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AN EXPLICIT FORMULA OF ATKINSON TYPE FOR THE PRODUCT OF $\zeta(s)$ AND A DIRICHLET POLYNOMIAL. II

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Abstract. In our former paper, we proved a formula of Atkinson type for the mean square $I(T,A)$ of the product of the Riemann zeta-function and a Dirichlet polynomial $A(s)$. Using that formula, in the present paper, we prove an Ω-result on the difference between $I(T,A)$ and $I(T,\overline{A})$, where $\overline{A}(s)$ is the Dirichlet polynomial whose coeffecients are complex conjugates of those of $A(s)$.

Key words and phrases: Atkinson formula, Dirichlet polynomial, mean square, Riemann zeta-function, Ω-result.

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1. Introduction and statement of results

Let $s = \sigma + it$ be a complex variable, $\zeta(s)$ the Riemann zeta-function, and

$$A(s) = \sum_{m \leqslant M} a(m) m^{-s}$$

be a Dirichlet polynomial, where $M \geqslant 1$ and $a(m) \in \mathbb{C}$ with $a(m) = \mathrm{O}(m^{\varepsilon})$. Here, and in what follows, ε denotes an arbitrarily small positive number, not necessarily the same at each occurrence.

The mean value

$$I(T,A)=\int_0^T\left|\zeta\left(\frac{1}{2}+it\right)A\left(\frac{1}{2}+it\right)\right|^2\mathrm{d}t,\quad T\geqslant 2,$$

has been studied by many mathematicians, motivated by the theory of power moments or the distribution of zeros of $\zeta(s)$. For the history, see the introduction of [6]. It is known that $I(T,A)$ can be written as

$$I(T,A)=\mathcal{M}(T,A)+E(T,A), \tag{1.1}$$

where $\mathcal{M}(T,A)$ is the main term and $E(T,A)$ is the error term. The definition of $\mathcal{M}(T,A)$ is given by

$$\mathcal{M}(T,A)=\sum_{k\leqslant M}\sum_{l\leqslant M}\frac{a(k)\overline{a(l)}}{[k,l]}\left(\log\frac{(k,l)^2T}{2\pi kl}+2\gamma-1\right)T,$$

where (k,l) is the greatest common divisor of k and l, $[k,l]=\frac{kl}{(k,l)}$ is the least common multiple of k and l, $\overline{a(l)}$ is the complex conjugate of $a(l)$, γ is Euler's constant. Formula (1.1) is a generalization of the asymptotic formula for the mean square of the Riemann zeta-function

$$\int_0^T\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\mathrm{d}t=T\log T+(2\gamma-1-\log 2\pi)T+E(T), \tag{1.2}$$

where $E(T)$ denotes the error term.

In our former paper [6], we proved an explicit formula of Atkinson type for $E(T,A)$. Atkinson's original formula [1] is an explicit formula for $E(T)$, and it is known that a lot of detailed information on the behaviour of $E(T)$ can be deduced from Atkinson's formula. Therefore, it is natural to expect that various information on the behaviour of $E(T,A)$ can be obtained by using our explicit formula for $E(T,A)$ in [6].

Let $L(s,\chi)$ be the Dirichlet L-function attached to a Dirichlet character χ. In [4], the first author proved a generalization of Atkinson's formula for $L(s,\chi)$. As an application, he proved that if χ_1 and χ_2 are distinct primitive characters of the same modulus, then

$$\int_0^T\left|L\left(\frac{1}{2}+it,\chi_1\right)\right|^2\mathrm{d}t-\int_0^T\left|L\left(\frac{1}{2}+it,\chi_2\right)\right|^2\mathrm{d}t=\Omega\big(T^{1/4}\big). \tag{1.3}$$

In particular, when χ is not a real character, then $\overline{\chi}\neq\chi$, and so

$$\int_0^T\left|L\left(\frac{1}{2}+it,\chi\right)\right|^2\mathrm{d}t-\int_{-T}^0\left|L\left(\frac{1}{2}+it,\chi\right)\right|^2\mathrm{d}t=\Omega\big(T^{1/4}\big). \tag{1.4}$$

Moreover, in [5], the first author studied the sign changes of the left-hand side of (1.3), (1.4) in detail, especially he showed that the right-hand sides of (1.3) and (1.4) can be replaced by $\Omega_\pm(T^{1/4})$. The method in [5] is inspired by the work of Heath-Brown and Tsang [3].

