

LATTICE SUMS OF HYPERPLANE ARRANGEMENTS

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ABSTRACT. We introduce certain lattice sums associated with hyperplane arrangements, which are (multiple) sums running over integers, and can be regarded as generalizations of certain linear combinations of zeta-functions of root systems. We also introduce generating functions of special values of those lattice sums, and study their properties by virtue of the theory of convex polytopes. Consequently we evaluate special values of those lattice sums, especially certain special values of zeta-functions of root systems and their affine analogues. In some special cases it is possible to treat sums running over positive integers, which may be regarded as zeta-functions associated with hyperplane arrangements.

1. INTRODUCTION

The notion of Witten zeta-functions associated with semisimple Lie algebras was introduced by Zagier [12], inspired by the work of Witten [11] in quantum gauge theory. Recently the authors have developed the theory of zeta-functions of root systems (e.g. [4, 5, 6, 7]), which are multi-variable generalizations of Witten zeta-functions. In particular, the ‘‘Weyl group symmetric’’ linear combinations of zeta-functions of root systems $S(\mathbf{s}, \mathbf{y}; \Delta)$ (where \mathbf{s} is a complex multi-variable, \mathbf{y} is a certain vector and Δ is a finite reduced root system) and the generating functions of special values of those linear combinations were introduced and studied in [4, 6, 7].

In the present paper, we will introduce certain lattice sums of hyperplane arrangements, which are generalizations of the above linear combinations of zeta-functions of root systems. We will also introduce the generating functions of special values of those lattice sums. It is to be stressed that those generating functions can describe not only values but also functional relations among zeta-functions of root systems. Furthermore if they are combined with Poincaré polynomials of Weyl groups, we obtain explicit formulas for special odd values of zeta-functions of root systems. These results will be treated in the forthcoming paper [8].

Another application is to calculate special values of affine analogue of zeta-functions of root systems. Although in the cases of affine root systems it is natural to work with the character formulas instead of the dimension formulas, a straightforward generalization is also interesting. We will present some examples in Section 3.

In the present paper, our consideration is not restricted to the case in the domain of absolute convergence; we will study the values of lattice sums outside the domain of absolute convergence. Here we explain this point by simple examples.

Let \mathbb{N} be the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, \mathbb{Z} the ring of rational integers, \mathbb{R} the field of real numbers, and \mathbb{C} the field of complex numbers. For any set S , the symbol $\sharp S$ denotes the cardinality of S .

Let $k \in \mathbb{N}$, and let $y \in \mathbb{R}$ with $y \notin \mathbb{Z}$ if $k = 1$. It is well-known (cf. [1, Theorem 12.19]) that

$$(1.1) \quad -\frac{(2\pi\sqrt{-1})^k}{k!} B_k(\{y\}) = \lim_{N \rightarrow \infty} \sum_{\substack{|m| \leq N \\ m \neq 0}} \frac{e^{2\pi\sqrt{-1}my}}{m^k},$$

where $\{y\} = y - [y]$ is the fractional part of y and $B_k(\cdot)$ is the k -th Bernoulli polynomial defined by

$$(1.2) \quad \frac{te^{t\{y\}}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(\{y\}) \frac{t^k}{k!}.$$

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In the case $k = 0$, then (1.1) does not hold straightforward. However this formula still holds in some sense via the following regularization. We see that the right-hand side of (1.1) is analytically continued to the whole space \mathbb{C} in the variable k and then it is evaluated as -1 at $k = 0$. This effect is (formally) realized in the series by replacing the condition $m \neq 0$ by $m = 0$ and $\sum_{m \neq 0}$ by $-\sum_{m=0}$ with $0^0 = 1$. Hence the sum consists of only one term. As a result, we may understand the case $k = 0$ as

$$(1.3) \quad -\frac{(2\pi\sqrt{-1})^k}{k!} B_k(\{y\}) = -\lim_{N \rightarrow \infty} \sum_{\substack{|m| \leq N \\ m=0}} e^{2\pi\sqrt{-1}my} = -e^{2\pi\sqrt{-1}my}|_{m=0} = -1,$$

where $B_0(\{y\}) = 1$.

This interpretation works well in the multi-dimensional cases. For example, let $\alpha, \beta, \gamma \in \mathbb{C}$ and $k_1, k_2, k_3 \in \mathbb{N}_0$, and consider the sum

$$(1.4) \quad S((k_1, k_2, k_3), (y_1, y_2)) = \lim_{N \rightarrow \infty} \sum_{\substack{m, n \in \mathbb{Z} \\ m+\alpha, n+\beta, m+n+\gamma \neq 0 \\ |m|, |n| \leq N}} \frac{e^{2\pi\sqrt{-1}(my_1+ny_2)}}{(m+\alpha)^{k_1}(n+\beta)^{k_2}(m+n+\gamma)^{k_3}}.$$

This is convergent if $k_1, k_2, k_3 \geq 1$. If some of k_i 's are 0, then we modify the series. In the case when only $k_1 = 0$, we replace the condition $m + \alpha \neq 0$ by $m + \alpha = 0$ in the sum with the minus sign and $0^0 = 1$, that is,

$$(1.5) \quad S((0, k_2, k_3), (y_1, y_2)) = -\lim_{N \rightarrow \infty} \sum_{\substack{m, n \in \mathbb{Z} \\ |m|, |n| \leq N \\ n+\beta, m+n+\gamma \neq 0 \\ m+\alpha=0}} \frac{e^{2\pi\sqrt{-1}(my_1+ny_2)}}{(n+\beta)^{k_2}(m+n+\gamma)^{k_3}}.$$

By the restriction $m + \alpha = 0$, this sum is 0 if $\alpha \notin \mathbb{Z}$. If $\alpha \in \mathbb{Z}$, then the sum reduces to the one-dimensional sum

$$(1.6) \quad S((0, k_2, k_3), (y_1, y_2)) = -\lim_{N \rightarrow \infty} \sum_{\substack{n \in \mathbb{Z} \\ |n| \leq N \\ n+\beta, n+\gamma-\alpha \neq 0}} \frac{e^{2\pi\sqrt{-1}(-\alpha y_1+ny_2)}}{(n+\beta)^{k_2}(n+\gamma-\alpha)^{k_3}}.$$

In the other cases, the sum is similarly modified. Then the special values $S((k_1, k_2, k_3), (y_1, y_2))$ for all $k_1, k_2, k_3 \in \mathbb{N}_0$ are explicitly given by coefficients of a generating function, which will be given in Example 3.1.

In the above arguments the sums are taken over all integers. However in some special cases, it is possible to treat sums running over only positive integers (Examples 3.2, 3.3), which may be regarded as zeta-functions associated with hyperplane arrangements.

In the next section we will introduce more general lattice sums, and their generating functions.

2. NOTATIONS AND STATEMENT OF MAIN RESULTS

We fix a positive integer r . Let $V = \mathbb{R}^r$ be a real vector space equipped with the standard inner product $\langle \cdot, \cdot \rangle$. We regard $f = (\vec{f}, \dot{f}) \in V \times \mathbb{C}$ with $\vec{f} \in V$ and $\dot{f} \in \mathbb{C}$ as an affine linear functional on V by $f(\mathbf{v}) = \langle \vec{f}, \mathbf{v} \rangle + \dot{f}$ for $\mathbf{v} \in V$.

We use the following notation: For $X \subset V$, put $\langle X \rangle = \sum_{\mathbf{v} \in X} \mathbb{Z}\mathbf{v}$. For $Y \subset V \times \mathbb{C}$, put $\vec{Y} = \{\vec{f} \mid f = (\vec{f}, \dot{f}) \in Y\}$.

Let $\Lambda \subset (\mathbb{Z}^r \setminus \{\vec{0}\}) \times \mathbb{C}$ with $\#\Lambda < \infty$ such that $\text{rank}\langle \vec{\Lambda} \rangle = r$. Put $\tilde{\Lambda} = \{f \in \Lambda \mid \text{rank}\langle \vec{\Lambda} \setminus \{\vec{f}\} \rangle \neq r\}$. For each $f \in \Lambda$ we associate a number $k_f \in \mathbb{N}_0$, and put $\mathbf{k} = (k_f)_{f \in \Lambda} \in \mathbb{N}_0^{\#\Lambda}$. For $k \in \mathbb{N}_0$, define

$$\begin{aligned} \Lambda_k &= \Lambda_k(\mathbf{k}) = \{f \in \Lambda \mid k_f = k\}, \\ \Lambda_+ &= \Lambda_+(\mathbf{k}) = \{f \in \Lambda \mid k_f > 0\}. \end{aligned}$$

Obviously

$$\Lambda_+ = \bigcup_{k \geq 1} \Lambda_k \quad \text{and} \quad \Lambda = \Lambda_+ \cup \Lambda_0.$$

For $H \subset \Lambda$ such that $\text{rank}\langle \vec{H} \rangle = r - 1$, let $\mathfrak{H}_H = \sum_{g \in H} \mathbb{R} \vec{g}$ be the hyperplane passing through $\vec{H} \cup \{\vec{0}\}$.

The following is the main object in the present paper, a lattice sum over the hyperplane arrangement given by linear functionals belonging to Λ .

