ON MORDELL-TORNHEIM AND OTHER MULTIPLE ZETA-FUNCTIONS

KOHJI MATSUMOTO

1. Introduction

In 1950, Tornheim [14] introduced the double series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{-s_1}n^{-s_2}(m+n)^{-s_3} \tag{1.1}$$

of three variables, and studied its values when s_1, s_2, s_3 are integers in the region of absolute convergence. Later Mordell [12] independently considered the special case $s_1 = s_2 = s_3$, and also studied the multiple sum

$$\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\frac{1}{m_1\cdots m_r(m_1+\cdots+m_r+a)} \tag{1.2}$$

with $a > -r$. Mordell's result for (1.2) was used by Hoffman [5] to obtain a formula for the value of

$$\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\frac{1}{m_1\cdots m_r(m_1+\cdots+m_r)^s} \tag{1.3}$$

when s is a positive integer.

One of the purposes of the present paper is to consider the multiple sum

$$\begin{aligned}&\zeta_{MT,r}(s_1,\ldots,s_r;s_{r+1})\\ &=\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty} m_1^{-s_1}\cdots m_r^{-s_r}(m_1+\cdots+m_r)^{-s_{r+1}},\end{aligned} \tag{1.4}$$

which is a generalization of both (1.1) and (1.3). Here $s_1,\ldots,s_{r+1}$ are complex variables. We call this the Mordell-Tornheim r-ple zeta-function. The series (1.4) is convergent absolutely when $\Re s_j > 1$ $(1 \le j \le r)$ and $\Re s_{r+1} > 0$, but we can prove the following theorem, a part of which was announced in [8].

Theorem 1. *The series (1.4) can be continued meromorphically to the whole $\mathbb{C}^{r+1}$ space, and the possible singularities are located only on the subsets of $\mathbb{C}^{r+1}$*

defined by one of the following equations:

$$
\begin{aligned}
&s_j + s_{r+1} = 1 - \ell \quad (1 \le j \le r,\ \ell \in \mathbb{N}_0),\\
&s_{j_1} + s_{j_2} + s_{r+1} = 2 - \ell \quad (1 \le j_1 < j_2 \le r,\ \ell \in \mathbb{N}_0),\\
&\qquad\qquad \cdots\cdots\\
&s_{j_1} + \cdots + s_{j_{r-1}} + s_{r+1} = r - 1 - \ell \quad (1 \le j_1 < \cdots < j_{r-1} \le r,\ \ell \in \mathbb{N}_0),\\
&s_1 + \cdots + s_r + s_{r+1} = r,
\end{aligned}
$$

where $\mathbb{N}_0$ *denotes the set of non-negative integers.*

In the case $r = 2$ (that is, the series (1.1)), this result was proved in [8]. The analytic continuation of (1.1) was first obtained by S. Akiyama and also by S. Egami (both proofs are unpublished), but the argument in [8] is different from theirs.

The series

$$\sum_{m=1}^{\infty} \sum_{n<m} m^{-s_1} n^{-s_2} (m+n)^{-1}, \tag{1.5}$$

closely connected with (1.1), was introduced by Apostol and Vu [3]. They were inspired by the work of Sitaramachandrarao and Sivaramasarma [13]. Partial results on the analytic continuation of (1.5) was obtained by Apostol and Vu themselves in [3], but the meromorphic continuation of (1.5), and of the more general series

$$\sum_{m=1}^{\infty} \sum_{n<m} m^{-s_1} n^{-s_2} (m+n)^{-s_3}, \tag{1.6}$$

to the whole space was first proved in [8]. In the present paper, we introduce the following generalization of (1.6), which we call the Apostol-Vu r-ple zeta-function:

$$
\begin{aligned}
&\zeta_{AV,r}(s_1, \ldots, s_r; s_{r+1})\\
&= \sum\cdots\sum_{1 \le m_1 < \cdots < m_r < \infty} m_1^{-s_1} \cdots m_r^{-s_r} (m_1 + \cdots + m_r)^{-s_{r+1}}.
\end{aligned} \tag{1.7}
$$

This series is convergent absolutely when $\Re s_j > 1$ $(1 \le j \le r)$ and $\Re s_{r+1} > 0$. Our second result is

Theorem 2. *The series (1.7) can be continued meromorphically to the whole* $\mathbb{C}^{r+1}$ *space, and the possible singularities are located only on the subsets of* $\mathbb{C}^{r+1}$ *defined by one of the following equations:*

$$s_i + \cdots + s_{r+1} = r + 1 - i - \ell \quad (1 \le i \le r,\ \ell \in \mathbb{N}_0).$$

Another type of multiple zeta-functions is the Euler-Zagier r-ple sum, which is

$$\zeta_r(s_1,\ldots,s_r)=\sum_{1\leq m_1<\cdots<m_r<\infty}\cdots\sum m_1^{-s_1}m_2^{-s_2}\cdots m_r^{-s_r} \tag{1.8}$$

(cf. Zagier [19]). The meromorphic continuation of (1.8) to the whole $\mathbb{C}^r$ space was recently established by various methods. One of them is due to the author [7], based on the Mellin-Barnes integral formula. Proofs of Theorems 1 and 2 in the present paper will be carried out under the same principle, in Section 3.

There are numerous papers on the values of various multiple zeta-functions in the domain of absolute convergence. Concerning multiple zeta-functions of the Mordell-Tornheim type, we mention here the recent work of Tsumura [15][16][17]. On the other hand, after proving the analytic continuation, we can discuss the values outside the domain of absolute convergence. As far as the author knows, the first work studying such a problem is a paper of Matsuoka [11], who discussed the value of $\zeta_2(1,s_2)$ when s_2 is a negative integer. Apostol and Vu [3] also studied some special values of $\zeta_2(s_1,s_2)$ and of (1.5) outside the domain of absolute convergence. Recently, special values of (1.8) at negative integers were studied extensively by Akiyama, Egami and Tanigawa ([1], [2]). Now our Theorems 1 and 2 give the full meromorphic continuation of the Mordell-Tornheim and the Apostol-Vu multiple zeta-functions. It is therefore desirable to develop a study on the values of these multiple zeta-functions at negative integers.