It is the purpose of the present paper to consider the same problem for $I(T,A)$, that is, to study the sign changes of

$$\begin{aligned}\Lambda(T,A) &= \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)A\left(\frac{1}{2}+it\right)\right|^2 \mathrm{d}t \\ &\quad - \int_{-T}^0 \left|\zeta\left(\frac{1}{2}+it\right)A\left(\frac{1}{2}+it\right)\right|^2 \mathrm{d}t. \qquad (1.5)\end{aligned}$$

Since the second integral on the right-hand side is equal to

$$\int_T^0 \left|\zeta\left(\frac{1}{2}-it\right)\sum_{m\leqslant M}\frac{a(m)}{m^{1/2-it}}\right|^2 \mathrm{d}(-t) = \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\sum_{m\leqslant M}\frac{\overline{a(m)}}{m^{1/2+it}}\right|^2 \mathrm{d}t,$$

we see that (1.5) can be rewritten as

$$\Lambda(T,A) = I(T,A) - I(T,\overline{A}), \qquad (1.6)$$

where the Dirichlet polynomial $\overline{A}(s)$ is defined by

$$\overline{A}(s) = \sum_{m\leqslant M} \overline{a(m)} m^{-s}.$$

Since $\mathcal{M}(T,\overline{A}) = \mathcal{M}(T,A)$, combining (1.1) and (1.6) we find that

$$\Lambda(T,A) = E(T,A) - E(T,\overline{A}). \qquad (1.7)$$

From this expression of $\Lambda(T,A)$, it is clear why the explicit formula for $E(T,A)$ is useful to our problem.

In order to state our main result, we first prepare notation. Let

$$\alpha(k,l) = a(k)\overline{a(l)} - \overline{a(k)}a(l) = 2i\mathrm{Im}\,(a(k)\overline{a(l)})$$

for $k,l \leqslant M$, $\kappa = k/(k,l)$, $\lambda = l/(k,l)$, and let

$$\delta_0 = \max_{\substack{k,l\leqslant M \\ \alpha(k,l)\neq 0}} \kappa\lambda. \qquad (1.8)$$

(If $\alpha(k,l)=0$ for all k and l, then $\Lambda(T,A)=0$, so our present problem has no meaning. Therefore, hereafter we assume $M \geqslant 2$ and that some $\alpha(k,l)\neq 0$, and hence, δ_0 exists.)

Define

$$C_{\cos}(M) = M^{-3/2} \sum_{\substack{k,l \leqslant M \\ \kappa\lambda = \delta_0}} \frac{|\alpha(k,l)|}{[k,l]} (\kappa\lambda)^{3/4} \cos\left(2\pi \frac{\kappa\overline{\kappa}}{\delta_0} + \arg\alpha(k,l)\right)$$

and

$$C_{\sin}(M) = M^{-3/2} \sum_{\substack{k,l \leqslant M \\ \kappa\lambda = \delta_0}} \frac{|\alpha(k,l)|}{[k,l]} (\kappa\lambda)^{3/4} \sin\left(2\pi \frac{\kappa\overline{\kappa}}{\delta_0} + \arg\alpha(k,l)\right).$$

Then our main theorem can be stated as follows.
THEOREM 1. *Let $M \geqslant 2$. Assume δ_0 exists and*

$$\sqrt{C_{\cos}(M)^2 + C_{\sin}(M)^2} \neq 0. \tag{1.9}$$

Then there exist positive constants T_0, c_1 and c_2 satisfying the property that, for any $T > T_0$, there exist t_1 and t_2 in the interval $[T, T + c_2\sqrt{T}]$ for which

$$\Lambda(t_1, A) > c_1 t_1^{1/4}, \quad \Lambda(t_2, A) < -c_1 t_2^{1/4}$$

hold. In particular, $\Lambda(T, A) = \Omega_{\pm}(T^{1/4})$.

REMARK 1. The constants T_0, c_1 and c_2 depend on M and $\{a(m)\}$, and can be computed effectively. This situation is different from the case of Dirichlet L-functions, where the corresponding constants cannot be effectively computed (see Remark 1 on p. 2 of [5]).

REMARK 2. In the case of $E(T)$, the omega result $E(T) = \Omega(T^{1/4})$ was proved by Good [2], and it is widely believed that $E(T)$ would be $\mathrm{O}(T^{1/4+\varepsilon})$. Therefore, it is also plausible to conjecture that $E(T, A) = \mathrm{O}(T^{1/4+\varepsilon})$ (with respect to T). Then $\Lambda(T, A) = \mathrm{O}(T^{1/4+\varepsilon})$ would follow by (1.7). This implies that the Omega-result given in the above theorem would be best-possible (up to ε-factor).