Definition 2.1. For $\mathbf{k} = (k_f)_{f \in \Lambda} \in \mathbb{N}_0^{\#\Lambda}$ and $\mathbf{y} \in V \setminus \bigcup_{f \in \tilde{\Lambda} \cap \Lambda_1} (\mathfrak{H}_{\Lambda \setminus \{f\}} + \mathbb{Z}^r)$, we define

$$(2.1) \quad S(\mathbf{k}, \mathbf{y}; \Lambda) = \lim_{N \rightarrow \infty} Z(N; \mathbf{k}, \mathbf{y}; \Lambda),$$

where

$$(2.2) \quad Z(N; \mathbf{k}, \mathbf{y}; \Lambda) = (-1)^{\#\Lambda_0} \sum_{\substack{\mathbf{v}=(v_1, \dots, v_r) \in \mathbb{Z}^r \\ |v_j| \leq N \quad (1 \leq j \leq r) \\ f(\mathbf{v}) \neq 0 \quad (f \in \Lambda_+) \\ f(\mathbf{v}) = 0 \quad (f \in \Lambda_0)}} e^{2\pi\sqrt{-1}\langle \mathbf{y}, \mathbf{v} \rangle} \prod_{f \in \Lambda_+} \frac{1}{f(\mathbf{v})^{k_f}}$$

for $N > 0$.

This $S(\mathbf{k}, \mathbf{y}; \Lambda)$ is a generalization of the notion of "Weyl group symmetric" linear combinations of zeta-functions of root systems $S(\mathbf{s}, \mathbf{y}; \Delta)$ mentioned in the Introduction (in the case $\mathbf{s} = \mathbf{k}$); cf. [6, (3.3)], [7, (110)]. The first main result in the present paper is as follows.

Theorem 2.2. *The series $S(\mathbf{k}, \mathbf{y}; \Lambda)$ converges and is continuous in \mathbf{y} on $V \setminus \bigcup_{f \in \tilde{\Lambda} \cap \Lambda_1} (\mathfrak{H}_{\Lambda \setminus \{f\}} + \mathbb{Z}^r)$.*

In order to define the generating function of $S(\mathbf{k}, \mathbf{y}; \Lambda)$, we need some more notations. Let $\mathcal{B} = \mathcal{B}(\Lambda)$ be the set of all subsets $B = \{f_1, \dots, f_r\} \subset \Lambda$ such that \vec{B} forms a basis of V . For $B \in \mathcal{B}$, let $\vec{B}^* = \{\vec{f}_1^B, \dots, \vec{f}_r^B\}$ be the dual basis of $\vec{B} = \{\vec{f}_1, \dots, \vec{f}_r\}$ in V . It should be noted that for each $B \in \mathcal{B}$, we have

$$(2.3) \quad \tilde{\Lambda} \subset B,$$

because all elements of $\tilde{\Lambda}$ are indispensable for constructing a basis.

Next we define a multi-dimensional generalization of fractional part $\{\cdot\}$ for real numbers, which was first introduced in [6, Section 4]. Let $\mathcal{R} = \mathcal{R}(\Lambda)$ be the set of all subsets $R = \{g_1, \dots, g_{r-1}\} \subset \Lambda$ such that $\vec{R} = \{\vec{g}_1, \dots, \vec{g}_{r-1}\}$ is linearly independent set. We need to fix a vector

$$(2.4) \quad \phi \in V \setminus \bigcup_{R \in \mathcal{R}} \mathfrak{H}_R$$

so that $\langle \phi, \vec{f}^B \rangle \neq 0$ for all $B \in \mathcal{B}$ and $f \in B$ (because if $\langle \phi, \vec{f}^B \rangle = 0$ for some $B \in \mathcal{B}$ and $f \in B$, then $\phi \in \mathfrak{H}_R$ with $R = B \setminus \{f\} \in \mathcal{R}$).

For $\mathbf{y} \in V$, $B \in \mathcal{B}$ and $f \in B$, we define the multi-dimensional fractional part by

$$(2.5) \quad \{\mathbf{y}\}_{B, f} = \begin{cases} \{\langle \mathbf{y}, \vec{f}^B \rangle\} & (\langle \phi, \vec{f}^B \rangle > 0), \\ 1 - \{-\langle \mathbf{y}, \vec{f}^B \rangle\} & (\langle \phi, \vec{f}^B \rangle < 0). \end{cases}$$

It should be noted that

$$(2.6) \quad \{a\} = 1 - \{-a\}$$

for $a \in \mathbb{R} \setminus \mathbb{Z}$.

Now we define the generating function of $S(\mathbf{k}, \mathbf{y}; \Lambda)$ and state its properties.

Definition 2.3. For $\mathbf{y} \in V$ and $\mathbf{t} = (t_f)_{f \in \Lambda} \in \mathbb{C}^{\#\Lambda}$, we define

$$(2.7) \quad F(\mathbf{t}, \mathbf{y}; \Lambda) = \sum_{B \in \mathcal{B}(\Lambda)} \left(\prod_{g \in \Lambda \setminus B} \frac{t_g}{t_g - 2\pi\sqrt{-1}\dot{g} - \sum_{f \in B} (t_f - 2\pi\sqrt{-1}\dot{f}) \langle \vec{g}, \vec{f}^B \rangle} \right) \\ \times \frac{1}{\#\langle \mathbb{Z}^r / \langle \vec{B} \rangle \rangle} \sum_{\mathbf{w} \in \mathbb{Z}^r / \langle \vec{B} \rangle} \left(\prod_{f \in B} \frac{t_f \exp((t_f - 2\pi\sqrt{-1}\dot{f}) \{ \mathbf{y} + \mathbf{w} \}_{B, f})}{\exp(t_f - 2\pi\sqrt{-1}\dot{f}) - 1} \right).$$

Theorem 2.4. (i) The function $F(\mathbf{t}, \mathbf{y}; \Lambda)$ has one-sided continuity in $\mathbf{y} \in V$ in the direction ϕ , that is

$$(2.8) \quad \lim_{c \rightarrow 0^+} F(\mathbf{t}, \mathbf{y} + c\phi; \Lambda) = F(\mathbf{t}, \mathbf{y}; \Lambda).$$

(ii) $F(\mathbf{t}, \mathbf{y}; \Lambda)$ is continuous in \mathbf{y} on $V \setminus \bigcup_{f \in \tilde{\Lambda}} (\mathfrak{H}_{\Lambda \setminus \{f\}} + \mathbb{Z}^r)$. In particular if $\tilde{\Lambda}$ is empty, then $F(\mathbf{t}, \mathbf{y}; \Lambda)$ is continuous on the whole V and is independent of the choice of ϕ .

(iii) $F(\mathbf{t}, \mathbf{y}; \Lambda)$ is holomorphic in the neighborhood of the origin in \mathbf{t} .

Write the Taylor expansion of $F(\mathbf{t}, \mathbf{y}; \Lambda)$ around the origin in \mathbf{t} as

$$(2.9) \quad F(\mathbf{t}, \mathbf{y}; \Lambda) = \sum_{\mathbf{k} \in \mathbb{N}_0^{\#\Lambda}} C(\mathbf{k}, \mathbf{y}; \Lambda) \prod_{f \in \Lambda} \frac{t_f^{k_f}}{k_f!}.$$

Theorem 2.5. We have

$$(2.10) \quad S(\mathbf{k}, \mathbf{y}; \Lambda) = \left(\prod_{f \in \Lambda} -\frac{(2\pi\sqrt{-1})^{k_f}}{k_f!} \right) C(\mathbf{k}, \mathbf{y}; \Lambda)$$

for $\mathbf{k} = (k_f)_{f \in \Lambda} \in \mathbb{N}_0^{\#\Lambda}$ and $\mathbf{y} \in V \setminus \bigcup_{f \in \tilde{\Lambda} \cap \Lambda_1} (\mathfrak{H}_{\Lambda \setminus \{f\}} + \mathbb{Z}^r)$.

The above results are again generalizations of the results proved in [6], [7]. In fact, the form of $F(\mathbf{t}, \mathbf{y}; \Lambda)$ in Definition 2.3 is the generalization of [6, Theorem 4.1], Theorem 2.4 is the generalization of the facts mentioned in [7, p.252], and Theorem 2.5 is the generalization of [6, (3.10)].

Before going into the proofs of the main theorems, in the next section we will give several examples. Then we will start the proofs of main theorems from Section 4. Section 4 is devoted to the proof of Theorem 2.2. Then from Section 5 to Section 8 we will describe the proof of Theorem 2.4 and Theorem 2.5. In the final section we will mention that there is some hierarchy among generating functions.

3. EXAMPLES

In this section we apply our theorems to some special cases, and to state explicit expressions of $F(\mathbf{t}, \mathbf{y}; \Lambda)$, $C(\mathbf{k}, \mathbf{y}; \Lambda)$ and $S(\mathbf{k}, \mathbf{y}; \Lambda)$ for those examples.