It is possible to consider the following more general multiple zeta-functions. Let $A_{nr}=(a_{ij})_{1\leq i\leq n,1\leq j\leq r}$ be an (n,r)-matrix, where a_{ij} are non-negative real numbers. Assume that all rows and all columns of A_{nr} include at least one non-zero element. Define

$$\begin{aligned}\zeta_r(s_1,\ldots,s_n;A_{nr})=\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}&(a_{11}m_1+\cdots+a_{1r}m_r)^{-s_1}\\ &\times\cdots\times(a_{n1}m_1+\cdots+a_{nr}m_r)^{-s_n}.\end{aligned} \tag{1.9}$$

This multiple series is absolutely convergent when $\Re s_i>r$ $(1\leq i\leq n)$. We have

Theorem 3. *The multiple zeta-function (1.9) can be continued meromorphically to the whole $\mathbb{C}^n$ space.*

Our proof of Theorem 3, presented in Section 2, also depends on the Mellin-Barnes formula. The assertion of the analytic continuation of $\zeta_{MT,r}$ and $\zeta_{AV,r}$ is clearly special cases of Theorem 3, but our treatment of $\zeta_{MT,r}$ and $\zeta_{AV,r}$ in Section 3 will give more explicit information. It is likely that Theorem 3 can be proved by various other ways. Indeed, in some cases, the assertion follows from a general result of Lichtin [6]. An advantage of our present method is that it shows a recursive structure of the theory, which will be embodied in (2.4), (3.2), and (3.6) below. Similar recursive structure also exists for other classes of multiple zeta-functions, such as those of Barnes, of Shintani, and of Witten. In view of the Mellin-Barnes induction argument in the present paper, we can find that all of the multiple zeta-functions mentioned above are members of a single family. In the final section we will discuss this unified viewpoint, which might be useful for further systematic study of multiple zeta-functions.

In the following sections, we write $\sigma_j = \Re s_j$. The Riemann zeta-function is denoted by $\zeta(s)$. The letter ε denotes an arbitrarily small positive number.

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2. Proof of Theorem 3

For each row $\mathbf{a}_i = (a_{i1}, \ldots, a_{ir})$ of A_{nr}, let $\rho(\mathbf{a}_i)$ be the number of non-zero elements of $\mathbf{a}_i$, and define

$$\rho(A_{nr}) = \prod_{i=1}^{n} \rho(\mathbf{a}_i).$$

By induction on $\rho(A_{nr})$, we prove Theorem 3 and the assertions that $\zeta_r(s_1, \ldots, s_n; A_{nr})$ is of polynomial order with respect to $\Im s_j$ $(1 \le j \le n)$ and possible singularities of $\zeta_r(s_1, \ldots, s_n; A_{nr})$ are located only on hyperplanes of the form

$$c_1 s_1 + \cdots + c_n s_n = u(c_1, \ldots, c_n) - \ell, \tag{2.1}$$

where $c_1, \ldots, c_n$ are non-negative integers, $u(c_1, \ldots, c_n)$ is an integer determined by $c_1, \ldots, c_n$, and $\ell \in \mathbb{N}_0$.

First consider the case $\rho(A_{nr}) = 1$. Each row includes only one non-zero element, which we denote by $a_{i,h(i)}$. Then

$$\begin{aligned}
\zeta_r(s_1, \ldots, s_n; A_{nr}) &= \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} (a_{1,h(1)} m_{h(1)})^{-s_1} \cdots (a_{n,h(n)} m_{h(n)})^{-s_n} \\
&= (a_{1,h(1)})^{-s_1} \cdots (a_{n,h(n)})^{-s_n} \left(\sum_{m_1=1}^{\infty} m_1^{-s(1)} \right) \cdots \left(\sum_{m_r=1}^{\infty} m_r^{-s(r)} \right) \\
&= (a_{1,h(1)})^{-s_1} \cdots (a_{n,h(n)})^{-s_n} \prod_{j=1}^{r} \zeta(s(j)), \qquad (2.2)
\end{aligned}$$

where

$$s(j) = \sum_{h(i)=j} s_i.$$

Hence $\zeta_r(s_1,\ldots,s_n;A_{nr})$ is clearly meromorphic in the whole $\mathbb{C}^n$ space, and the assertions on the order and on the location of singularities are obvious.

Now consider the case $\rho(A_{nr})\geq 2$. Then $r\geq 2$, and at least one row of A_{nr} has at least two non-zero elements. Changing the parameters if necessary, we may assume that $a_{n,r-1}\neq 0$, $a_{nr}\neq 0$.

The Mellin-Barnes integral formula can be stated as follows. Let s and λ be complex numbers with $\Re s>0$, $|\arg\lambda|<\pi$, $\lambda\neq 0$, and c be real with $-\Re s<c<0$. Then

$$(1+\lambda)^{-s}=\frac{1}{2\pi\sqrt{-1}}\int_{(c)}\frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)}\lambda^z dz, \tag{2.3}$$

where the path of integration is the vertical line from $c-\sqrt{-1}\infty$ to $c+\sqrt{-1}\infty$. A simple proof of (2.3) is mentioned in Section 4 of [8].

Let $r^*\geq r$ and assume that $(s_1,\ldots,s_n)$ is in the region

$$\mathcal{B}^*=\{(s_1,\ldots,s_n)\mid \sigma_i>r^*\ (1\leq i\leq n)\}.$$

Then the series (1.9) is absolutely convergent. Rewrite the right-hand side of (1.9) as

$$\begin{aligned}&\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}(a_{11}m_1+\cdots+a_{1r}m_r)^{-s_1}\cdots(a_{n-1,1}m_1+\cdots+a_{n-1,r}m_r)^{-s_{n-1}}\\ &\quad\times(a_{n1}m_1+\cdots+a_{n,r-1}m_{r-1})^{-s_n}\left(1+\frac{a_{nr}m_r}{a_{n1}m_1+\cdots+a_{n,r-1}m_{r-1}}\right)^{-s_n},\end{aligned}$$

and apply (2.3) with $s=s_n$ and $\lambda=a_{nr}m_r/(a_{n1}m_1+\cdots+a_{n,r-1}m_{r-1})$. Then we have