We will prove Theorem 1 in Sections 2 and 3. The structure of the proof is the same as in [5], so we only show a brief outline. Then in Section 4, we will discuss condition (1.9). This condition seems natural, and we may expect that this condition is valid in many practical cases. In Section 4, we will give a sufficient condition for (1.9), and will also give some examples which satisfy, or do not satisfy, condition (1.9).

2. The main lemma

Our basic tool is the explicit formula of Atkinson type for $E(T,A)$ proved in [6], so we first state it. Let T, Y be positive numbers satisfying $C_1T < Y < C_2T$ and $T \geqslant C^* = \max\{e, C_1^{-1}\}$ (where C_1, C_2 are fixed constants with $0 < C_1 < C_2$ and $e = 2.71828...$),

$$\begin{aligned}
\mathrm{arcsinh} x &= \log(x+\sqrt{x^2+1}), \quad \xi(T,u) = \frac{T}{2\pi} + \frac{u}{2} - \sqrt{\frac{u^2}{4} + \frac{uT}{2\pi}},\\
f(T,u) &= 2T \mathrm{arcsinh}\sqrt{\frac{\pi u}{2T}} + \sqrt{2\pi uT + \pi^2u^2} - \frac{\pi}{4},\\
g(T,u) &= T\log\frac{T}{2\pi u} - T + 2\pi u + \frac{\pi}{4},
\end{aligned}$$

and define

$$\begin{aligned}
\Sigma_1(T,Y) = &\sum_{k,l\leqslant M}\sum_{n\leqslant \kappa\lambda Y} \mathrm{Im}\left\{ \frac{a(k)\overline{a(l)}}{[k,l]}(\kappa\lambda)^{1/2}\frac{d(n)}{n^{1/2}}\mathrm{e}^{2\pi i n\overline{\kappa}/\lambda}\right.\\
&\times\left(\mathrm{arcsinh}\sqrt{\frac{\pi n}{2T\kappa\lambda}}\right)^{-1}\left(1+\frac{2T\kappa\lambda}{\pi n}\right)^{-1/4}\\
&\left.\times\exp\left(i\left(f\left(T,\frac{n}{\kappa\lambda}\right) - \frac{\pi n}{\kappa\lambda} + \frac{\pi}{2}\right)\right)\right\}
\end{aligned}$$

and

$$\begin{aligned}
\Sigma_2(T,Y) = &\sum_{k,l\leqslant M}\sum_{n\leqslant (\lambda/\kappa)Y} \mathrm{Re}\left\{ \frac{a(k)\overline{a(l)}}{[k,l]}(\kappa\lambda)^{1/2}\frac{d(n)}{n^{1/2}}\mathrm{e}^{-2\pi i n\kappa/\lambda}\right.\\
&\left.\times\left(\log\frac{T\lambda}{2\pi n\kappa}\right)^{-1}\exp\left(ig\left(T,\frac{\kappa n}{\lambda}\right)\right)\right\},
\end{aligned}$$

where $d(n)$ denotes the number of positive divisors of n, and $\overline{\kappa}$ is defined by $\kappa\overline{\kappa} \equiv 1 \pmod{\lambda}$. Then Theorem 1.2 of [6] asserts that

$$E(T,A) = \Sigma_1(T,Y) + \Sigma_2(T,\xi(T,Y)) + R(T,A), \tag{2.1}$$

where $R(T,A)$ is the error term, satisfying the estimate

$$\begin{aligned}
R(T,A) \ll\, &M^{1+\varepsilon}(\log T)^3 + M^{2+\varepsilon}(\log T)^{3/2-\varepsilon}(\log\log T)^2\\
&+ M^{5/2+\varepsilon}(\log T)^{-3/4+\varepsilon}.
\end{aligned} \tag{2.2}$$

Now we explain our strategy to the proof of Theorem 1. We first introduce the "weight"

$$K(u) = K_{\tau,\theta,\nu}(u) = (1-|u|)\left(1+\tau\sin\left(4\pi\frac{\nu}{M}\theta u\right)\right) \tag{2.3}$$

for $-1 \leqslant u \leqslant 1$, where θ (> 1) is a large positive constant, τ takes the value 1 or -1, and $\nu = \delta_0^{-1/2}$. The special case $\nu = M = 1$ of this weight was introduced by Heath-Brown and Tsang [3] (inspired by the work of Mueller [7]) to consider the sign changes of $E(T)$. In [5], to study the sign changes of (1.3), (1.4), the first author defined a more general form of the weight. The above (2.3) is an analogue of the weight in [5]. The choice of the above value of ν is one of the essential point of our proof.