Example 3.1. Let $V = \mathbb{R}^2$. Let $\alpha, \beta, \gamma \in \mathbb{C}$,

$$(3.1) \quad \Lambda = \{f_1 = ((1, 0), \alpha), f_2 = ((0, 1), \beta), f_3 = ((1, 1), \gamma)\},$$

$$(3.2) \quad \mathcal{B} = \{\{f_1, f_2\}, \{f_1, f_3\}, \{f_2, f_3\}\},$$

which corresponds to the series in (1.4), (1.5) and so on. Then the generating function is given by

$$(3.3) \quad F((t_1, t_2, t_3), (y_1, y_2); \Lambda) = \\ \frac{t_3}{t_3 - 2\pi\sqrt{-1}\gamma - (t_1 - 2\pi\sqrt{-1}\alpha) - (t_2 - 2\pi\sqrt{-1}\beta)} \frac{t_1 e^{(t_1 - 2\pi\sqrt{-1}\alpha)\{y_1\}} t_2 e^{(t_2 - 2\pi\sqrt{-1}\beta)\{y_2\}}}{e^{(t_1 - 2\pi\sqrt{-1}\alpha)} - 1} \frac{t_2 e^{(t_2 - 2\pi\sqrt{-1}\beta)\{y_2\}}}{e^{(t_2 - 2\pi\sqrt{-1}\beta)} - 1} \\ + \frac{t_2}{t_2 - 2\pi\sqrt{-1}\beta + (t_1 - 2\pi\sqrt{-1}\alpha) - (t_3 - 2\pi\sqrt{-1}\gamma)} \frac{t_1 e^{(t_1 - 2\pi\sqrt{-1}\alpha)\{y_1 - y_2\}} t_3 e^{(t_3 - 2\pi\sqrt{-1}\gamma)\{y_2\}}}{e^{(t_1 - 2\pi\sqrt{-1}\alpha)} - 1} \frac{t_3 e^{(t_3 - 2\pi\sqrt{-1}\gamma)\{y_2\}}}{e^{(t_3 - 2\pi\sqrt{-1}\gamma)} - 1} \\ + \frac{t_1}{t_1 - 2\pi\sqrt{-1}\alpha + (t_2 - 2\pi\sqrt{-1}\beta) - (t_3 - 2\pi\sqrt{-1}\gamma)} \frac{t_2 e^{(t_2 - 2\pi\sqrt{-1}\beta)(1 - \{y_1 - y_2\})} t_3 e^{(t_3 - 2\pi\sqrt{-1}\gamma)\{y_1\}}}{e^{(t_2 - 2\pi\sqrt{-1}\beta)} - 1} \frac{t_3 e^{(t_3 - 2\pi\sqrt{-1}\gamma)\{y_1\}}}{e^{(t_3 - 2\pi\sqrt{-1}\gamma)} - 1}.$$

In particular, if $\alpha, \beta, \gamma \notin \mathbb{Z}$ with $\alpha + \beta \neq \gamma$, we have

$$\begin{aligned}
(3.4) \quad C((2, 1, 1), (y_1, y_2); \Lambda) = & - \frac{16\sqrt{-1}\pi^3\{y_1 - y_2\}e^{2\sqrt{-1}\pi(\alpha - \alpha\{y_1 - y_2\} - \gamma\{y_2\})}}{(-1 + e^{2\sqrt{-1}\pi\alpha})^2(-1 + e^{-2\sqrt{-1}\pi\gamma})(\alpha + \beta - \gamma)} \\
& + \frac{16\sqrt{-1}\pi^3\{y_1 - y_2\}e^{2\sqrt{-1}\pi(2\alpha - \alpha\{y_1 - y_2\} - \gamma\{y_2\})}}{(-1 + e^{2\sqrt{-1}\pi\alpha})^2(-1 + e^{-2\sqrt{-1}\pi\gamma})(\alpha + \beta - \gamma)} \\
& + \frac{16\sqrt{-1}\pi^3e^{2\sqrt{-1}\pi(\alpha - \alpha\{y_1 - y_2\} - \gamma\{y_2\})}}{(-1 + e^{2\sqrt{-1}\pi\alpha})^2(-1 + e^{-2\sqrt{-1}\pi\gamma})(\alpha + \beta - \gamma)} \\
& - \frac{8\pi^2e^{2\sqrt{-1}\pi(-\beta + \beta\{y_1 - y_2\} - \gamma\{y_1\})}}{(-1 + e^{-2\sqrt{-1}\pi\beta})(-1 + e^{-2\sqrt{-1}\pi\gamma})(\alpha + \beta - \gamma)^2} \\
& + \frac{16\sqrt{-1}\pi^3\{y_1\}e^{2\sqrt{-1}\pi(\alpha - \alpha\{y_1\} - \beta\{y_2\})}}{(-1 + e^{2\sqrt{-1}\pi\alpha})^2(-1 + e^{-2\sqrt{-1}\pi\beta})(\alpha + \beta - \gamma)} \\
& - \frac{16\sqrt{-1}\pi^3\{y_1\}e^{2\sqrt{-1}\pi(2\alpha - \alpha\{y_1\} - \beta\{y_2\})}}{(-1 + e^{2\sqrt{-1}\pi\alpha})^2(-1 + e^{-2\sqrt{-1}\pi\beta})(\alpha + \beta - \gamma)} \\
& - \frac{16\sqrt{-1}\pi^3e^{2\sqrt{-1}\pi(\alpha - \alpha\{y_1\} - \beta\{y_2\})}}{(-1 + e^{2\sqrt{-1}\pi\alpha})^2(-1 + e^{-2\sqrt{-1}\pi\beta})(\alpha + \beta - \gamma)} \\
& + \frac{8\pi^2e^{-2\sqrt{-1}\pi(\alpha\{y_1\} + \beta\{y_2\})}}{(-1 + e^{-2\sqrt{-1}\pi\alpha})(-1 + e^{-2\sqrt{-1}\pi\beta})(\alpha + \beta - \gamma)^2} \\
& - \frac{8\pi^2e^{-2\sqrt{-1}\pi(\alpha\{y_1 - y_2\} + \gamma\{y_2\})}}{(-1 + e^{-2\sqrt{-1}\pi\alpha})(-1 + e^{-2\sqrt{-1}\pi\gamma})(\alpha + \beta - \gamma)^2}
\end{aligned}$$

and

$$\begin{aligned}
(3.5) \quad S((2, 1, 1), (y_1, y_2); \Lambda) &= \lim_{N \rightarrow \infty} \sum_{\substack{m, n \in \mathbb{Z} \\ |m|, |n| \leq N}} \frac{e^{2\pi\sqrt{-1}(my_1 + ny_2)}}{(m + \alpha)^2(n + \beta)^1(m + n + \gamma)^1} \\
&= \frac{-(2\pi\sqrt{-1})^2}{2!} - \frac{(2\pi\sqrt{-1})^1}{1!} - \frac{(2\pi\sqrt{-1})^1}{1!} C((2, 1, 1), (y_1, y_2); \Lambda).
\end{aligned}$$

If $\alpha = 0$ and $\beta, \gamma \notin \mathbb{Z}$ with $\beta \neq \gamma$, we have

$$\begin{aligned}
(3.6) \quad C((0, 1, 2), (y_1, y_2); \Lambda) &= \frac{\sqrt{-1}\{y_2\}e^{-2\sqrt{-1}\pi\gamma\{y_2\}}}{2(-1 + e^{-2\sqrt{-1}\pi\gamma})\pi(\beta - \gamma)} - \frac{\sqrt{-1}e^{-2\sqrt{-1}\pi\gamma(1 + \{y_2\})}}{2(-1 + e^{-2\sqrt{-1}\pi\gamma})^2\pi(\beta - \gamma)} \\
&+ \frac{e^{-2\sqrt{-1}\pi\beta\{y_2\}}}{4(-1 + e^{-2\sqrt{-1}\pi\beta})\pi^2(\beta - \gamma)^2} - \frac{e^{-2\sqrt{-1}\pi\gamma\{y_2\}}}{4(-1 + e^{-2\sqrt{-1}\pi\gamma})\pi^2(\beta - \gamma)^2}
\end{aligned}$$

and

$$\begin{aligned}
(3.7) \quad S((0, 1, 2), (y_1, y_2); \Lambda) &= - \lim_{N \rightarrow \infty} \sum_{\substack{m, n \in \mathbb{Z} \\ |m|, |n| \leq N \\ m=0}} \frac{e^{2\pi\sqrt{-1}(my_1 + ny_2)}}{(n + \beta)^1(m + n + \gamma)^2} \\
&= - \lim_{N \rightarrow \infty} \sum_{\substack{n \in \mathbb{Z} \\ |n| \leq N}} \frac{e^{2\pi\sqrt{-1}ny_2}}{(n + \beta)^1(n + \gamma)^2} \\
&= \frac{-(2\pi\sqrt{-1})^0}{0!} \frac{-(2\pi\sqrt{-1})^1}{1!} \frac{-(2\pi\sqrt{-1})^2}{2!} C((0, 1, 2), (y_1, y_2); \Lambda).
\end{aligned}$$

Example 3.2. Let $V = \mathbb{R}$. Let

$$(3.8) \quad \Lambda = \Lambda_\alpha = \{f_{-1} = (-1, \alpha), f_0 = (1, 0), f_1 = (1, \alpha)\},$$

$$(3.9) \quad \mathcal{B} = \{\{f_{-1}\}, \{f_0\}, \{f_1\}\},$$

where $\alpha \in \mathbb{C} \setminus \{0\}$, which corresponds to the series

$$(3.10) \quad S(\mathbf{k}, y; \Lambda_\alpha) = \lim_{N \rightarrow \infty} \sum_{\substack{m \in \mathbb{Z} \setminus \{0, \pm\alpha\} \\ |m| \leq N}} \frac{e^{2\pi\sqrt{-1}my}}{(-m + \alpha)^{k_{-1}} m^{k_0} (m + \alpha)^{k_1}},$$

where $k_{-1}, k_0, k_1 \in \mathbb{N}$. Then the generating function is given by

$$\begin{aligned}
(3.11) \quad F((t_{-1}, t_0, t_1), y; \Lambda_\alpha) &= \\
&\frac{t_0}{t_0 + (t_{-1} - 2\pi\sqrt{-1}\alpha)} \frac{t_1}{t_1 - 2\pi\sqrt{-1}\alpha + (t_{-1} - 2\pi\sqrt{-1}\alpha)} \frac{t_{-1}e^{(t_{-1} - 2\pi\sqrt{-1}\alpha)(1 - \{y\})}}{e^{t_{-1} - 2\pi\sqrt{-1}\alpha} - 1} \\
&+ \frac{t_{-1}}{t_{-1} - 2\pi\sqrt{-1}\alpha + t_0} \frac{t_1}{t_1 - 2\pi\sqrt{-1}\alpha - t_0} \frac{t_0 e^{t_0 \{y\}}}{e^{t_0} - 1} \\
&+ \frac{t_{-1}}{t_{-1} - 2\pi\sqrt{-1}\alpha + (t_1 - 2\pi\sqrt{-1}\alpha)} \frac{t_0}{t_0 - (t_1 - 2\pi\sqrt{-1}\alpha)} \frac{t_1 e^{(t_1 - 2\pi\sqrt{-1}\alpha)\{y\}}}{e^{t_1 - 2\pi\sqrt{-1}\alpha} - 1}.
\end{aligned}$$