$$\begin{aligned}&\zeta_r(s_1,\ldots,s_n;A_{nr})\\ &=\frac{1}{2\pi\sqrt{-1}}\int_{(c)}\frac{\Gamma(s_n+z)\Gamma(-z)}{\Gamma(s_n)}\sum_{m_1}\cdots\sum_{m_r}(a_{11}m_1+\cdots+a_{1r}m_r)^{-s_1}\\ &\quad\times\cdots\times(a_{n-1,1}m_1+\cdots+a_{n-1,r}m_r)^{-s_{n-1}}\\ &\quad\times(a_{n1}m_1+\cdots+a_{n,r-1}m_{r-1})^{-s_n-z}(a_{nr}m_r)^z dz\\ &=\frac{1}{2\pi\sqrt{-1}}\int_{(c)}\frac{\Gamma(s_n+z)\Gamma(-z)}{\Gamma(s_n)}\zeta_r(s_1,\ldots,s_{n-1},s_n+z,-z;A'_{n+1,r})dz,\end{aligned} \tag{2.4}$$

where

$$A'_{n+1,r}=\begin{pmatrix} a_{11} & \cdots & a_{1,r-1} & a_{1r}\\ \cdots & \cdots & \cdots & \cdots\\ a_{n-1,1} & \cdots & a_{n-1,r-1} & a_{n-1,r}\\ a_{n1} & \cdots & a_{n,r-1} & 0\\ 0 & \cdots & 0 & a_{nr}\end{pmatrix}.$$

To assure the absolute convergence of the multiple series in the integrand, we have to choose a c satisfying $-\sigma_n + r - 1 < c < -1$. Since $\rho(A'_{n+1,r}) < \rho(A_{nr})$, we can use the induction assumption to find that $\zeta_r(s_1, \ldots, s_{n-1}, s_n + z, -z; A'_{n+1,r})$ is continued meromorphically, and possible singularities are located on hyperplanes of one of the following forms:

(i) $c_1 s_1 + \cdots + c_n s_n = u(c_1, \ldots, c_n) - \ell$,

(ii) $d_1 s_1 + \cdots + d_n s_n + d_0 z = u(d_1, \ldots, d_n, d_0) - \ell$

or

(iii) $e_1 s_1 + \cdots + e_n s_n - e_0 z = u(e_1, \ldots, e_n, e_0) - \ell$,

where c_i, d_i, e_i are non-negative integers ($1 \leq i \leq n$), d_0 and e_0 are positive integers, and $\ell \in \mathbb{N}_0$. Hence the poles of the integrand on the right-hand side of (2.5) with respect to z are

(I) $z = -d_1 d_0^{-1} s_1 - \cdots - d_n d_0^{-1} s_n + d_0^{-1} u(d_1, \ldots, d_n, d_0) - d_0^{-1} \ell$,

(II) $z = e_1 e_0^{-1} s_1 + \cdots + e_n e_0^{-1} s_n - e_0^{-1} u(e_1, \ldots, e_n, e_0) + e_0^{-1} \ell$,

(III) $z = -s_n - \ell$

and

(IV) $z = \ell$,

where $\ell \in \mathbb{N}_0$. Since $\sigma_i > r^*$ ($1 \leq i \leq n$), choosing r^* sufficiently large if necessary, we can assume that all poles of types I and III are on the left of the line $\Re z = c$, while all poles of types II and IV are on the right of $\Re z = c$. For brevity we write

$$-d_1 d_0^{-1} s_1 - \cdots - d_n d_0^{-1} s_n + d_0^{-1} u(d_1, \ldots, d_n, d_0) = D(s_1, \ldots, s_n),$$

$$e_1 e_0^{-1} s_1 + \cdots + e_n e_0^{-1} s_n - e_0^{-1} u(e_1, \ldots, e_n, e_0) = E(s_1, \ldots, s_n).$$

Let $(s_1^0, \ldots, s_n^0)$ be any point in the space $\mathbb{C}^n$. We show that the right-hand side of (2.4) can be continued meromorphically to $(s_1^0, \ldots, s_n^0)$.

First of all, remove the singularities of type (i) from the integral on the right-hand side of (2.4). This type of singularity is cancelled by the factor

$$(c_1 s_1 + \cdots + c_n s_n - u(c_1, \ldots, c_n) + \ell)^{v(c_1, \ldots, c_n)},$$

where $v(c_1, \ldots, c_n)$ is a positive integer. Let L be a sufficiently large positive integer for which

$$c_1 s_1 + \cdots + c_n s_n = u(c_1, \ldots, c_n) - L$$

does not hold for any $(c_1, \ldots, c_n)$ appearing in (i), if $\Re s_i \geq \Re s_i^0$ ($1 \leq i \leq n$). Define

$$\begin{aligned}&\Phi(s_1, \ldots, s_n)\\ &= \prod_{c_i} \prod_{\ell=0}^{L-1} (c_1 s_1 + \cdots + c_n s_n - u(c_1, \ldots, c_n) + \ell)^{v(c_1, \ldots, c_n)},\end{aligned}$$

where the first product runs over all $(c_1, \ldots, c_n)$ appearing in (i). Rewrite the right-hand side of (2.4) as

$$\Phi(s_1, \ldots, s_n)^{-1} J(s_1, \ldots, s_n),$$

where

$$J(s_1,\ldots,s_n)=\frac{1}{2\pi\sqrt{-1}}\int_{(c)}\frac{\Gamma(s_n+z)\Gamma(-z)}{\Gamma(s_n)}\Phi(s_1,\ldots,s_n)\\ \times\zeta_r(s_1,\ldots,s_{n-1},s_n+z,-z;A'_{n+1,r})dz.$$

Then the integrand of $J(s_1,\ldots,s_n)$ does not have singularities of type (i) in the region $\Re s_i\geq\Re s_i^0$ $(1\leq i\leq n)$.

Let $M>0$ be sufficiently large integer for which $\Re s_i^0+M>r^*$ $(1\leq i\leq n)$ holds. Put $s_i^*=s_i^0+M$, and consider the poles of types I, II, III and IV for $(s_1,\ldots,s_n)=(s_1^*,\ldots,s_n^*)\in\mathcal{B}^*$. Let $\mathcal{I}(\mathrm{I},\mathrm{III})$ be the set of all imaginary parts of the poles of type I and of type III, and similarly define $\mathcal{I}(\mathrm{II},\mathrm{IV})$.