Let $Q(t)$ be a real-valued function defined on $t \geqslant 0$, satisfying

$$|Q(t)| \leqslant c_1 t^{1/4}, \quad t \geqslant 0, \tag{2.4}$$

where c_1 is a positive constant. Define

$$\Lambda^*(t) = \frac{1}{\sqrt{2M^2 t}}\left(\Lambda\left(\frac{2\pi t^2}{M^2}\right) + Q\left(\frac{2\pi t^2}{M^2}\right)\right).$$

The following is an analogue of Lemma 1 of [3] or Lemma 1 of [5], and is the main body of our proof of the theorem.

Lemma 1. *Let $M \geqslant 2$, and assume* (1.9). *Then we have*

$$\int_{-1}^{1}\Lambda^*(t+\theta u)K(u)\mathrm{d}u \tag{2.5}$$
$$= -\frac{\tau}{2}\sqrt{C_{\cos}(M)^2 + C_{\sin}(M)^2}\sin\left(\frac{4\pi t}{M\delta_0^{1/2}} - \frac{\pi}{4} - \frac{\pi}{\delta_0} + \beta\right) + R^*,$$

where β is defined by

$$\sin\beta = \frac{C_{\sin}(M)}{\sqrt{C_{\cos}(M)^2 + C_{\sin}(M)^2}}$$

and R^* is the error term written as $R^* = R_1 + I_2 + I_3 + I_4$ with

$$R_1 \ll \frac{1}{M^{3/2}}\sum_{k,l\leqslant M}\frac{|\alpha(k,l)|}{[k,l]}(\kappa\lambda)^{3/4}\left\{\frac{\theta}{t^{1/2-\varepsilon}} + \frac{M^{5/6}}{(\kappa\lambda)^{5/12-\varepsilon}t^{5/6-\varepsilon}}\right.$$

$$+(\kappa\lambda)^{1/4+\varepsilon}\frac{M^{3/8-\varepsilon}}{t^{3/8-\varepsilon}}\left(\frac{M}{\theta}+\frac{1}{\delta_0^{1/2}}\right)+(\kappa\lambda)^{1/4+\varepsilon}\frac{M^{3/2-\varepsilon}}{t^{3/2-\varepsilon}}$$

$$\left.+(\kappa\lambda)^{1/4+\varepsilon}\frac{\theta^2}{M^{13/8+\varepsilon}t^{3/8+\varepsilon}}+\kappa\lambda\frac{M^2}{\theta^2}\right\}$$

$$+\frac{1}{M^{3/2}}\sum_{\substack{k,l\leqslant M\\ \kappa\lambda=\delta_0}}\frac{|\alpha(k,l)|}{[k,l]}\delta_0^{3/4}$$

$$\times\left(\frac{\theta^2M^2}{\delta_0^3t^4}+\frac{\delta_0M^2}{\theta^2}+\frac{M^2}{\delta_0t^2}+\frac{M}{t\delta_0^{3/2}}\right), \tag{2.6}$$

$$I_2 \ll \frac{1}{MT^{1/2-\varepsilon}}\sum_{k.l\leqslant M}\frac{|\alpha(k,l)|}{[k,l]}(\kappa\lambda)^{1/2}\left(\frac{M^{3/2-\varepsilon}}{\theta}+\frac{M^{1/2-\varepsilon}}{\delta_0^{1/2}}\right.$$

$$\left.+\frac{M^{3/2-\varepsilon}}{t^{1-\varepsilon}}+\theta\right), \tag{2.7}$$

and $I_3 \ll M^{3/2+\varepsilon}t^{-1/2+\varepsilon}$, $I_4 \ll c_1M^{-3/2}$.

From this lemma, just similarly to Section 4 of [5], we can easily deduce Theorem 1. The argument runs as follows. Choose $c_1 = c_1(M,\{a(m)\})$ sufficiently small, $\theta = \theta(M,\{a(m)\},\delta_0)$ sufficiently large, and then choose $t = t(M,\{a(m)\},\delta_0,\theta)$ sufficiently large such that R^* is sufficiently small, compared with the main term on the right-hand side of (2.5).