Then for $\alpha \notin \mathbb{Z}$,

$$\begin{aligned}
(3.12) \quad C((2, 2, 2), y; \Lambda_\alpha) &= -\frac{1}{4\pi^6\alpha^6} + \frac{1}{24\pi^4\alpha^4} - \frac{\{y\}}{4\pi^4\alpha^4} + \frac{\{y\}^2}{4\pi^4\alpha^4} - \frac{3\sqrt{-1}e^{2\pi\sqrt{-1}\alpha\{y\}}}{16\pi^5\alpha^5(-1 + e^{2\pi\sqrt{-1}\alpha})^2} \\
&- \frac{3\sqrt{-1}e^{2\pi\sqrt{-1}\alpha(1 - \{y\})}}{16\pi^5\alpha^5(-1 + e^{2\pi\sqrt{-1}\alpha})^2} + \frac{3\sqrt{-1}e^{2\pi\sqrt{-1}\alpha(2 - \{y\})}}{16\pi^5\alpha^5(-1 + e^{2\pi\sqrt{-1}\alpha})^2} + \frac{3\sqrt{-1}e^{2\pi\sqrt{-1}\alpha(\{y\} + 1)}}{16\pi^5\alpha^5(-1 + e^{2\pi\sqrt{-1}\alpha})^2} \\
&- \frac{\{y\}e^{2\pi\sqrt{-1}\alpha\{y\}}}{8\pi^4\alpha^4(-1 + e^{2\pi\sqrt{-1}\alpha})^2} + \frac{\{y\}e^{2\pi\sqrt{-1}\alpha(1 - \{y\})}}{8\pi^4\alpha^4(-1 + e^{2\pi\sqrt{-1}\alpha})^2} - \frac{\{y\}e^{2\pi\sqrt{-1}\alpha(2 - \{y\})}}{8\pi^4\alpha^4(-1 + e^{2\pi\sqrt{-1}\alpha})^2} \\
&+ \frac{\{y\}e^{2\pi\sqrt{-1}\alpha(\{y\} + 1)}}{8\pi^4\alpha^4(-1 + e^{2\pi\sqrt{-1}\alpha})^2} - \frac{e^{2\pi\sqrt{-1}\alpha(1 - \{y\})}}{8\pi^4\alpha^4(-1 + e^{2\pi\sqrt{-1}\alpha})^2} - \frac{e^{2\pi\sqrt{-1}\alpha(\{y\} + 1)}}{8\pi^4\alpha^4(-1 + e^{2\pi\sqrt{-1}\alpha})^2}
\end{aligned}$$

and for $\alpha \in \mathbb{Z}$,

$$(3.13) \quad \begin{aligned} C((2, 2, 2), y; \Lambda_\alpha) = & -\frac{1}{4\pi^6\alpha^6} + \frac{1}{24\pi^4\alpha^4} - \frac{\{y\}}{4\pi^4\alpha^4} + \frac{\{y\}^2}{4\pi^4\alpha^4} - \frac{3\sqrt{-1}\{y\}e^{-2\pi\sqrt{-1}\alpha\{y\}}}{16\pi^5\alpha^5} \\ & + \frac{3\sqrt{-1}\{y\}e^{2\pi\sqrt{-1}\alpha\{y\}}}{16\pi^5\alpha^5} + \frac{3\sqrt{-1}e^{-2\pi\sqrt{-1}\alpha\{y\}}}{32\pi^5\alpha^5} - \frac{3\sqrt{-1}e^{2\pi\sqrt{-1}\alpha\{y\}}}{32\pi^5\alpha^5} \\ & + \frac{\{y\}^2e^{-2\pi\sqrt{-1}\alpha\{y\}}}{16\pi^4\alpha^4} + \frac{\{y\}^2e^{2\pi\sqrt{-1}\alpha\{y\}}}{16\pi^4\alpha^4} - \frac{\{y\}e^{-2\pi\sqrt{-1}\alpha\{y\}}}{16\pi^4\alpha^4} - \frac{\{y\}e^{2\pi\sqrt{-1}\alpha\{y\}}}{16\pi^4\alpha^4} \\ & - \frac{23e^{-2\pi\sqrt{-1}\alpha\{y\}}}{128\pi^6\alpha^6} - \frac{23e^{2\pi\sqrt{-1}\alpha\{y\}}}{128\pi^6\alpha^6} + \frac{e^{-2\pi\sqrt{-1}\alpha\{y\}}}{96\pi^4\alpha^4} + \frac{e^{2\pi\sqrt{-1}\alpha\{y\}}}{96\pi^4\alpha^4}. \end{aligned}$$

For example, setting $y = 0$ and $\alpha = 1, 2, 3$, we obtain

$$\begin{aligned} S((2, 2, 2), 0; \Lambda_1) &= \sum_{m \in \mathbb{Z} \setminus \{0, \pm 1\}} \frac{1}{(-m+1)^2 m^2 (m+1)^2} = \frac{1}{2}\pi^2 - \frac{39}{8}, \\ S((2, 2, 2), 0; \Lambda_2) &= \sum_{m \in \mathbb{Z} \setminus \{0, \pm 2\}} \frac{1}{(-m+2)^2 m^2 (m+2)^2} = \frac{1}{32}\pi^2 - \frac{39}{512}, \\ S((2, 2, 2), 0; \Lambda_3) &= \sum_{m \in \mathbb{Z} \setminus \{0, \pm 3\}} \frac{1}{(-m+3)^2 m^2 (m+3)^2} = \frac{1}{162}\pi^2 - \frac{13}{1944}. \end{aligned}$$

Similarly, computing $C((2k, 2k, 2k), 0; \Lambda_\alpha)$, we can obtain

$$\begin{aligned} S((4, 4, 4), 0; \Lambda_1) &= \frac{1}{40}\pi^4 + \frac{35}{16}\pi^2 - \frac{3075}{128}, \\ S((6, 6, 6), 0; \Lambda_2) &= \frac{11}{20643840}\pi^6 + \frac{21}{2097152}\pi^4 + \frac{3003}{16777216}\pi^2 - \frac{137067}{268435456}, \\ S((8, 8, 8), 0; \Lambda_3) &= \frac{43}{8678218953600}\pi^8 + \frac{367}{7810397058240}\pi^6 + \frac{581}{1983592903680}\pi^4 \\ &+ \frac{46189}{21422803359744}\pi^2 - \frac{2864587}{1028294561267712}. \end{aligned}$$

Here we define the zeta-function associated with Λ_α by

$$(3.14) \quad \zeta((s_1, s_2, s_3); \Lambda_\alpha) = \sum_{\substack{m=1 \\ m \neq \pm\alpha}}^{\infty} \frac{1}{(-m+\alpha)^{s_1} m^{s_2} (m+\alpha)^{s_3}},$$

which can be regarded as a Hurwitz-type analogue of the Riemann zeta-function, that is, with a shifting parameter α . We can easily check that $S((2k, 2k, 2k), 0; \Lambda_\alpha) = 2\zeta((2k, 2k, 2k); \Lambda_\alpha)$ for $k \in \mathbb{N}$. Therefore we obtain from the above results that, for example,

$$\begin{aligned} \zeta((2, 2, 2); \Lambda_1) &= \frac{1}{4}\pi^2 - \frac{39}{16}, \\ \zeta((4, 4, 4); \Lambda_1) &= \frac{1}{80}\pi^4 + \frac{35}{32}\pi^2 - \frac{3075}{256}, \\ \zeta((6, 6, 6); \Lambda_2) &= \frac{11}{41287680}\pi^6 + \frac{21}{4194304}\pi^4 + \frac{3003}{33554432}\pi^2 - \frac{137067}{536870912}. \end{aligned}$$

Example 3.3. Let $V = \mathbb{R}^2$, and $\alpha \in \mathbb{C} \setminus \{0\}$. Let

$$(3.15) \quad \Lambda = \Lambda_\alpha = \{\{f_{1j}\}_j, \{f_{2j}\}_j, \{f_{3j}\}_j\} \\ = \{ \{(-1, 0, \alpha), (1, 0, 0), (1, 0, \alpha)\}, \{(0, -1, \alpha), (0, 1, 0), (0, 1, \alpha)\}, \{(-1, -1, \alpha), (1, 1, 0), (1, 1, \alpha)\} \},$$

$$(3.16) \quad \mathcal{B} = \{\{f_{1j}, f_{2l}\}_{j,l}, \{f_{1j}, f_{3l}\}_{j,l}, \{f_{2j}, f_{3l}\}_{j,l}\}.$$

