w is the mapping defined by $w(s) = s - (\kappa - 1)/2$.

Let q be the least common multiple of $q_1, \ldots, q_m$, χ_j^* be the character mod q induced by χ_j $(1 \leqslant j \leqslant m)$, and h be a positive integer satisfying $1 \leqslant h \leqslant q$, $(h, q) = 1$. If $p \equiv h \pmod{q}$, then $\chi_j(p) = \chi_j^*(p) = \chi_j^*(h)$, hence from (4.4) we have

$$\sum_{p \equiv h \pmod{q}} \left| \tilde{c}(p) \sum_{j=1}^{m} \chi_j^*(h) \int_{\mathbb{C}} p^{-s} \, d\mu_j w^{-1}(s) \right| < \infty,$$

or equivalently,

$$\sum_{p\equiv h \pmod q} |\tilde{c}(p)||\rho(\log p)| < \infty, \tag{4.5}$$

where

$$\rho(z) = \int_{\mathbb{C}} \mathrm{e}^{-sz}\, d\nu_h(s) \tag{4.6}$$

and

$$d\nu_h = \sum_{j=1}^{m} \chi_j^*(h)\, d\mu_j w^{-1}. \tag{4.7}$$

In the next section we will show that (4.5) implies $\rho(z) \equiv 0$. Then, differentiating r-times the equality $\rho(z) \equiv 0$ and putting $z = 0$, we obtain

$$\int_{\mathbb{C}} s^r\, d\nu_h(s) \equiv 0 \quad (r = 0, 1, 2, \ldots).$$

Hence

$$\sum_{j=1}^{m} \chi_j^*(h) \int_{\mathbb{C}} s^r\, d\mu_j(s) \equiv 0 \quad (r = 0, 1, 2, \ldots)$$

for $1 \leqslant h \leqslant q$, $(h, q) = 1$. Multiplying the both sides of the above equality by $\bar{\chi}_i^*(h)$ and summing with respect to h, we have

$$\begin{aligned} 0 &= \sum_{\substack{1\leqslant h\leqslant q\\(h,q)=1}} \sum_{j=1}^{m} \chi_j^*(h)\bar{\chi}_i^*(h) \int_{\mathbb{C}} s^r\, d\mu_j(s) \\ &= \sum_{j=1}^{m} \left\{ \sum_{\substack{1\leqslant h\leqslant q\\(h,q)=1}} \chi_j^*(h)\bar{\chi}_i^*(h) \right\} \int_{\mathbb{C}} s^r\, d\mu_j(s) \\ &= \varphi(q) \int_{\mathbb{C}} s^r\, d\mu_i(s) \end{aligned}$$

for $1 \leqslant i \leqslant m$, hence (4.1) follows. All the conditions of Lemma 6 are now verified, hence, applying Lemma 6, we obtain the assertion of Lemma 5.

The above argument of deducing (4.1) from (4.3), by using the orthogonality of Dirichlet characters, is analogous to the proof of Lemma 4.9 of Bagchi [2]. Bagchi assumed that all the characters are of the same modulus, but this restriction can be easily removed as above. This point was inspired by Section 7.3 of Karatsuba–Voronin [10].

5 A vanishing lemma

To complete the proof of Lemma 4, now the only remaining task is to establish

Lemma 7 *The function $\rho(z)$ defined by (4.6) is identically equal to zero.*

This is a generalization of Lemma 6 in [18]. The positive density method mentioned in Section 1 is used in the proof of this lemma. Let $0 < \theta < 1$, and $\mathcal{P}_\theta(h)$ be the set of primes satisfying $p \equiv h \pmod q$ and $|\tilde{c}(p)| > \theta$. From (4.5) we have

$$\sum_{p \in \mathcal{P}_\theta(h)} |\rho(\log p)| < \infty. \tag{5.1}$$

Let β be a positive number satisfying

$$\beta\lambda < \pi, \tag{5.2}$$

and put $a = a(m) = \exp\left((m - \frac{1}{4})\beta\right)$, $b = b(m) = \exp\left((m + \frac{1}{4})\beta\right)$. Denote by $\mathcal{A}$ the set of all integers m such that there exists an $r \in (\log a, \log b]$ with $|\rho(r)| \leqslant \mathrm{e}^{-r}$. Then

$$\sum_{p \in \mathcal{P}_\theta(h)} |\rho(\log p)| \geqslant \sum_{m \notin \mathcal{A}} \sum_{\substack{p \in \mathcal{P}_\theta(h) \\ a < p \leqslant b}} |\rho(\log p)| \geqslant \sum_{m \notin \mathcal{A}} \sum_{\substack{p \in \mathcal{P}_\theta(h) \\ a < p \leqslant b}} p^{-1},$$

hence with (5.1) we obtain

$$\sum_{m \notin \mathcal{A}} \sum_{\substack{p \in \mathcal{P}_\theta(h) \\ a < p \leqslant b}} p^{-1} < \infty. \tag{5.3}$$