First consider the case

$$\mathcal{I}(\mathrm{I},\mathrm{III})\cap\mathcal{I}(\mathrm{II},\mathrm{IV})=\emptyset, \tag{2.5}$$

and join $D(s_1^*,\ldots,s_n^*)$ and $D(s_1^0,\ldots,s_n^0)$ by the segment $S(D)$ which is parallel to the real axis. Similarly join $E(s_1^*,\ldots,s_n^*)$ and $E(s_1^0,\ldots,s_n^0)$ by the segment $S(E)$, and join $-s_n^*$ and $-s_n^0$ by the segment $S(n)$. By the assumption (2.5), we can modify the path $\Re z=c$ to obtain the new path $\mathcal{C}$, from $c-\sqrt{-1}\infty$ to $c+\sqrt{-1}\infty$, such that all segments of the form $S(D)$ and $S(n)$ are on the left of $\mathcal{C}$, and all segments of the form $S(E)$ are on the right of $\mathcal{C}$. We have

$$J(s_1,\ldots,s_n)=\frac{1}{2\pi\sqrt{-1}}\int_{\mathcal{C}}\frac{\Gamma(s_n+z)\Gamma(-z)}{\Gamma(s_n)}\Phi(s_1,\ldots,s_n)\\ \times\zeta_r(s_1,\ldots,s_{n-1},s_n+z,-z;A'_{n+1,r})dz \tag{2.6}$$

in a sufficiently small neighbourhood of $(s_1^*,\ldots,s_n^*)$. When we move $(s_1,\ldots,s_n)$ from $(s_1^*,\ldots,s_n^*)$ to $(s_1^0,\ldots,s_n^0)$ with keeping the values of imaginary parts of each s_i, the path $\mathcal{C}$ does not cross any poles of the integrand. Therefore the integral (2.6) can be continued holomorphically to $(s_1^0,\ldots,s_n^0)$. This implies the continuation of $\zeta_r(s_1,\ldots,s_n;A_{nr})$ to the point $(s_1^0,\ldots,s_n^0)$ where (2.5) holds.

Next we consider the case when (2.5) does not hold. We describe the method for this case by discussing a typical example, that is, there are $(d_1,\ldots,d_n,d_0)$ appearing in I and $(e_1,\ldots,e_n,e_0)$ appearing in II such that

$$\Im D(s_1^*,\ldots,s_n^*)=\Im E(s_1^*,\ldots,s_n^*). \tag{2.7}$$

The associated poles are $D(s_1^*,\ldots,s_n^*)-d_0^{-1}\ell_1$ and $E(s_1^*,\ldots,s_n^*)+e_0^{-1}\ell_2$ $(\ell_1,\ell_2\in\mathbb{N}_0)$. When $(s_1^*,\ldots,s_n^*)$ is moved to $(s_1^0,\ldots,s_n^0)$, these poles are moved to $D(s_1^0,\ldots,s_n^0)-d_0^{-1}\ell_1$ and $E(s_1^0,\ldots,s_n^0)+e_0^{-1}\ell_2$. If

$$\Re D(s_1^0,\ldots,s_n^0)-d_0^{-1}\ell_1\neq\Re E(s_1^0,\ldots,s_n^0)+e_0^{-1}\ell_2 \tag{2.8}$$

for any ℓ_1 and ℓ_2, then the above argument (in the case (2.5) holds) is still valid with a slight modification. In fact, let η be a small positive number, and consider the oriented polygonal path $S'(D)$ joining the points $D(s_1^*,\ldots,s_n^*)$, $D(s_1^*+\sqrt{-1}\eta,\ldots,s_n^*+\sqrt{-1}\eta)$, $D(s_1^0+\sqrt{-1}\eta,\ldots,s_n^0+\sqrt{-1}\eta)$, and then $D(s_1^0,\ldots,s_n^0)$

in that order. Similarly define the path $S'(E)$ which joins $E(s_1^*,\ldots,s_n^*)$, $E(s_1^*+\sqrt{-1}\eta,\ldots,s_n^*+\sqrt{-1}\eta)$, $E(s_1^0+\sqrt{-1}\eta,\ldots,s_n^0+\sqrt{-1}\eta)$, and then $E(s_1^0,\ldots,s_n^0)$. Then $S'(D)$ lies on the lower side of the line

$$\mathcal{L}=\{z \mid \Im z=\Im D(s_1^*,\ldots,s_n^*)=\Im E(s_1^*,\ldots,s_n^*)\},$$

while $S'(E)$ lies on the upper side of $\mathcal{L}$. Because of (2.8), we can define the path $\mathcal{C}'$, which is almost the same as $\mathcal{C}$, but near the line $\mathcal{L}$ we draw $\mathcal{C}'$ such that it separates

$$\bigcup_{\ell_1\in\mathbb{N}_0}(S'(D)-d_0^{-1}\ell_1) \qquad \text{and} \qquad \bigcup_{\ell_2\in\mathbb{N}_0}(S'(E)+e_0^{-1}\ell_2).$$

Then the expression (2.6), with replacing $\mathcal{C}$ by $\mathcal{C}'$, is valid in a sufficiently small neighbourhood of $(s_1^*,\ldots,s_n^*)$. When $(s_1,\ldots,s_n)$ moves along the polygonal path joining $(s_1^*,\ldots,s_n^*)$, $(s_1^*+\sqrt{-1}\eta,\ldots,s_n^*+\sqrt{-1}\eta)$, $(s_1^0+\sqrt{-1}\eta,\ldots,s_n^0+\sqrt{-1}\eta)$, and then $(s_1^0,\ldots,s_n^0)$ in that order, the path $\mathcal{C}'$ encounters no pole, hence we obtain the analytic continuation.