Set $\mathbf{y} = 0$ and

$$S(\{k_j\}_{1 \leq j \leq 9}, 0; \Lambda_\alpha) = \lim_{N \rightarrow \infty} \sum_{\substack{m, n \in \mathbb{Z} \setminus \{0, \pm\alpha\} \\ m+n \neq 0, \pm\alpha \\ |m|, |n| \leq N}} \frac{1}{(-m + \alpha)^{k_1} m^{k_2} (m + \alpha)^{k_3}} \\ \times \frac{1}{(-n + \alpha)^{k_4} n^{k_5} (n + \alpha)^{k_6} (-m + n + \alpha)^{k_7} (m + n)^{k_8} (m + n + \alpha)^{k_9}}.$$

Then, computing $C(\{k_j\}, 0; \Lambda_\alpha)$, we obtain, for example,

$$S((1, 2, 2, 2, 2, 1, 1, 1, 2, 2), 0; \Lambda_1) = \frac{1}{1890} \pi^6 + \frac{701}{2160} \pi^4 - \frac{1841}{108} \pi^2 + \frac{2822557}{20736}, \\ S((2, 2, 2, 2, 2, 2, 2, 2, 2, 2), 0; \Lambda_2) = \frac{11}{15482880} \pi^6 + \frac{4901}{70778880} \pi^4 - \frac{26747}{28311552} \pi^2 + \frac{20643217}{10871635968}, \\ S((1, 1, 1, 2, 2, 2, 1, 1, 1), 0; \Lambda_3) = \frac{2}{295245} \pi^4 - \frac{227}{6377292} \pi^2 + \frac{14183}{459165024}.$$

Similarly to Example 3.2, we define the zeta-function associated with Λ by

$$(3.17) \quad \zeta_2(\{s_j\}_{1 \leq j \leq 9}; \Lambda_\alpha) = \sum_{\substack{m, n=1 \\ m \neq \pm\alpha \\ n \neq \pm\alpha \\ m+n \neq \pm\alpha}}^{\infty} \frac{1}{(-m + \alpha)^{s_1} m^{s_2} (m + \alpha)^{s_3}} \\ \times \frac{1}{(-n + \alpha)^{s_4} n^{s_5} (n + \alpha)^{s_6} (-m + n + \alpha)^{s_7} (m + n)^{s_8} (m + n + \alpha)^{s_9}},$$

which can be regarded as a Hurwitz-type analogue of the zeta-function of the root system of type A_2 defined by

$$(3.18) \quad \zeta_2((s_1, s_2, s_3); A_2) = \sum_{m, n=1}^{\infty} \frac{1}{m^{s_1} n^{s_2} (m + n)^{s_3}}$$

(see [5, Section 2] [7, Section 11.7, Example 2]). Note that (3.18) is also called the Tornheim double sum or the Mordell-Tornheim double zeta-function (see, for example, [9, 10]). We already studied certain Hurwitz-type analogues of zeta-functions of root systems in [6, Section 8]. From the viewpoint of root systems, we can regard $S(\{2k\}_{1 \leq j \leq 9}, 0; \Lambda_\alpha)$ is the sum of zeta values $\zeta_2(\{2k\}_{1 \leq j \leq 9}; \Lambda_\alpha)$ under the action of the Weyl group of type A_2 ($\simeq S_3$). This implies that

$$S(\{2k\}_{1 \leq j \leq 9}, 0; \Lambda_\alpha) = 6\zeta_2(\{2k\}_{1 \leq j \leq 9}; \Lambda_\alpha) \quad (k \in \mathbb{N}).$$

Therefore, as an analogue of $\zeta_2((2, 2, 2); A_2) = \pi^6/2835$, we obtain from the above result that

$$\zeta_2((2, 2, 2, 2, 2, 2, 2, 2, 2); \Lambda_2) = \frac{11}{92897280} \pi^6 + \frac{4901}{424673280} \pi^4 - \frac{26747}{169869312} \pi^2 + \frac{20643217}{65229815808}.$$

Remark 3.4. We give another interpretation of the series (3.14) and (3.17), that is, we regard each term of these series as a product of positive roots of affine root system $A_1^{(1)}$ and $A_2^{(1)}$ respectively (for the theory of affine root systems, see [3]). Since there are infinitely many positive roots in affine root systems, the product consists of infinitely many factors. In order for the infinite product to make sense, we understand that infinitely many variables are set to be zero and hence the product is truncated.

4. PROOF OF THEOREM 2.2

Now we start the proofs of the main theorems. First of all, in this section, we prove Theorem 2.2. The main body of the argument is the proof of an evaluation formula (Proposition 4.1) for $S(\mathbf{k}, \mathbf{y}; \Lambda)$.

For $t, b \in \mathbb{C}$ and $y \in \mathbb{R}$ let

$$(4.1) \quad F(t, y; b) = \frac{t e^{(t-2\pi\sqrt{-1}b)y}}{e^{t-2\pi\sqrt{-1}b} - 1} = \sum_{k=0}^{\infty} C(k, y; b) \frac{t^k}{k!},$$

where the right-hand side converges when $|t|$ is sufficiently small.

It is to be noted that $F(t, \{y\}; b)$ (resp. $C(k, \{y\}; b)$) is just the special case $r = 1$, $\Lambda = \{(1, b)\} = B$ of $F(\mathbf{t}, \mathbf{y}; \Lambda)$ defined by (2.7) (resp. $C(\mathbf{k}, \mathbf{y}; \Lambda)$ defined by (2.9)).

Proposition 4.1. *For $\mathbf{k} = (k_f)_{f \in \Lambda} \in \mathbb{N}_0^{\#\Lambda}$, $\mathbf{y} \in V \setminus \bigcup_{f \in \tilde{\Lambda} \cap \Lambda_1} (\mathfrak{H}_{\Lambda \setminus \{f\}} + \mathbb{Z}^r)$, the series (2.1) converges. For a fixed decomposition $\Lambda = B_0 \cup L_0$ with $B_0 = \{f_1, \dots, f_r\} \in \mathcal{B}$, we have*

$$(4.2) \quad S(\mathbf{k}, \mathbf{y}; \Lambda) = \frac{1}{\#(\mathbb{Z}^r / \langle \vec{B}_0 \rangle)} \prod_{f \in \Lambda} \left(-\frac{(2\pi\sqrt{-1})^{k_f}}{k_f!} \right) \sum_{\mathbf{w} \in \mathbb{Z}^r / \langle \vec{B}_0 \rangle} \int_0^1 \cdots \int_0^1 \left(\prod_{g \in L_0} C(k_g, x_g; \dot{g}) dx_g \right) \\ \times \prod_{f \in B_0} \left(C(k_f, \{\mathbf{y} + \mathbf{w} - \sum_{g \in L_0} x_g \vec{g}\}_{B_0, f}; \dot{f}) \right).$$

This is a generalization of [7, Theorem 6] (for integral values of \mathbf{k}). Only the case in the domain of absolute convergence was considered in [7, Theorem 6], so there was no problem of convergence. In our present situation, if $k_f \geq 2$ for all $f \in B$ with some fixed $B \in \mathcal{B}$, then the matter of convergence is again obvious, so it is easy to prove our claims. However if $k_f = 1$ for sufficiently many $f \in \Lambda$, then there are subtle problems on convergence, and the proof becomes much more complicated. It should be remarked that the key of the convergence of $S(\mathbf{k}, \mathbf{y}; \Lambda)$ is the condition $\text{rank} \langle \vec{\Lambda} \rangle = r$.

Since it is difficult to treat (2.1) directly, in the following we consider a little modified sum

$$(4.3) \quad S_1(\mathbf{k}, \mathbf{y}; \Lambda; B_0) = \lim_{N \rightarrow \infty} Z_1(N; \mathbf{k}, \mathbf{y}; \Lambda; B_0),$$

where

$$(4.4) \quad Z_1(N; \mathbf{k}, \mathbf{y}; \Lambda; B_0) = (-1)^{\#\Lambda_0} \sum_{\substack{\mathbf{v} \in \mathbb{Z}^r \\ |\text{Re } f(\mathbf{v})| \leq N \\ f(\mathbf{v}) \neq 0 \quad (f \in \Lambda_+) \\ f(\mathbf{v}) = 0 \quad (f \in \Lambda_0)}} e^{2\pi\sqrt{-1}\langle \mathbf{y}, \mathbf{v} \rangle} \prod_{f \in \Lambda_+} \frac{1}{f(\mathbf{v})^{k_f}}.$$

That is, the condition $|v_j| \leq N$ for $1 \leq j \leq r$ in the definition of $S(\mathbf{k}, \mathbf{y}; \Lambda)$ is replaced by $|\text{Re } f(\mathbf{v})| \leq N$ for $f \in B_0$. At the last stage of the proof we will show that $S(\mathbf{k}, \mathbf{y}; \Lambda) = S_1(\mathbf{k}, \mathbf{y}; \Lambda; B_0)$. In particular, we will find that $S_1(\mathbf{k}, \mathbf{y}; \Lambda; B_0)$ actually does not depend on the choice of B_0 .

The proof of Proposition 4.1 consists of three steps.

The first step. We first consider the simplest case of (4.3), which corresponds to $r = 1$ and $\Lambda = \{(1, b)\} = B$ with $\mathcal{B} = \{B\}$, in Lemmas 4.2 and 4.4.

Lemma 4.2. *For $b \in \mathbb{C}$, $y \in \mathbb{R}$ and $k \in \mathbb{N}$ with $k \geq 2$, we have*

$$(4.5) \quad \lim_{N \rightarrow \infty} \left(\sum_{\substack{n \in \mathbb{Z} \\ |n + \text{Re } b| \leq N \\ n + b \neq 0}} \frac{e^{2\pi\sqrt{-1}ny}}{(n + b)^k} \right) = -\frac{(2\pi\sqrt{-1})^k}{k!} C(k, \{y\}; b),$$

$$(4.6) \quad \lim_{N \rightarrow \infty} \left(- \sum_{\substack{n \in \mathbb{Z} \\ |n + \text{Re } b| \leq N \\ n + b = 0}} e^{2\pi\sqrt{-1}ny} \right) = -C(0, \{y\}; b).$$

(Actually the sum on the left-hand side of (4.6) consists of at most one term.) The series above converge absolutely uniformly in y and hence $C(k, \{y\}; b)$ and $C(0, \{y\}; b)$ are continuous in y .