Let S be the set of large real number t such that the distance between

$$\frac{4\pi t}{M\delta_0^{1/2}}-\frac{\pi}{4}-\frac{\pi}{\delta_0}+\beta$$

and its nearest integer is $\geqslant \frac{1}{6}$. Then, for $t\in S$, by suitable choices of τ we can show that the right-hand side of (2.5) can be positive, and also be negative, and hence, $\Lambda^*(w)$ should change its sign in the interval $w\in[t-\theta,t+\theta]$. If $t\notin S$, then $t-\frac{M\delta_0^{1/2}}{24}$ or $t+\frac{M\delta_0^{1/2}}{24}$ belongs to S, and hence, $\Lambda^*(w)$ changes its sign in the interval $\left[t-\theta-\frac{M\delta_0^{1/2}}{24},t+\theta+\frac{M\delta_0^{1/2}}{24}\right]$. Therefore, putting $x=\frac{2\pi w^2}{M^2}$ we find that, for any large t, the quantity $\Lambda(x)+Q(x)$ changes its sign in the interval

$$\frac{2\pi}{M^2}\xi^2\leqslant x\leqslant\frac{2\pi}{M^2}\left(\xi^2+4\left(\theta+\frac{M\delta_0^{1/2}}{24}\right)\xi+4\left(\theta+\frac{M\delta_0^{1/2}}{24}\right)^2\right),$$

where $\xi=t-\theta-\frac{M\delta_0^{1/2}}{24}$. Writing $\frac{2\pi\xi^2}{M^2}=T$ and choosing $Q(t)=\pm c_1t^{1/4}$, we obtain the assertion of Theorem 1.

3. Proof of Lemma 1

In this section, we give a brief sketch of the proof of Lemma 1. Let

$$\begin{aligned} e(t,y) &= \left(1+\frac{y}{4t^2}\right)^{-1/4}\left(\frac{2t}{\sqrt{y}}\operatorname{arcsinh}\frac{\sqrt{y}}{2t}\right)^{-1}, \\ f_M(t,y) &= \frac{2\pi t\sqrt{y}}{M^2}\left(\frac{2t}{\sqrt{y}}\operatorname{arcsinh}\frac{\sqrt{y}}{2t}+\sqrt{1+\frac{y}{4t^2}}\right)-\frac{\pi}{4}, \end{aligned}$$

and

$$g_M(t,y)=\frac{4\pi t^2}{M^2}\log\frac{t}{\sqrt{y\mathrm{e}}}+\frac{\pi}{4}+\frac{2\pi y}{M^2}.$$

For sufficiently large t, using (2.1) with $T=\frac{2\pi(t+\theta u)^2}{M^2}$ and $Y=\frac{(t+\theta u)^2}{M^2}$, we obtain

$$\Lambda^*(t+\theta u)=\Lambda_1^*-\Lambda_2^*+\Lambda_3^*+\Lambda_4^* \tag{3.1}$$

with

$$\begin{aligned} \Lambda_1^* &= \frac{1}{M^{3/2}}\sum_{k,l\leqslant M}\frac{|\alpha(k,l)|}{[k,l]}(\kappa\lambda)^{3/4}\sum_{n\leqslant\frac{\kappa\lambda(t+\theta u)^2}{M^2}}\frac{d(n)}{n^{3/4}}e\left(t+\theta u,\frac{nM^2}{\kappa\lambda}\right) \\ &\quad\times\cos\left(2\pi n\frac{\overline{\kappa}}{\lambda}+\arg\alpha(k,l)+f_M\left(t+\theta u,\frac{nM^2}{\kappa\lambda}\right)-\frac{\pi n}{\kappa\lambda}\right), \\ \Lambda_2^* &= \frac{1}{\sqrt{2M^2(t+\theta u)}}\sum_{k,l\leqslant M}\frac{|\alpha(k,l)|}{[k,l]}(\kappa\lambda)^{1/2}\sum_{n\leqslant\frac{((3-\sqrt{5})\lambda/2\kappa)(t+\theta u)^2}{M^2}}\frac{d(n)}{n^{1/2}} \\ &\quad\times\frac{1}{\log((\lambda/n\kappa)^{1/2}(t+\theta u)/M)} \\ &\quad\times\cos\left(-2\pi n\frac{\kappa}{\lambda}+g_M\left(t+\theta u,\frac{n\kappa M^2}{\lambda}\right)+\arg\alpha(k,l)\right), \\ \Lambda_3^* &= \frac{1}{\sqrt{2M^2(t+\theta u)}}\left(R\left(\frac{2\pi(t+\theta u)^2}{M^2},A\right)-R\left(\frac{2\pi(t+\theta u)^2}{M^2},\overline{A}\right)\right), \end{aligned}$$

and

$$\Lambda_4^*=\frac{1}{\sqrt{2M^2(t+\theta u)}}Q\left(\frac{2\pi(t+\theta u)^2}{M^2}\right).$$