The remaining case is that

$$D(s_1^0,\ldots,s_n^0)-d_0^{-1}\ell_1=E(s_1^0,\ldots,s_n^0)+e_0^{-1}\ell_2 \tag{2.9}$$

holds for some ℓ_1 and ℓ_2. Then this might hold for some other pairs of (ℓ_1,ℓ_2). In this case we consider the path $\mathcal{C}''$ which is almost the same as $\mathcal{C}$, but near the line $\mathcal{L}$ we only require that $S(D)$ is on the left of $\mathcal{C}''$. When we deform the path $\Re z=c$ on the right-hand side of (2.4) to $\mathcal{C}''$, we might encounter several poles of type II. Then we move $(s_1,\ldots,s_n)$ from $(s_1^*,\ldots,s_n^*)$ to $(s_1^0,\ldots,s_n^0)$; again the path might encounter several poles of the same type. Hence, in a sufficiently small neighbourhood U of $(s_1^0,\ldots,s_n^0)$, the integral $J(s_1,\ldots,s_n)$ has the expression

$$\begin{aligned} R(s_1,\ldots,s_n)+\frac{1}{2\pi\sqrt{-1}}\int_{\mathcal{C}''}&\frac{\Gamma(s_n+z)\Gamma(-z)}{\Gamma(s_n)}\Phi(s_1,\ldots,s_n)\\ &\times\zeta_r(s_1,\ldots,s_{n-1},s_n+z,-z;A'_{n+1,r})dz, \end{aligned} \tag{2.10}$$

where $R(s_1,\ldots,s_n)$ is the sum of residues of the above poles. The expression (2.10) gives the analytic continuation to U.

From the condition (2.9) we find that possible polar sets of $R(s_1,\ldots,s_n)$ are of the form

$$c_1s_1+\cdots+c_ns_n=u(c_1,\ldots,c_n)-\ell,$$

where $c_1,\ldots,c_n\in\mathbb{N}_0$, $u(c_1,\ldots,c_n)$ is an integer and $\ell\in\mathbb{N}_0$. The polar sets of $\Phi(s_1,\ldots,s_n)^{-1}$ are clearly of the same form. Hence the proof of Theorem 3 is now complete.

3. Proofs of Theorems 1 and 2

The analytic continuation of the Mordell-Tornheim and the Apostol-Vu multiple zeta-functions is now established by Theorem 3. In this section, however, we prove Theorems 1 and 2 by a little different argument, which gives more explicit information.

First we prove, by induction on r, the assertions of Theorem 1 and the fact that $\zeta_{MT,r}(s_1,\ldots,s_r;s_{r+1})$ is of polynomial order with respect to $|\Im s_{r+1}|$, uniformly in any vertical strip $-\infty<\sigma_1\leq\Re s_{r+1}\leq\sigma_2<\infty$.

When $r=1$, $\zeta_{MT,1}(s_1;s_2)$ is nothing but the Euler sum $\zeta_2(s_1,s_2)$, and the desired assertions were proved in [7] or [8]. The case $r=2$ was also established in [8].

Now let $r\geq 3$, and assume $\sigma_j>1$ $(1\leq j\leq r)$, $\sigma_{r+1}>0$. Rewrite the right-hand side of (1.4) as

$$\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}m_1^{-s_1}\cdots m_r^{-s_r}(m_1+\cdots+m_{r-1})^{-s_{r+1}}\left(1+\frac{m_r}{m_1+\cdots+m_{r-1}}\right)^{-s_{r+1}},$$

and apply (2.4) to obtain

$$\begin{aligned}&\zeta_{MT,r}(s_1,\ldots,s_r;s_{r+1})\\&=\frac{1}{2\pi\sqrt{-1}}\int_{(c)}\frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})}\\&\qquad\times\zeta_{MT,r-1}(s_1,\ldots,s_{r-1};s_{r+1}+z)\zeta(s_r-z)dz,\end{aligned}\tag{3.1}$$

where $-\sigma_{r+1}<c<0$. By the induction assumption we find that the poles of $\zeta_{MT,r-1}(s_1,\ldots,s_{r-1};s_{r+1}+z)$ as a function in z are

$$\begin{aligned}&z=-s_j-s_{r+1}+1-\ell\quad(1\leq j\leq r-1,\ell\in\mathbb{N}_0),\\&z=-s_{j_1}-s_{j_2}-s_{r+1}+2-\ell\quad(1\leq j_1<j_2\leq r-1,\ell\in\mathbb{N}_0),\\&\qquad\cdots\cdots\\&z=-s_{j_1}-\cdots-s_{j_{r-2}}-s_{r+1}+r-2-\ell\\&\qquad\qquad(1\leq j_1<\cdots<j_{r-2}\leq r-1,\ell\in\mathbb{N}_0),\\&z=-s_1-\cdots-s_{r-1}-s_{r+1}+r-1,\end{aligned}$$

all of which are located to the left of $\Re z=c$. The other poles of the integrand on the right-hand side of (2.2) are $z=-s_{r+1}-\ell$ $(\ell\in\mathbb{N}_0)$, $z=\ell$ $(\ell\in\mathbb{N}_0)$, and $z=s_r-1$. When we shift the path of integration to the right to $\Re z=K-\varepsilon$, where K is a positive integer, the relevant poles are $z=\ell$ $(0\leq\ell\leq K-1)$ and $z=s_r-1$. If we assume that s_r is not a positive integer, then all of these poles are simple. By the induction assumption on the order of $\zeta_{MT,r-1}$ and Stirling's formula we see that the integrand is of exponential decay with respect to $|\Im z|$, hence this shifting is possible. we obtain

$$
\begin{aligned}
&\zeta_{MT,r}(s_1,\ldots,s_r;s_{r+1})\\
&=\frac{\Gamma(s_r+s_{r+1}-1)\Gamma(1-s_r)}{\Gamma(s_{r+1})}\zeta_{MT,r-1}(s_1,\ldots,s_{r-1};s_r+s_{r+1}-1)\\
&+\sum_{k=0}^{K-1}\binom{-s_{r+1}}{k}\zeta_{MT,r-1}(s_1,\ldots,s_{r-1};s_{r+1}+k)\zeta(s_r-k)\\
&+\frac{1}{2\pi\sqrt{-1}}\int_{(K-\varepsilon)}\frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})}\\
&\qquad\times\zeta_{MT,r-1}(s_1,\ldots,s_{r-1};s_{r+1}+z)\zeta(s_r-z)dz.
\end{aligned}
\tag{3.2}
$$

The poles of the integrand of the last integral are listed above, hence we see that this integral is holomorphic at any points satisfying all of the following inequalities:

$$
\begin{aligned}
&\sigma_{r+1}>-K+\varepsilon,\\
&\sigma_j+\sigma_{r+1}>1-K+\varepsilon \quad (1\leq j\leq r-1),\\
&\sigma_{j_1}+\sigma_{j_2}+\sigma_{r+1}>2-K+\varepsilon \quad (1\leq j_1<j_2\leq r-1),\\
&\qquad\qquad\cdots\cdots\\
&\sigma_{j_1}+\cdots+\sigma_{j_{r-2}}+\sigma_{r+1}>r-2-K+\varepsilon \quad (1\leq j_1<\cdots<j_{r-2}\leq r-1),\\
&\sigma_1+\cdots+\sigma_{r-1}+\sigma_{r+1}>r-1-K+\varepsilon,\\
&\sigma_r<1+K-\varepsilon.
\end{aligned}
$$