Next, let δ be a positive number with $1 + \delta < \min\{3/2, \mathrm{e}^{\beta/2}\}$, and assume $0 < \varepsilon < \delta/100$. Using Lemma 1, analogously to (4.9) of [18], we obtain

$$\sum_{\substack{p \in \mathcal{P}_\theta(h) \\ a < p \leqslant b}} p^{-1} \geqslant \frac{1 - \theta^2}{4 - \theta^2} \left(\sum_{\substack{a(1+\delta) < p \leqslant b \\ p \equiv h \ (\mathrm{mod}\ q)}} p^{-1} \right) (1 + \mathrm{o}(1)) \tag{5.4}$$

as $m \to \infty$. Here we mention that there are misprints on the right-hand side of the first line of (4.9) of [18]; it is to be read as

$$\left(\sum_{\substack{p \in \mathcal{P}_\mu \\ a < p \leqslant b}} 1 \right) \frac{1}{b} + \int_a^b \left(\sum_{\substack{p \in \mathcal{P}_\mu \\ a < p \leqslant u}} 1 \right) \frac{du}{u^2}.$$

Let $\pi(x; q, h)$ be the number of primes up to x satisfying $p \equiv h \pmod q$. Then the Siegel–Walfisz theorem implies

$$\pi(x; q, h) = \frac{1}{\varphi(q)} \int_0^x \frac{dt}{\log t} + \mathrm{O}\left(x \exp(-c_5(\varepsilon)\sqrt{\log x}) \right) \tag{5.5}$$

for $x \geqslant \exp(q^\varepsilon)$ with a constant $c_5(\varepsilon) > 0$. From (5.5) it can be easily deduced that

$$\sum_{\substack{p \leqslant x \\ p \equiv h \ (\mathrm{mod}\ q)}} p^{-1} = \frac{1}{\varphi(q)} \log\log x + B + \mathrm{O}(\exp(-c_6\sqrt{\log x})), \tag{5.6}$$

where B, $c_6(>0)$, and the implied constant depend on q. Using (5.6) we obtain

$$\sum_{\substack{a(1+\delta)<p\leqslant b\\ p\equiv h(q)}} p^{-1} = \frac{1}{\varphi(q)}\left(\frac{1}{2} - \frac{\log(1+\delta)}{\beta}\right)\frac{1}{m} + \mathrm{O}\left(\frac{1}{m^2}\right). \tag{5.7}$$

Collecting (5.3), (5.4) and (5.7) we find

$$\sum_{m\notin\mathcal{A}} \frac{1}{m} < \infty.$$

Hence, if we write $\mathcal{A} = \{a_m \mid m = 1, 2, \ldots\}$, $a_1 < a_2 < \cdots$, then

$$\lim_{m\to\infty} \frac{a_m}{m} = 1.$$

The definition of $\mathcal{A}$ implies that there exists a ξ_m for each m such that $(a_m - \frac{1}{4})\beta < \xi_m \leqslant (a_m + \frac{1}{4})\beta$ and $|\rho(\xi_m)| \leqslant \exp(-\xi_m)$. Hence

$$\lim_{m\to\infty} \frac{\xi_m}{m} = \beta \tag{5.8}$$

and

$$\limsup_{m\to\infty} \frac{\log|\rho(\xi_m)|}{\xi_m} \leqslant -1. \tag{5.9}$$

Now we quote the following

Lemma 8 *Let $f(s)$ be an entire function of exponential type, and let $\{\xi_m\}$ be a sequence of complex numbers. Let λ, η and ω be positive numbers such that*

$$\limsup_{y\to\infty} \frac{\log|f(\pm iy)|}{y} \leqslant \lambda,$$
$$|\xi_m - \xi_n| \geqslant \omega|m-n|,$$
$$\lim_{m\to\infty} \frac{\xi_m}{m} = \beta,$$

and $\lambda\beta < \pi$. Then

$$\limsup_{m\to\infty} \frac{\log|f(\xi_m)|}{|\xi_m|} = \limsup_{r\to\infty} \frac{\log|f(r)|}{r}.$$

For the proof of this lemma, see Theorem 6.4.12 of [13].