Proof. Let $\gamma_{X, Y}$ be the counterclockwise rectangle contour with vertices at $\pm X \pm 2\pi\sqrt{-1}Y$. Applying the Cauchy theorem to the integral

$$(4.7) \quad \lim_{N \rightarrow \infty} \lim_{M \rightarrow \infty} \int_{\gamma_{M, N+\epsilon}} F(t, \{y\}; b) t^{-k-1} dt \quad (k \geq 2)$$

with a sufficiently small $\epsilon > 0$, we see that the sum of all the residues vanishes, namely,

$$(4.8) \quad \frac{C(k, \{y\}; b)}{k!} + \frac{1}{(2\pi\sqrt{-1})^k} \lim_{N \rightarrow \infty} \sum_{\substack{n \in \mathbb{Z} \\ |n + \operatorname{Re} b| \leq N \\ n + b \neq 0}} \frac{e^{2\pi\sqrt{-1}ny}}{(n+b)^k} = 0,$$

and hence (4.5).

The left-hand side of (4.6) consists of only one term $e^{-2\pi\sqrt{-1}by}$ if $b \in \mathbb{Z}$, and vanishes if $b \notin \mathbb{Z}$, while

$$(4.9) \quad C(0, \{y\}; b) = F(0, \{y\}; b) = \begin{cases} e^{-2\pi\sqrt{-1}by} & (b \in \mathbb{Z}), \\ 0 & (b \notin \mathbb{Z}) \end{cases}$$

and hence (4.6).

Both in (4.5) and (4.6), the absolute uniform convergence of the series in y is clear. \square

The case $k = 1$ is more subtle. We first prepare the following

Lemma 4.3. *For $\mu > 0$ there exists $K > 0$ such that for $a, z > 0$*

$$(4.10) \quad \int_0^\infty \frac{e^{-xz}}{\sqrt{x^2 + a^2}} dx \leq K(az)^{-\frac{1}{\mu+1}}.$$

Proof. Rewrite

$$(4.11) \quad \int_0^\infty \frac{e^{-xz}}{\sqrt{x^2 + a^2}} dx = \int_0^\infty \frac{e^{-x}}{\sqrt{x^2 + (az)^2}} dx.$$

By the inequality of weighted arithmetic and geometric means

$$(4.12) \quad \mu A + B \geq (\mu + 1)(A^\mu B)^{\frac{1}{\mu+1}} \quad (\mu, A, B > 0),$$

we have

$$(4.13) \quad \sqrt{x^2 + (az)^2} \geq \sqrt{\mu + 1} \mu^{-\frac{\mu}{2(\mu+1)}} x^{\frac{\mu}{\mu+1}} (az)^{\frac{1}{\mu+1}}$$

and

$$(4.14) \quad \begin{aligned} \int_0^\infty \frac{e^{-x}}{\sqrt{x^2 + (az)^2}} dx &\leq \frac{\mu^{\frac{\mu}{2(\mu+1)}}}{\sqrt{\mu + 1}} (az)^{-\frac{1}{\mu+1}} \int_0^\infty e^{-x} x^{-\frac{\mu}{\mu+1}} dx \\ &\leq \left(\frac{\mu^{\frac{\mu}{2(\mu+1)}}}{\sqrt{\mu + 1}} \Gamma\left(\frac{1}{\mu + 1}\right) \right) (az)^{-\frac{1}{\mu+1}}. \end{aligned}$$

\square

Let

$$(4.15) \quad \delta_{N < 0 < M} = \begin{cases} 1 & (N < 0 < M), \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.4. *For $b \in \mathbb{C}$ and $y \in \mathbb{R} \setminus \mathbb{Z}$ we have*

$$(4.16) \quad \lim_{N \rightarrow \infty} \left(\sum_{\substack{n \in \mathbb{Z} \\ |n + \operatorname{Re} b| \leq N \\ n + b \neq 0}} \frac{e^{2\pi\sqrt{-1}ny}}{n+b} \right) = -2\pi\sqrt{-1}C(1, \{y\}; b)$$

and $C(1, \{y\}; b)$ is continuous in y . Moreover for any $\mu > 0$ there exists $K > 0$ such that for all $y \in \mathbb{R} \setminus \mathbb{Z}$ and all $M, N \in \mathbb{R}$ with sufficiently large $|M|, |N|$ and $M \geq |N|$

$$(4.17) \quad \left| \sum_{\substack{n \in \mathbb{Z} \\ N \leq n + \operatorname{Re} b \leq M \\ n + b \neq 0}} \frac{e^{2\pi\sqrt{-1}ny}}{n+b} \right| \leq K|N|^{-\frac{1}{\mu+1}} ((1 - \{y\})^{-\frac{1}{\mu+1}} + \{y\}^{-\frac{1}{\mu+1}}) + \delta_{N < 0 < M} K.$$

Proof. From (4.1) we can easily see that

$$(4.18) \quad C(1, \{y\}; b) = \begin{cases} \left(\{y\} - \frac{1}{2}\right) e^{-2\pi\sqrt{-1}b\{y\}} & (b \in \mathbb{Z}), \\ \frac{e^{-2\pi\sqrt{-1}b\{y\}}}{e^{-2\pi\sqrt{-1}b} - 1} & (b \notin \mathbb{Z}), \end{cases}$$

from which the continuity of $C(1, \{y\}; b)$ follows.

Let γ_Y be the horizontal path from $-\infty + 2\pi\sqrt{-1}Y$ to $\infty + 2\pi\sqrt{-1}Y$. Then for all $y \in \mathbb{R} \setminus \mathbb{Z}$, we have

$$(4.19) \quad \begin{aligned} & \frac{1}{2\pi\sqrt{-1}} \left(- \int_{\gamma_{M+\epsilon}} F(t, \{y\}; b) t^{-2} dt + \int_{\gamma_{N-\epsilon}} F(t, \{y\}; b) t^{-2} dt \right) - \delta_{N < 0 < M} C(1, \{y\}; b) \\ &= \frac{1}{2\pi\sqrt{-1}} \sum_{\substack{n \in \mathbb{Z} \\ N \leq n + \operatorname{Re} b \leq M \\ n + b \neq 0}} \frac{e^{2\pi\sqrt{-1}ny}}{n + b}. \end{aligned}$$

On the other hand, for $L \in \mathbb{Z}$

$$(4.20) \quad \begin{aligned} \left| \int_{\gamma_{L \pm \epsilon}} F(t, \{y\}; b) t^{-2} dt \right| &\leq \int_{-\infty}^{\infty} \left| \frac{e^{(x+2\pi\sqrt{-1}(L \pm \epsilon) - 2\pi\sqrt{-1}b)\{y\}}}{e^{x+2\pi\sqrt{-1}(L \pm \epsilon) - 2\pi\sqrt{-1}b} - 1} \frac{1}{x + 2\pi\sqrt{-1}(L \pm \epsilon)} \right| dx \\ &\leq \int_{-\infty}^{\infty} \frac{e^{x\{y\}} e^{2\pi \operatorname{Im} b\{y\}}}{|e^{x-2\pi\sqrt{-1}(b \pm \epsilon)} - 1| |x + 2\pi\sqrt{-1}(L \pm \epsilon)|} dx \\ &= \int_{-\infty}^{\infty} \frac{e^{x\{y\}} e^{2\pi |\operatorname{Im} b|}}{|e^{x-2\pi\sqrt{-1}(b \pm \epsilon)} - 1| \sqrt{x^2 + 4\pi^2(L \pm \epsilon)^2}} dx. \end{aligned}$$

It is easy to see that there exists $K' > 0$ independent of y such that

$$(4.21) \quad \frac{e^{x\{y\}} e^{2\pi |\operatorname{Im} b|}}{|e^{x-2\pi\sqrt{-1}(b \pm \epsilon)} - 1|} \leq g(x, y) := \begin{cases} K' e^{x\{y\}} & (x < 0), \\ K' e^{x(\{y\}-1)} & (x \geq 0). \end{cases}$$

Applying (4.21) and Lemma 4.3 to (4.20), we have

$$(4.22) \quad \begin{aligned} \left| \int_{\gamma_{L \pm \epsilon}} F(t, \{y\}; b) t^{-2} dt \right| &\leq \int_{-\infty}^{\infty} \frac{g(x, y)}{\sqrt{x^2 + 4\pi^2(L \pm \epsilon)^2}} dx \\ &\leq K'' |L|^{-\frac{1}{\mu+1}} ((1 - \{y\})^{-\frac{1}{\mu+1}} + \{y\}^{-\frac{1}{\mu+1}}) \end{aligned}$$

for some $K'' > 0$. Therefore, choosing $N = -L$ and $M = L$ in (4.19) and taking the limit $L \rightarrow \infty$, we obtain (4.16). Moreover, since $C(1, \{y\}; b)$ is bounded in y , we obtain (4.17). \square

The second step. Secondly we consider the higher rank case of (4.3) under the special condition $\Lambda = B$ with $\mathcal{B} = \{B\}$ in Lemmas 4.7, 4.8 and 4.10. We first prepare the following algebraic lemma. This statement is included in [2, Chapitre 6, Section 1, 9], but here we supply a proof.