Since K can be taken arbitrarily large, (3.2) implies the meromorphic continuation of $\zeta_{MT,r}(s_1,\ldots,s_r;s_{r+1})$ to the whole $\mathbb{C}^{r+1}$ space, including the case when $s_r=m$, a positive integer. When $s_r=m$, the first and the second terms on the right-hand side of (3.2) are singular (when $K\geq m$), but these singularities cancel each other. Indeed we can easily show

$$\begin{aligned}
&\zeta_{MT,r}(s_1,\ldots,s_{r-1},m;s_{r+1})\\
&=\binom{-s_{r+1}}{m-1}\Bigg\{\zeta'_{MT,r-1}(s_1,\ldots,s_{r-1};s_{r+1}+m-1)\\
&\quad+\left(\frac{\Gamma'}{\Gamma}(s_{r+1}+m-1)-\left(1+\frac{1}{2}+\cdots+\frac{1}{m-1}\right)\right)\\
&\qquad\times\zeta_{MT,r-1}(s_1,\ldots,s_{r-1};s_{r+1}+m-1)\Bigg\}\\
&+\sum_{\substack{0\le k\le K-1\\ k\ne m-1}}\binom{-s_{r+1}}{k}\zeta_{MT,r-1}(s_1,\ldots,s_{r-1};s_{r+1}+k)\zeta(s_r-k)\\
&+\frac{1}{2\pi\sqrt{-1}}\int_{(K-\varepsilon)}\frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})}\\
&\qquad\times\zeta_{MT,r-1}(s_1,\ldots,s_{r-1};s_{r+1}+z)\zeta(s_r-z)dz, \qquad (3.3)
\end{aligned}$$

where $\zeta'_{MT,r-1}$ means the derivative with respect to the last variable.

From (3.2) and (3.3) we can see that the location of possible singularities are as stated in the statement of the theorem. Also from (3.2) and (3.3) we can prove the assertion on the order of $\zeta_{MT,r}$. As for the last integral on the right-hand sides of (3.2) and (3.3), split it at $z=0$ and $z=-\Im s_{r+1}$, and estimate each part separately. The factor $\zeta'_{MT,r-1}$ in (3.3) can be estimated by using Cauchy's integral formula. Then we find that $\zeta_{MT,r}(s_1,\ldots,s_r;s_{r+1})$ is of polynomial order with respect to $|\Im s_{r+1}|$, uniformly in any vertical strip. Hence the proof of Theorem 1 is complete.

Now we proceed to the proof of Theorem 2. We introduce the auxiliary multiple series

$$\begin{aligned}
&\varphi_{j,r}(s_1,\ldots,s_j;s_{j+1},\ldots,s_r;s_{r+1})\\
&=\sum_{1\le m_1<\cdots<m_r<\infty}\cdots\sum m_1^{-s_1}\cdots m_r^{-s_r}(m_1+\cdots+m_j)^{-s_{r+1}} \qquad (3.4)
\end{aligned}$$

($1\le j\le r$), which is convergent absolutely when $\sigma_j>1$ ($1\le j\le r$) and $\sigma_{r+1}>0$. It is clear that $\varphi_{r,r}=\zeta_{AV,r}$, and

$$\varphi_{1,r}(s_1;s_2,\ldots,s_r;s_{r+1})=\zeta_r(s_1+s_{r+1},s_2,\ldots,s_r), \qquad (3.5)$$

where the right-hand side is the r-ple sum of the Euler-Zagier type. We prove

Theorem 4. *For $1\le j\le r$, we have*

(i) the function $\varphi_{j,r}(s_1,\ldots,s_j;s_{j+1},\ldots,s_r;s_{r+1})$ can be continued meromorphically to the whole $\mathbb{C}^{r+1}$ space,

(ii) the possible singularities of $\varphi_{j,r}$ are located only on the subsets of $\mathbb{C}^{r+1}$ defined by one of the following equations:

$$\begin{aligned}
&s_r = 1,\\
&s_i + \cdots + s_r = r + 1 - i - \ell \quad (j+1 \le i \le r-1,\ \ell \in \mathbb{N}_0),\\
&s_i + \cdots + s_r + s_{r+1} = r + 1 - i - \ell \quad (1 \le i \le j,\ \ell \in \mathbb{N}_0).
\end{aligned}$$

(iii) each of these singularities can be cancelled by the corresponding linear factor, and

(iv) $\varphi_{j,r}$ is of polynomial order with respect to $|\Im s_i|$ $(1 \le i \le r+1)$.

When $j = 1$, that is the case of the Euler-Zagier r-ple sum (3.2), the assertion (i) was proved by various methods, as was mentioned in Section 1. In [9], (i) and (ii) were proved in a more generalized form, and (iii) and (iv) can be easily shown from (4.4) of [9].

Now let $j \ge 2$, assume that Proposition 1 is true for $j-1$. Also assume $\sigma_j > 1$ $(1 \le j \le r)$ and $\sigma_{r+1} > 0$. By using (2.4) with $s = s_{r+1}$, $\lambda = m_j/(m_1+\cdots+m_{j-1})$ we have

$$\begin{aligned}
&\varphi_{j,r}(s_1,\ldots,s_j;s_{j+1},\ldots,s_r;s_{r+1})\\
&= +\frac{1}{2\pi\sqrt{-1}}\int_{(c)}\frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})}\\
&\qquad \times \varphi_{j-1,r}(s_1,\ldots,s_{j-1};s_j - z,s_{j+1},\ldots,s_r;s_{r+1}+z)dz,
\end{aligned}\tag{3.6}$$

where $-\sigma_{r+1} < c < 0$. The singularities of $\varphi_{j-1,r}$ in the integrand are, by the induction assumption, only on

$$\begin{aligned}
&s_r = 1,\\
&s_i + \cdots + s_r = r + 1 - i - \ell \quad (j+1 \le i \le r-1,\ \ell \in \mathbb{N}_0),\\
&s_i + \cdots + s_r + s_{r+1} = r + 1 - i - \ell \quad (1 \le i \le j-1,\ \ell \in \mathbb{N}_0),
\end{aligned}$$

or

$$s_j + \cdots + s_r - z = r + 1 - j - \ell \quad (\ell \in \mathbb{N}_0). \tag{3.7}$$