Therefore,

$$\int_{-1}^{1}\Lambda^*(t+\theta u)K(u)\mathrm{d}u=I_1-I_2+I_3+I_4, \tag{3.2}$$

with

$$I_j=\int_{-1}^{1}\Lambda_j^* K(u)\mathrm{d}u,\quad 1\leqslant j\leqslant 4.$$

Using the fact $|K(u)|\leqslant 2$ and (2.2) (or, more roughly, $R(T,A)\ll M^{5/2+\varepsilon}(\log T)^3$), we find that I_3 satisfies the estimate given in the statement of Lemma 1. The estimate $I_4\ll c_1M^{-3/2}$ follows from $|K(u)|\leqslant 2$ and (2.4). (It is to be noted that, if the exponent on the right-hand side of (2.4) is larger than $\frac{1}{4}$, then a positive power of t remains in the estimate of I_4, which invalidates our argument. This explains the optimality of the value $\frac{1}{4}$ of the exponent appearing in our theorem.)

The method of estimating I_2 is similar to that in pp.5–6 of [5]. The formula corresponding to (9) of [5] is

$$\begin{aligned}I_2 &= \frac{1}{\sqrt{2M^2t}}\sum_{k,l\leqslant M}\frac{|\alpha(k,l)|}{[k,l]}(\kappa\lambda)^{1/2}\sum_{n\leqslant\frac{((3-\sqrt{5})\lambda/2\kappa)t^2}{M^2}}\frac{d(n)}{n^{1/2}}\\ &\quad\times\frac{1}{\log((\lambda/n\kappa)^{1/2}t/M)}\\ &\quad\times\int_{-1}^{1}\cos\left(-2\pi n\frac{\kappa}{\lambda}+g_M\left(t+\theta u,\frac{n\kappa M^2}{\lambda}\right)+\arg\alpha(k,l)\right)K_\tau(u)\mathrm{d}u\\ &\quad+\mathrm{O}\left(\frac{\theta}{Mt^{1/2-\varepsilon}}\sum_{k,l\leqslant M}\frac{|\alpha(k,l)|}{[k,l]}(\kappa\lambda)^{1/2}\right).\end{aligned}$$

We evaluate the integral on the right-hand side by integration by parts, similarly to the argument in [5]. Here we note a misprint in [5]; on line 2, p. 6 of [5], the estimate $K'_{\tau,n_0}(u)\ll q^{-1}n_0^{1/2}\theta$ is to be read as $K'_{\tau,n_0}(u)\ll 1+q^{-1}n_0^{1/2}\theta$. The corresponding estimate in our present situation is $K'_\tau(u)\ll 1+\frac{\nu\theta}{M}$. Applying this estimate, we obtain (2.7).

Now we treat I_1. First, as an analogue of (11) of [5], we obtain

$$\begin{aligned}I_1 &= \frac{1}{M^{3/2}}\sum_{k,l\leqslant M}\frac{|\alpha(k,l)|}{[k,l]}(\kappa\lambda)^{3/4}\sum_{n\leqslant\frac{\kappa\lambda t^2}{M^2}}\frac{d(n)}{n^{3/4}}e\left(t,\frac{nM^2}{\kappa\lambda}\right)J_1\\ &\quad+\mathrm{O}\left(\frac{1}{M^{3/2}}\sum_{k,l\leqslant M}\frac{|\alpha(k,l)|}{[k,l]}(\kappa\lambda)^{3/4}\right.\\ &\quad\times\left.\left(\frac{\theta}{t^{1/2-\varepsilon}}+\frac{M^{5/6}}{(\kappa\lambda)^{5/12-\varepsilon}t^{5/6-\varepsilon}}\right)\right),\end{aligned}\tag{3.3}$$

where

$$J_1 = \int_{-1}^{1} \cos\left(2\pi n\frac{\overline{\kappa}}{\lambda} + \arg\alpha(k,l) + f_M\left(t+\theta u, \frac{nM^2}{\kappa\lambda}\right) - \frac{\pi n}{\kappa\lambda}\right) K(u)\mathrm{d}u.$$