Lemma 4.5. *Let Q, P be free \mathbb{Z} -modules of rank r with $Q \subset P$ so that P/Q is a finite abelian group. Then*

$$(4.23) \quad \operatorname{Hom}(P/Q, \mathbb{Q}/\mathbb{Z}) \simeq \operatorname{Hom}(Q, \mathbb{Z}) / \operatorname{Hom}(P, \mathbb{Z})$$

and for $\bar{\lambda} \in P/Q$, we have

$$(4.24) \quad \frac{1}{\sharp(P/Q)} \sum_{\bar{f} \in \operatorname{Hom}(Q, \mathbb{Z}) / \operatorname{Hom}(P, \mathbb{Z})} e^{2\pi\sqrt{-1}\bar{f}(\bar{\lambda})} = \delta_{\bar{\lambda}, 0},$$

where the right-hand side denotes Kronecker's delta.

Proof. First we note that an element of $\operatorname{Hom}(P, \mathbb{Z})$ can be naturally regarded as an element of $\operatorname{Hom}(Q, \mathbb{Z})$. Denote this injection by ι . Next, let $f \in \operatorname{Hom}(Q, \mathbb{Z})$. It is well-known that there exist a basis $\{\lambda_i\}_{i=1}^r$ of P and a basis $\{\lambda'_i\}_{i=1}^r$ of Q such that $\lambda'_i = k_i \lambda_i$ with $k_i \in \mathbb{N}$ and hence

$$(4.25) \quad P/Q = \langle \bar{\lambda}_1 \rangle \oplus \cdots \oplus \langle \bar{\lambda}_r \rangle,$$

where each $\langle \bar{\lambda}_i \rangle$ is a cyclic group of order k_i with $\bar{\lambda}_i \in P/Q$. Define $\varphi(f)$ by the linear extension of

$$(4.26) \quad \varphi(f)(\bar{\lambda}_i) = p(f(\lambda'_i)/k_i),$$

where p denotes the natural projection $\mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}$. Then $\varphi(f) \in \text{Hom}(P/Q, \mathbb{Q}/\mathbb{Z})$ is well-defined.

We show that the sequence

$$(4.27) \quad 0 \rightarrow \text{Hom}(P, \mathbb{Z}) \xrightarrow{\iota} \text{Hom}(Q, \mathbb{Z}) \xrightarrow{\varphi} \text{Hom}(P/Q, \mathbb{Q}/\mathbb{Z}) \rightarrow 0$$

is exact. First show the surjectivity of φ . Let $g \in \text{Hom}(P/Q, \mathbb{Q}/\mathbb{Z})$. Then choose a representative $a_i \in \mathbb{Q}$ of $g(\bar{\lambda}_i)$. Since $k_i g(\bar{\lambda}_i) = g(\bar{\lambda}'_i) = g(0) = 0$, we have $k_i a_i \in \mathbb{Z}$. Define $f \in \text{Hom}(Q, \mathbb{Z})$ by the linear extension of $f(\lambda'_i) = k_i a_i$. Then $\varphi(f)(\bar{\lambda}_i) = p(a_i) = g(\bar{\lambda}_i)$, so $g = \varphi(f)$. Therefore φ is surjective. Next, let $f \in \ker \varphi$. Then $\varphi(f)(\bar{\lambda}_i) = p(f(\lambda'_i)/k_i) = 0$ and so $f(k_i \lambda_i) \in k_i \mathbb{Z}$, that is, $f(\lambda_i) \in \mathbb{Z}$. This implies $f \in \text{Hom}(P, \mathbb{Z})$, and hence the exactness at $\text{Hom}(Q, \mathbb{Z})$ is proved. The assertion (4.23) immediately follows from (4.27), and (4.24) follows from the orthogonality relations of group characters (cf. Apostol [1, Theorem 6.13]). \square

Lemma 4.6. *Let $B \in \mathcal{B}$ and $f \in B$. Then*

$$(4.28) \quad \mathfrak{H}_{B \setminus \{f\}} + \mathbb{Z}^r = \{\mathbf{y} \in V \mid \langle \mathbf{y} + \mathbf{w}, \vec{f}^B \rangle \in \mathbb{Z} \text{ for some } \mathbf{w} \in \mathbb{Z}^r\}.$$

Proof. We denote the left-hand side by P and the right-hand side by Q respectively. If $\mathbf{y} \in P$, then $\mathbf{y} = \mathbf{y}_0 + \mathbf{w}_0$ with $\langle \mathbf{y}_0, \vec{f}^B \rangle = 0$ and $\mathbf{w}_0 \in \mathbb{Z}^r$. By setting $\mathbf{w} = -\mathbf{w}_0$ we have

$$(4.29) \quad \langle \mathbf{y} + \mathbf{w}, \vec{f}^B \rangle = \langle \mathbf{y}_0, \vec{f}^B \rangle = 0 \in \mathbb{Z},$$

and $\mathbf{y} \in Q$. Conversely, if $\mathbf{y} \in Q$, then

$$(4.30) \quad \mathbf{y} + \mathbf{w} \in \mathfrak{H}_{B \setminus \{f\}} + \mathbb{Z} \vec{f}^B \subset \mathfrak{H}_{B \setminus \{f\}} + \mathbb{Z}^r$$

because $\mathfrak{H}_{B \setminus \{f\}}$ is orthogonal to \vec{f}^B . Hence we have $\mathbf{y} \in P$. \square

Lemma 4.7. *Assume $\Lambda = B$ with $\mathcal{B} = \{B\}$. For $\mathbf{k} = (k_f)_{f \in \Lambda} \in \mathbb{N}_0^r$ and $\mathbf{y} \in V \setminus \bigcup_{f \in \Lambda_1} (\mathfrak{H}_{\Lambda \setminus \{f\}} + \mathbb{Z}^r)$, the limit (4.3) (with $B_0 = B$) converges, and we have*

$$(4.31) \quad S_1(\mathbf{k}, \mathbf{y}; \Lambda; B) = \frac{1}{\sharp(\mathbb{Z}^r / \langle \vec{B} \rangle)} \sum_{\mathbf{w} \in \mathbb{Z}^r / \langle \vec{B} \rangle} \prod_{f \in \Lambda} \left(-\frac{(2\pi\sqrt{-1})^{k_f}}{k_f!} C(k_f, \{\mathbf{y} + \mathbf{w}\}_{B, f}; \dot{f}) \right).$$

Proof. Let $A = {}^t(\vec{f})_{f \in B}$ be a regular matrix, where \vec{f} are regarded as column vectors. Then $A^{-1} = (\vec{f}^B)_{f \in B}$. For $\mathbf{v} \in \mathbb{Z}^r$, write $\mathbf{u} = (u_f)_{f \in B} = A\mathbf{v}$ so that $u_f = \langle \vec{f}, \mathbf{v} \rangle$ and $\mathbf{v} = A^{-1}\mathbf{u} = \sum_{f \in B} \vec{f}^B u_f$. We have

$$(4.32) \quad \begin{aligned} Z_1(N; \mathbf{k}, \mathbf{y}; \Lambda; B) &= (-1)^{\sharp \Lambda_0} \sum_{\substack{\mathbf{v} \in \mathbb{Z}^r \\ |\text{Re } f(\mathbf{v})| \leq N \\ f(\mathbf{v}) \neq 0 \\ f(\mathbf{v}) = 0}} (f \in B) \quad e^{2\pi\sqrt{-1}\langle \mathbf{y}, \mathbf{v} \rangle} \prod_{f \in \Lambda_+} \frac{1}{(\langle \vec{f}, \mathbf{v} \rangle + \dot{f})^{k_f}} \\ &= (-1)^{\sharp \Lambda_0} \sum_{\substack{u_f \in \mathbb{Z} \\ |u_f + \text{Re } \dot{f}| \leq N \\ u_f + \dot{f} \neq 0 \\ u_f + \dot{f} = 0}} (f \in B) \quad \iota(\mathbf{u}) e^{2\pi\sqrt{-1}\langle \mathbf{y}, A^{-1}\mathbf{u} \rangle} \prod_{f \in \Lambda_+} \frac{1}{(u_f + \dot{f})^{k_f}}, \end{aligned}$$

where

$$(4.33) \quad \iota(\mathbf{u}) = \begin{cases} 1 & (A^{-1}\mathbf{u} \in \mathbb{Z}^r), \\ 0 & (A^{-1}\mathbf{u} \notin \mathbb{Z}^r). \end{cases}$$

We prove

$$(4.34) \quad \iota(\mathbf{u}) = \frac{1}{\sharp(\mathbb{Z}^r / \langle \vec{B} \rangle)} \sum_{\mathbf{w} \in \mathbb{Z}^r / \langle \vec{B} \rangle} e^{2\pi\sqrt{-1}\langle \mathbf{w}, A^{-1}\mathbf{u} \rangle}.$$

In fact, using Lemma 4.5 with $Q = \mathbb{Z}^r$, $P = \langle \vec{B}^* \rangle$ and noting $\text{Hom}(\mathbb{Z}^r, \mathbb{Z}) \simeq \mathbb{Z}^r$, $\text{Hom}(\langle \vec{B}^* \rangle, \mathbb{Z}) \simeq \langle \vec{B} \rangle$ and $\sharp(\langle \vec{B}^* \rangle / \mathbb{Z}^r) = \sharp(\mathbb{Z}^r / \langle \vec{B} \rangle)$, we have

$$\frac{1}{\sharp(\mathbb{Z}^r / \langle \vec{B} \rangle)} \sum_{\mathbf{w} \in \mathbb{Z}^r / \langle \vec{B} \rangle} e^{2\pi\sqrt{-1}\langle \mathbf{w}, \bar{\lambda} \rangle} = \delta_{\bar{\lambda}, 0}$$

for $\bar{\lambda} \in \langle \vec{B}^* \rangle / \mathbb{Z}^r$. Since $A^{-1}\mathbf{u} = \sum_{f \in B} \vec{f}^B u_f \in \langle \vec{B}^* \rangle$, choosing $\bar{\lambda} = \overline{A^{-1}\mathbf{u}} \in \langle \vec{B}^* \rangle / \mathbb{Z}^r$ in the above, we obtain (4.34).