In view of (5.2) and (5.8), we can apply this lemma to $f = \rho$. From (5.9) we obtain

$$\limsup_{r\to\infty} \frac{\log|\rho(r)|}{r} \leqslant -1.$$

This corresponds to (4.12) of [18], and from this we can deduce the conclusion $\rho(z) \equiv 0$ in much the same way as in [18]. The proof of Lemma 7, hence of Lemma 4, is now complete.

6 Completion of the proof of Theorem 1

Let S_λ be the set of functions $f \in H(D_\lambda)$ such that $f(s) \neq 0$ for any $s \in D_\lambda$, or $f \equiv 0$. We first show

Lemma 9 *The support of the random element $\varphi(s,\omega)$ is S_λ^m.*

This is a generalization of Lemma 8 of [18]. Instead of Lemma 10 of [18], we use its multidimensional version, that is Lemma 2 of [17] (with replacing D by D_λ). Similarly to [18], we find that the support of the H^m-valued random element

$$\left(\sum_p f_{1p}(s,\omega(p)),\ldots,\sum_p f_{mp}(s,\omega(p))\right)$$

is the closure of the set of all convergent series of the form

$$\sum_p \mathbf{f}_p(s,a_p) \quad (a_p \in \gamma).$$

By Lemma 4 the latter set is dense in H^m. From this fact, again similarly to [18], we obtain the conclusion of Lemma 9.

Now we can complete the proof of Theorem 1 in a standard way. For any given K_j, we find a λ sufficiently large for which $K_j \subset D_\lambda$ holds $(1 \leqslant j \leqslant m)$. Let $f_j \in \mathcal{F}(K_j)$ $(1 \leqslant j \leqslant m)$, and first consider the case when all f_js have non-vanishing holomorphic continuation to D_λ. Let

$$G = \{(g_1,\ldots,g_m) \in H^m \mid \sup_{1\leqslant j\leqslant m}\ \sup_{s\in K_j} |g_j(s) - f_j(s)| < \varepsilon\}.$$

Then G is open, hence Lemma 3 implies

$$\liminf_{T\to\infty} P_T(G) \geqslant P_\varphi(G). \tag{6.1}$$

Since G is an open neighbourhood of $(f_1,\ldots,f_m)$, which is contained in the support of P_φ by Lemma 9, we have $P_\varphi(G) > 0$. This implies the assertion of Theorem 1, because

$$P_T(G) = \nu_T(\sup_{1\leqslant j\leqslant m}\ \sup_{s\in K_j} |L_j(s+i\tau,F) - f_j(s)| < \varepsilon).$$

When $f_j(s)$ cannot be continued to D_λ for some j, then we use Mergelyan's theorem to find a polynomial $q_j(s)$ for which $\exp(q_j(s))$ approximates $f_j(s)$ sufficiently (see the last section of [18]). Therefore we can reduce the problem to the already proved case. The proof of Theorem 1 is now complete.

7 Proof of Theorems 2, 3 and 4

The proofs presented in this section are simple modifications of known arguments, but we describe the details for the convenience of readers.

The following proofs of Theorems 2 and 3 are similar to the proofs of Theorem 6.6.2 and Theorem 6.6.3 of [13].

Let $s_{\nu j}$ be complex numbers ($0 \leqslant \nu \leqslant N-1,\ 1 \leqslant j \leqslant m$), and assume $s_{0j} \neq 0$ ($1 \leqslant j \leqslant m$). By Lemma 6.6.1 of [13] we can find polynomials $p_j(s) = \sum_{\nu=0}^{N-1} b_{\nu j} s^\nu$ such that

$$s_{\nu j} = (\exp(p_j(s)))^{(\nu)}|_{s=0} \quad (0 \leqslant \nu \leqslant N-1).$$

Let $\kappa/2 < \sigma_1 < (\kappa+1)/2$, and K be a compact subset of D with connected complement such that σ_1 is contained in the interior of K. By δ we denote the distance of σ_1 from the boundary of K. Then from Theorem 1 we find a real τ for which

$$\sup_{1 \leqslant j \leqslant m} \sup_{s \in K} |L_j(s+i\tau, F) - \mathrm{e}^{p_j(s-\sigma_1)}| < \frac{\varepsilon \delta^N}{2^N N!}$$

holds. Then, using Cauchy's integral formula we have

$$\begin{aligned} &|L_j^{(\nu)}(\sigma_1 + i\tau, F) - s_{\nu j}| \\ &= \frac{\nu!}{2\pi} \left| \int_{|s-\sigma_1|=\delta/2} \frac{|L_j(s+i\tau, F) - \mathrm{e}^{p_j(s-\sigma_1)}}{(s-\sigma_1)^{\nu+1}} ds \right| < \varepsilon \end{aligned}$$

for $0 \leqslant \nu \leqslant N-1$, which implies Theorem 2.