This J_1 is the analogue of J_A in [5]. In [5], J_A was evaluated in two ways. The result of the first evaluation is (12) of [5], while the second evaluation is presented by (13), (14), and (15) of [5]. Analogously to (12) of [5], we obtain

$$J_1 \ll \frac{M(\kappa\lambda)^{1/2}}{\theta n^{1/2}} + \frac{\nu(\kappa\lambda)^{1/2}}{n^{1/2}} + \frac{M^2}{t^2} \tag{3.4}$$

for $n \leqslant \frac{\kappa\lambda t^2}{M^2}$. On the other hand, the analogy of (13) of [5] is

$$J_1 = J_1^* + \mathrm{O}\left(\frac{\theta^2}{M^2 t^3}\left(\frac{nM^2}{\kappa\lambda}\right)^{3/2}\right), \tag{3.5}$$

where

$$J_1^* = \int_{-1}^{1} \cos\left(C + f_M\left(t, \frac{nM^2}{\kappa\lambda}\right) + f_M'\left(t, \frac{nM^2}{\kappa\lambda}\right)\theta u\right) K(u)du$$

with $C = \frac{2\pi n\overline{\kappa}}{\lambda} + \arg\alpha(k,l) - \frac{\pi n}{\kappa\lambda}$. As for J_1^*, analogously to (14) and (15) of [5], we obtain

$$J_1^* = -\frac{\tau}{2}\sin\left(C + f_M\left(t, \frac{nM^2}{\kappa\lambda}\right)\right) + \mathrm{O}\left(\frac{\theta^2 M^2}{\delta_0^3 t^4}\right) + \mathrm{O}\left(\frac{\delta_0 M^2}{\theta^2}\right) \tag{3.6}$$

when $n = \frac{\kappa\lambda}{\delta_0}$ (hence $n=1$ by the definition of δ_0), and

$$J_1^* \ll \left|\left(f_M'\left(t, \frac{nM^2}{\kappa\lambda}\right) - \frac{4\pi\nu}{M}\right)\theta\right|^{-2} + \frac{\kappa\lambda M^2}{n\theta^2} \tag{3.7}$$

otherwise. (3.6) and (3.7) can be shown by calculations similar to those in the upper-half of p.7 of [5]. During the calculations the quantity $f_M' \pm 4\pi\nu M^{-1}$ appears in denominators. Since f_M' is positive, $f_M' + 4\pi\nu M^{-1}$ is always positive. On the other hand, to avoid the possibility that $f_M' - 4\pi\nu M^{-1} = 0$, we have to assume that t is large and $n \leqslant \frac{\kappa\lambda t^{3/2}}{M^{3/2}}$. Therefore, (3.6) and (3.7) are valid under these assumptions.

The main term on the right-hand side of (3.6) produces the main term in the formula of Lemma 1. It is a key point of the proof that this main term appears only in the case $n = 1$ and $\kappa\lambda = \delta_0$. (We have chosen the values of δ_0 and ν carefully in order to produce this situation.)

Divide the sum with respect to n on the right-hand side of (3.3) into two parts, according as $n \leqslant \frac{\kappa\lambda t^{3/2}}{M^{3/2}}$ and $\frac{\kappa\lambda t^{3/2}}{M^{3/2}} < n \leqslant \frac{\kappa\lambda t^2}{M^2}$. We evaluate the first part by (3.5)–(3.7), and the second part by (3.4). We obtain

$$I_1 = -\frac{\tau}{2M^{3/2}} \sum_{\substack{k,l \leqslant M \\ \kappa\lambda = \delta_0}} \frac{|\alpha(k,l)|}{[k,l]} (\kappa\lambda)^{3/4} \\ \times \sin\left(\frac{2\pi\kappa\overline{\kappa}}{\delta_0} + \arg\alpha(k,l) - \frac{\pi}{\delta_0} + \frac{4\pi t}{M\delta_0^{1/2}} - \frac{\pi}{4}\right) + R_1, \qquad (3.8)$$

where R_1 is the error term satisfying the estimate (2.6). It is easy to see that the main term on the right-hand side of the above is equal to the main term on the right-hand side of (2.5) if (1.9) holds. Thus, the assertion of Lemma 1 follows.

4. On condition (1.9)

In this final section we discuss when condition (1.9) of Theorem 1 holds, or does not hold. Hereafter we assume that M is a positive integer ($\geqslant 2$), and first we prove the following simple criterion.

LEMMA 2. *If $\alpha(M-1, M) \neq 0$, then $C_{\sin}(M) \neq 0$, and hence* (1.9) *holds.*

Proof. If $\alpha(M-1, M) \neq 0$, then also $\alpha(M, M-1) \neq 0$. For $(k,l) = (M-1, M)$ we have $\kappa = M-1$, $\lambda = M$, and $\overline{\kappa} = -1$, while for $(k,l) = (M, M-1)$ we have $\kappa = M$, $\lambda = M-1$, and $\overline{\kappa} = 1$. Only these two pairs of (κ, λ) satisfy $\kappa\lambda = \delta_0 = M(M-1)$. Since $|\alpha(M-1, M)| = |\alpha(M, M-1)|$, we see that $C_{\sin}(M) \neq 0$ is equivalent to

$$\sin\left(\frac{-2\pi}{M} + \arg\alpha(M-1, M)\right) + \\ \sin\left(\frac{2\pi}{M-1} + \arg\alpha(M, M-1)\right) \neq 0. \qquad (4.1)$$