The poles of other factors of the integrand with respect to z are $z = -s_{r+1} - \ell$ $(\ell \in \mathbb{N}_0)$ and $z = \ell$ $(\ell \in \mathbb{N}_0)$. Therefore, when we shift the path of integration, this time to the left, to $\Re z = -\sigma_{r+1} - K + \varepsilon$ (which is possible because of the induction assumption (iv)), the relevant poles are $z = -s_{r+1} - \ell$ $(0 \le \ell \le K-1)$, and we obtain

$$
\begin{aligned}
&\varphi_{j,r}(s_1,\ldots,s_j;s_{j+1},\ldots,s_r;s_{r+1})\\
&=\sum_{k=0}^{K-1}\binom{-s_{r+1}}{k}\varphi_{j-1,r}(s_1,\ldots,s_{j-1};s_j+s_{r+1}+k,s_{j+1},\ldots,s_r;-k)\\
&\quad+\frac{1}{2\pi\sqrt{-1}}\int_{(-\sigma_{r+1}-K+\varepsilon)}\frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})}\\
&\qquad\times\varphi_{j-1,r}(s_1,\ldots,s_{j-1};s_j-z,s_{j+1},\ldots,s_r;s_{r+1}+z)dz. \qquad (3.8)
\end{aligned}
$$

Let

$$
\begin{aligned}
\Phi_L(s_1,\ldots,s_{r+1})=&(s_r-1)\prod_{\ell\le L}\left\{\prod_{i=j+1}^{r-1}(s_i+\ldots+s_r-r-1+i+\ell)\right.\\
&\left.\times\prod_{i=1}^{j-1}(s_i+\ldots+s_r+s_{r+1}-r-1+i+\ell)\right\},
\end{aligned}
$$

where L is a positive integer, and rewrite the integral on the right-hand side of (3.8) as

$$
\begin{aligned}
&\Phi_L(s_1,\ldots,s_{r+1})^{-1}\int_{(-\sigma_{r+1}-K+\varepsilon)}\frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})}\\
&\times\Phi_L(s_1,\ldots,s_{r+1})\varphi_{j-1,r}(s_1,\ldots,s_{j-1};s_j-z,s_{j+1},\ldots,s_r;s_{r+1}+z)dz. \qquad (3.9)
\end{aligned}
$$

By the induction assumption (iii), $\Phi_L(s_1,\ldots,s_{r+1})$ cancels the singularities of $\varphi_{j-1,r}$ for $\ell\le L$, except for those of the form (3.7). Hence the integral on (3.9) is holomorphic at any points satisfying all of the following inequalities:

$$
\begin{aligned}
&\sigma_{r+1}>-K+\varepsilon,\\
&\sigma_i+\cdots+\sigma_r>r-i-L \quad (j+1\le i\le r-1),\\
&\sigma_j+\cdots+\sigma_r+\sigma_{r+1}>-K+r+1-j+\varepsilon,\\
&\sigma_i+\cdots+\sigma_r+\sigma_{r+1}>r-i-L \quad (1\le i\le j-1).
\end{aligned}
$$

Since K and L can be arbitrarily large, this with (3.8) implies the meromorphic continuation of $\varphi_{j,r}(s_1,\ldots,s_j;s_{j+1},\ldots,s_r;s_{r+1})$ to the whole $\mathbb{C}^{r+1}$ space. The other assertions of Theorem 4 can also be shown from (3.8).

The case $j=r$ of Theorem 4 implies Theorem 2, because in this case the sum on the right-hand side of (3.8) is

$$
\sum_{k=0}^{K-1}\binom{-s_{r+1}}{k}\varphi_{r-1,r}(s_1,\ldots,s_{r-1};s_r+s_{r+1}+k;-k),
$$

hence the singularity $s_r=1$ does not appear.

4. The family of multiple zeta-functions

First we discuss the continuation of Witten multiple zeta-functions. Let $\mathfrak{g}$ be a semi-simple Lie algebra, and define

$$\zeta_{\mathfrak{g}}(s)=\sum_{\rho}(\dim\rho)^{-s}, \tag{4.1}$$

where ρ runs over all finite dimensional representations of $\mathfrak{g}$. This type of multiple series was introduced by Witten [18] in order to calculate the volumes of certain moduli spaces. In Zagier [19], explicit forms of (4.1) for some simple examples are given; $\zeta_{\mathfrak{sl}(2)}(s)=\zeta(s)$, $\zeta_{\mathfrak{sl}(3)}(s)=2^s\zeta_{MT,2}(s,s;s)$, and

$$\zeta_{\mathfrak{so}(5)}(s)=6^s\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}m^{-s}n^{-s}(m+n)^{-s}(m+2n)^{-s}. \tag{4.2}$$

From Theorem 1 (or [8]) we know that $\zeta_{\mathfrak{sl}(3)}(s)$ is meromorphic in the whole complex plane. Similarly we can show the meromorphic continuation of $\zeta_{\mathfrak{so}(5)}(s)$. In fact, Theorem 3 implies the meromorphic continuation of

$$\zeta_{\mathfrak{so}(5)}(s_1,s_2,s_3,s_4)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}m^{-s_1}n^{-s_2}(m+n)^{-s_3}(m+2n)^{-s_4} \tag{4.3}$$

to the whole $\mathbb{C}^4$ space. Or, similarly to the argument in Section 3, we can show

$$\begin{aligned}
&\zeta_{\mathfrak{so}(5)}(s_1,s_2,s_3,s_4)\\
&=\frac{1}{2\pi\sqrt{-1}}\int_{(c)}\frac{\Gamma(s_4+z)\Gamma(-z)}{\Gamma(s_4)}\sum_m\sum_n m^{-s_1}n^{-s_2}(m+n)^{-s_3}\\
&\qquad\times(m+n)^{-s_4}\left(\frac{n}{m+n}\right)^z dz\\
&=\frac{1}{2\pi\sqrt{-1}}\int_{(c)}\frac{\Gamma(s_4+z)\Gamma(-z)}{\Gamma(s_4)}\zeta_{MT,2}(s_1,s_2-z;s_3+s_4+z)dz,
\end{aligned} \tag{4.4}$$

where $-\Re s_4<c<0$, and can prove the meromorphic continuation by shifting the path to $\Re z=K-\varepsilon$.