Next we prove Theorem 3. Suppose $f_L \not\equiv 0$. Then there exists a bounded region $R \subset \mathbb{C}^{Nm}$ such that

$$|f_L| \geqslant B_0 > 0 \tag{7.1}$$

in R, where B_0 is a constant. Let $\kappa/2 < \sigma < (\kappa+1)/2$. From Theorem 2 we find a sequence of real numbers $\{\tau_k\}$ satisfying $\lim_{n \to \infty} \tau_k = +\infty$ and

$$\begin{aligned} \mathbf{x}_k &= (L_1(\sigma + i\tau_k, F), \ldots, L_m(\sigma + i\tau_k, F), L_1'(\sigma + i\tau_k, F), \ldots, L_m'(\sigma + i\tau_k, F), \\ &\quad \ldots, L_1^{(N-1)}(\sigma + i\tau_k, F), \ldots, L_m^{(N-1)}(\sigma + i\tau_k, F)) \in R. \end{aligned}$$

Substituting $s = \sigma + i\tau_k$ into (1.8) and dividing the both sides by $(\sigma + i\tau_k)^L$, we have

$$\sum_{l=0}^{L-1} (\sigma + i\tau_k)^{l-L} f_l(\mathbf{x}_k) = -f_L(\mathbf{x}_k). \tag{7.2}$$

Since R is bounded, $|f_l(\mathbf{x}_k)|$ is bounded ($0 \leqslant l \leqslant L-1$). Hence the left-hand side of (7.2) tends to zero as $k \to \infty$. On the other hand, $|f_L(\mathbf{x}_k)| \geqslant B_0 > 0$ by (7.1). This contradiction implies $f_L \equiv 0$. Similarly we obtain $f_{L-1} \equiv \cdots \equiv f_1 \equiv f_0 \equiv 0$.

Finally we prove Theorem 4. The following argument is analogous to the proof of Theorem 1 in Laurinčikas [15]. We may assume $u_1 \neq 0$ and $u_2 \neq 0$. Let $\eta > 0$, $\hat{\sigma} = (\sigma_1 + \sigma_2)/2$,

$$\begin{aligned} f_1(s) &= \frac{s - \hat{\sigma} + 3}{u_1}, \qquad f_2(s) = -\frac{3}{u_2}, \\ f_3(s) &= \cdots = f_m(s) = \eta, \end{aligned}$$

and

$$K = \left\{ s \in \mathbb{C} \mid \left| s - \left(\frac{\kappa}{2} + \frac{1}{4} \right) \right| \leqslant \max\left(\left| \sigma_1 - \left(\frac{\kappa}{2} + \frac{1}{4} \right) \right|, \left| \sigma_2 - \left(\frac{\kappa}{2} + \frac{1}{4} \right) \right| \right) \right\}.$$

Applying Theorem 1 we find a τ for which

$$\sup_{1\leqslant j\leqslant m}\sup_{s\in K}|L_j(s+i\tau,F)-f_j(s)|<\eta. \tag{7.3}$$

Then

$$\sup_{s\in K_j}\left|V(s+i\tau)-\sum_{j=1}^m u_jf_j(s)\right|<\eta\sum_{j=1}^m|u_j|.$$

Since

$$\sum_{j=1}^m u_jf_j(s)=s-\hat{\sigma}+(u_3+\cdots+u_m)\eta,$$

we have

$$\sup_{s\in K}|V(s+i\tau)-(s-\hat{\sigma})|<2\eta\sum_{j=1}^m|u_j|. \tag{7.4}$$

Let C be the circle $\{s\in\mathbb{C}\mid|s-\hat{\sigma}|=(\sigma_2-\sigma_1)/2\}$. Then C is included in K. Choosing η so small that it satisfies

$$2\eta\sum_{j=1}^m|u_j|<\frac{\sigma_2-\sigma_1}{4},$$

we find from (7.4) that

$$|s-\hat{\sigma}|>|V(s+i\tau)-(s-\hat{\sigma})|$$

for $s\in C$. Hence by Rouché's theorem we can conclude that $V(s+i\tau)$ has a zero in $|s-\hat{\sigma}|\leqslant(\sigma_2-\sigma_1)/2$. Since Theorem 1 further implies that the set of $\tau\in[0,T]$ satisfying (7.3) has a positive lower density, we obtain the assertion of Theorem 4.

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