Since $\arg\alpha(M-1, M) = \pm\frac{\pi}{2}$ and $\arg\alpha(M, M-1) = \mp\frac{\pi}{2}$, we see that there is no M ($\geqslant 2$) for which both of the terms on the left-hand side of (4.1) are 0. (In fact, the first term is 0 only when $M = 4$, while the second term is 0 only when $M=5$.) Moreover, since $\frac{2M-1}{M(M-1)}$ is not an integer for any $M\geqslant 2$,

$$\left(\frac{-2\pi}{M}+\arg\alpha(M-1,M)\right)-\left(\frac{2\pi}{M-1}+\arg\alpha(M,M-1)\right)$$
$$=-2\pi\frac{2M-1}{M(M-1)}\pm\pi$$

is not congruent to π (mod 2π). Therefore, (4.1) holds for any $M\geqslant 2$. □

Remark 3. Under the assumption $\alpha(M-1,M)\neq 0$, we see similarly that $C_{\cos}(M)\neq 0$ is equivalent to

$$\cos\left(\frac{-2\pi}{M}+\arg\alpha(M-1,M)\right)+\cos\left(\frac{2\pi}{M-1}+\arg\alpha(M,M-1)\right)\neq 0. \tag{4.2}$$

But when $M=2$, this does not hold; in fact, both of the terms on the right-hand side of (4.2) are 0. For $M\geqslant 3$, we can show $C_{\cos}(M)\neq 0$ similarly as above.

We discuss some examples for small values of M.

Example 1. When $M=2$, the assumption of the existence of δ_0 implies that $\alpha(1,2)\neq 0$. Therefore, from Lemma 2 we see that (1.9) holds.

Example 2. When $M=3$, the existence of δ_0 implies that at least one of $\alpha(2,3)$, $\alpha(1,3)$, or $\alpha(1,2)$ is not 0. If $\alpha(2,3)\neq 0$, then by Lemma 2 it follows that (1.9) holds. If $\alpha(2,3)=0$ but $\alpha(1,3)\neq 0$, then $\delta_0=3$ and the pairs of (k,l) which attains $\kappa\lambda=\delta_0$ are $(k,l)=(1,3)$ and $(3,1)$. By simple calculations, we see that $C_{\cos}(3)\neq 0$, $C_{\sin}(3)\neq 0$, and so (1.9) holds. Lastly, if $\alpha(2,3)=\alpha(1,3)=0$ but $\alpha(1,2)\neq 0$, then $\delta_0=2$, and $C_{\cos}(3)=0$ but $C_{\sin}(3)\neq 0$. Therefore, when $M=3$, condition (1.9) always holds if δ_0 exists.

Example 3. Finally, we consider the case $M=4$. In this case, we can construct an example for which δ_0 exists but (1.9) does not hold. Let $a(1)=1$, $a(2)=-2i$, $a(3)=0$, and $a(4)=2$. Then, obviously, $\alpha(1,3)=\alpha(2,3)=\alpha(3,4)=0$. Also we have $\alpha(1,4)=0$. However, $\alpha(1,2)=4i$ and $\alpha(2,4)=-8i$, both are not 0, hence δ_0 exists. The value of δ_0 is 2, and the pairs of (k,l) which attains $\kappa\lambda=\delta_0$ are $(k,l)=(1,2)$, $(2,4)$ (with $(\kappa,\lambda)=(1,2)$), and $(k,l)=(2,1)$, $(4,2)$ (with $(\kappa,\lambda)=(2,1)$). Using these data, we can calculate

$$C_{\sin}(4) \quad = \quad 4^{-3/2}2^{3/4}\left\{\frac{|4i|}{2}\sin\frac{3\pi}{2}+\frac{|-8i|}{4}\sin\frac{\pi}{2}\right.$$

$$
\begin{aligned}
& + \frac{|-4i|}{2} \sin \frac{3\pi}{2} + \frac{|8i|}{4} \sin \frac{5\pi}{2} \Big\} \\
= \; & 4^{-3/2} 2^{3/4} \{ 2 \cdot (-1) + 2 \cdot 1 + 2 \cdot (-1) + 2 \cdot 1 \} = 0,
\end{aligned}
$$

and, more obviously, $C_{\cos}(4) = 0$. Therefore, (1.9) does not hold.

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