On the other hand, the author [7] [8] introduced the generalized r-ple zeta-function

$$\begin{aligned}
&\tilde{\zeta}_r(s_1,\ldots,s_r;\alpha_1,\ldots,\alpha_r;w_1,\ldots,w_r)\\
&=\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}(\alpha_1+m_1w_1)^{-s_1}(\alpha_2+m_1w_1+m_2w_2)^{-s_2}\\
&\qquad\times\cdots\times(\alpha_r+m_1w_1+\cdots+m_rw_r)^{-s_r},
\end{aligned} \tag{4.5}$$

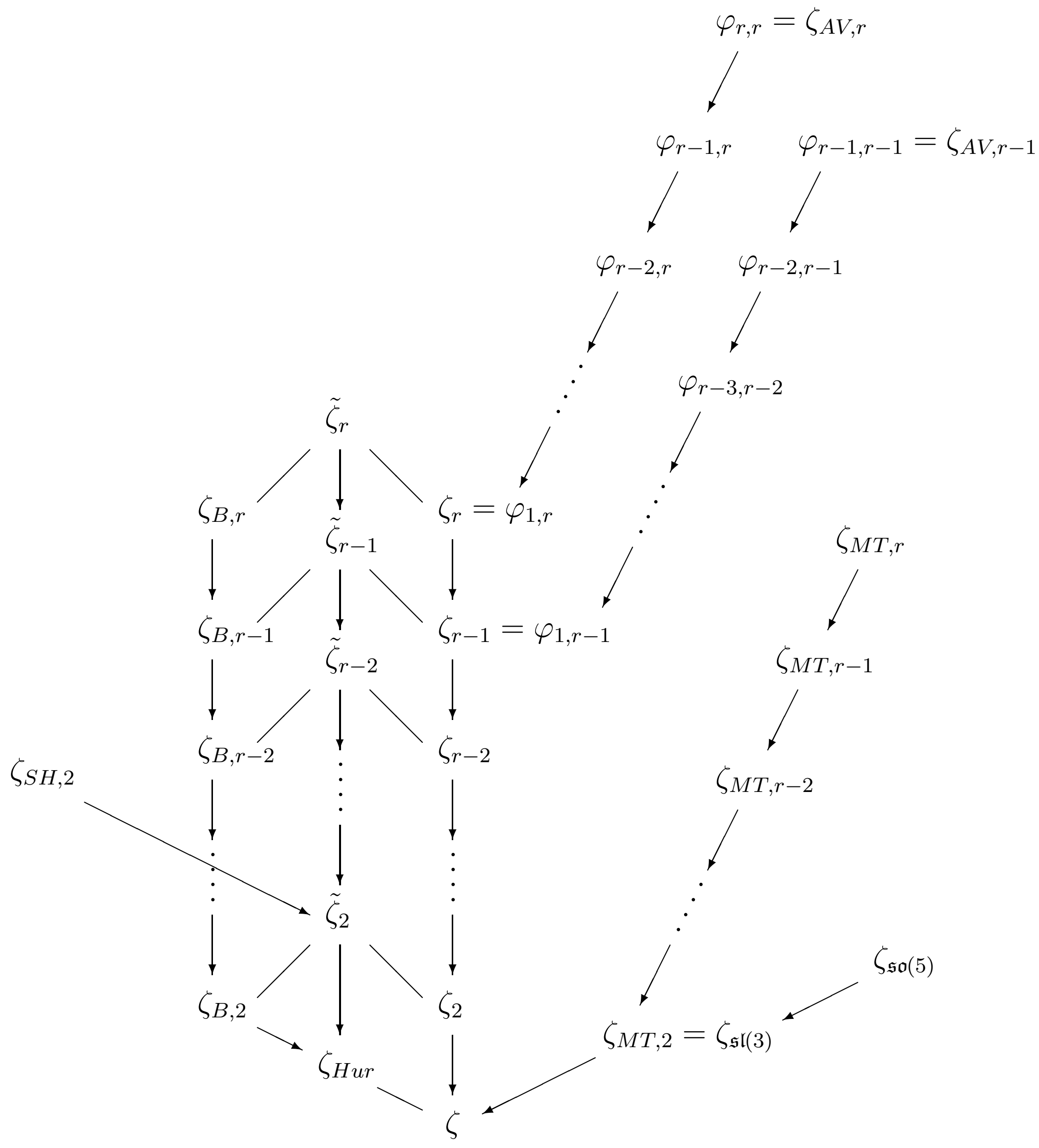
where $\alpha_1,\ldots,\alpha_r$, $w_1,\ldots,w_r$ are complex parameters. This function has been studied in detail in [7] [8] [9] [10]. It is to be stressed that (4.5) includes both the

Euler-Zagier r-ple sum and the Barnes r-ple zeta-function (Barnes [4]) as special cases. The latter is defined by

$$\zeta_{B,r}(s)=\tilde{\zeta}_r(0,\ldots,0,s;1,\ldots,1,\alpha_r;w_1,\ldots,w_r),$$

and so $\zeta_{B,1}$ is essentially equal to the Hurwitz zeta-function ζ_{Hur}. Moreover, at least in the case $r=2$, we can show that the Shintani double zeta-function $\zeta_{SH,2}(s_1,s_2)$ can be written as an integral of Mellin-Barnes type including the generalized double zeta $\tilde{\zeta}_2$ as a factor of the integrand (see Section 8 of [7]).

Therefore, based on the Mellin-Barnes induction argument, we now have a unified view of the family of various multiple zeta-functions mentioned above, introduced historically under various motivations different from each other. The whole situation may be illustrated in the following figure. In this figure, $\zeta_X \longrightarrow \zeta_Y$ means that ζ_X can be expressed as a Mellin-Barnes integral involving ζ_Y, and ζ_X —— ζ_Y means that ζ_X is a generalization of ζ_Y.



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Kohji MATSUMOTO
Graduate School of Mathematics, Nagoya University,
Chikusa-ku, Nagoya 464-8602, Japan