Using (4.34) we find that (4.32) is equal to

$$\begin{aligned} & \frac{1}{\sharp(\mathbb{Z}^r / \langle \vec{B} \rangle)} (-1)^{\sharp\Lambda_0} \sum_{\mathbf{w} \in \mathbb{Z}^r / \langle \vec{B} \rangle} \sum_{\substack{u_f \in \mathbb{Z} \\ |u_f + \text{Re } \dot{f}| \leq N \quad (f \in B) \\ u_f + \dot{f} \neq 0 \quad (f \in \Lambda_+) \\ u_f + \dot{f} = 0 \quad (f \in \Lambda_0)}} e^{2\pi\sqrt{-1}\langle \mathbf{y} + \mathbf{w}, A^{-1}\mathbf{u} \rangle} \left(\prod_{f \in \Lambda_+} \frac{1}{(u_f + \dot{f})^{k_f}} \right) \\ (4.35) \quad & = \frac{1}{\sharp(\mathbb{Z}^r / \langle \vec{B} \rangle)} \sum_{\mathbf{w} \in \mathbb{Z}^r / \langle \vec{B} \rangle} \prod_{f \in \Lambda_0} \left(- \sum_{\substack{u_f \in \mathbb{Z} \\ |u_f + \text{Re } \dot{f}| \leq N \\ u_f + \dot{f} = 0}} e^{2\pi\sqrt{-1}\langle \mathbf{y} + \mathbf{w}, \vec{f}^B \rangle u_f} \right) \\ & \quad \times \prod_{f \in \Lambda_+} \left(\sum_{\substack{u_f \in \mathbb{Z} \\ |u_f + \text{Re } \dot{f}| \leq N \\ u_f + \dot{f} \neq 0}} \frac{e^{2\pi\sqrt{-1}\langle \mathbf{y} + \mathbf{w}, \vec{f}^B \rangle u_f}}{(u_f + \dot{f})^{k_f}} \right), \end{aligned}$$

where we have used $A^{-1}\mathbf{u} = \sum_{f \in B} \vec{f}^B u_f$. (Note that at present $\Lambda = \Lambda_0 \cup \Lambda_+ = B$.)

By Lemma 4.6 we see that the assumption $\mathbf{y} \in V \setminus \bigcup_{f \in \Lambda_1} (\mathfrak{H}_{\Lambda \setminus \{f\}} + \mathbb{Z}^r)$ implies that $\langle \mathbf{y} + \mathbf{w}, \vec{f}^B \rangle \notin \mathbb{Z}$ for all $f \in \Lambda_1$ and $\mathbf{w} \in \mathbb{Z}^r$. Therefore the condition of Lemma 4.4 is satisfied (for $y = \langle \mathbf{y} + \mathbf{w}, \vec{f}^B \rangle$). Therefore letting $N \rightarrow \infty$ on the right-hand side of (4.35) and applying Lemmas 4.2 and 4.4, we obtain

$$(4.36) \quad S_1(\mathbf{k}, \mathbf{y}; \Lambda; B) = \frac{1}{\sharp(\mathbb{Z}^r / \langle \vec{B} \rangle)} \sum_{\mathbf{w} \in \mathbb{Z}^r / \langle \vec{B} \rangle} \prod_{f \in \Lambda} \left(- \frac{(2\pi\sqrt{-1})^{k_f}}{k_f!} C(k_f, \{\langle \mathbf{y} + \mathbf{w}, \vec{f}^B \rangle\}; \dot{f}) \right).$$

Lastly we note that the factor $\{\langle \mathbf{y} + \mathbf{w}, \vec{f}^B \rangle\}$ on the right-hand side of the above can be replaced by $\{\mathbf{y} + \mathbf{w}\}_{B, \dot{f}}$. This is because $C(k, \{y\}; b) = C(k, 1 - \{-y\}; b)$ for $k \geq 2$ or $k = 0$, while $y \notin \mathbb{Z}$ when $k = 1$ (cf. (2.6)). This completes the proof of the lemma. \square

Lemma 4.7 gives the right-hand side of (4.2) in the special case $\Lambda = B$ and $\mathcal{B} = \{B\}$, but for $S_1(\mathbf{k}, \mathbf{y}; \Lambda; B)$ instead of $S(\mathbf{k}, \mathbf{y}; \Lambda)$. In order to use Lemma 4.7 in the proof of the general case, we need the following inequality.

Lemma 4.8. *Assume $\Lambda = B$ with $\mathcal{B} = \{B\}$. For $\mu > 0$ and $\mathbf{k} = (k_f)_{f \in \Lambda} \in \mathbb{N}_0^r$ there exists $K > 0$ such that for all $\mathbf{y} \in V \setminus \bigcup_{f \in \Lambda_1} (\mathfrak{H}_{\Lambda \setminus \{f\}} + \mathbb{Z}^r)$ and all sufficiently large $N > 0$,*

$$(4.37) \quad |Z_1(N; \mathbf{k}, \mathbf{y}; \Lambda; B)| \leq K \sum_{\mathbf{w} \in \mathbb{Z}^r / \langle \vec{B} \rangle} \prod_{f \in \Lambda_1} (1 + (1 - \{\langle \mathbf{y} + \mathbf{w}, \vec{f}^B \rangle\})^{-\frac{1}{\mu+1}} + \{\langle \mathbf{y} + \mathbf{w}, \vec{f}^B \rangle\}^{-\frac{1}{\mu+1}}).$$

Proof. From the proof of Lemma 4.7, we have

$$(4.38) \quad |Z_1(N; \mathbf{k}, \mathbf{y}; \Lambda; B)| \leq \frac{1}{\#\langle \mathbb{Z}^r / \langle \vec{B} \rangle \rangle} \sum_{\mathbf{w} \in \mathbb{Z}^r / \langle \vec{B} \rangle} \prod_{f \in \Lambda_0} \left| - \sum_{\substack{u_f \in \mathbb{Z} \\ |u_f + \operatorname{Re} \dot{f}| \leq N \\ u_f + \dot{f} = 0}} e^{2\pi\sqrt{-1}\langle \mathbf{y} + \mathbf{w}, \vec{f}^B \rangle u_f} \right| \\ \times \prod_{f \in \Lambda_+} \left| \sum_{\substack{u_f \in \mathbb{Z} \\ |u_f + \operatorname{Re} \dot{f}| \leq N \\ u_f + \dot{f} \neq 0}} \frac{e^{2\pi\sqrt{-1}\langle \mathbf{y} + \mathbf{w}, \vec{f}^B \rangle u_f}}{(u_f + \dot{f})^{k_f}} \right|.$$

On the right-hand side of (4.38), each sum corresponding to $f \in \Lambda_0$ consists of just one term, and each sum corresponding to $k_f \geq 2$ is convergent absolutely as $N \rightarrow \infty$, so all of them are bounded. For $f \in \Lambda_1$, we apply Lemma 4.4 to obtain that the right-hand side of (4.38) is

$$(4.39) \quad \leq K \sum_{\mathbf{w} \in \mathbb{Z}^r / \langle \vec{B} \rangle} \prod_{f \in \Lambda_1} (1 + (1 - \langle \mathbf{y} + \mathbf{w}, \vec{f}^B \rangle)^{-\frac{1}{\mu+1}} + \langle \mathbf{y} + \mathbf{w}, \vec{f}^B \rangle^{-\frac{1}{\mu+1}}).$$

□

To evaluate the difference between $Z(N; \mathbf{k}, \mathbf{y}; \Lambda; B)$ and $Z_1(N; \mathbf{k}, \mathbf{y}; \Lambda; B)$ in the final step of the proof, the following two lemmas are necessary. For $B \in \mathcal{B}$ and $R > 0$, let

$$(4.40) \quad U_R = U_R(B) = \{\mathbf{x} \in \mathbb{R}^r \mid |\operatorname{Re} f(\mathbf{x})| \leq R \quad (f \in B)\},$$

$$(4.41) \quad W_R = \{\mathbf{x} \in \mathbb{R}^r \mid |x_j| \leq R \quad (1 \leq j \leq r)\}.$$

Lemma 4.9. *There exist positive numbers c, d with $c > d > 0$ such that*

$$(4.42) \quad U_{dR}(B) \subset W_R \subset U_{cR}(B)$$

for all sufficiently large $R > 0$.

Proof. We show the first inclusion. Each vertex \mathbf{x} of the parallelotope $U_R(B)$ satisfies one of the following equations

$$(4.43) \quad A\mathbf{x} + (\operatorname{Re} \dot{f})_{f \in B} = (R_f)_{f \in B} \quad (R_f \in \{R, -R\}),$$

where we regard $(\operatorname{Re} \dot{f})_{f \in B}$, $(R_f)_{f \in B}$ and \vec{f} as column vectors respectively and $A = {}^t(\vec{f})_{f \in B}$. Hence we see that the Euclid norm $\|\mathbf{x}\|$ of each vertex $\mathbf{x} = A^{-1}(R_f - \operatorname{Re} \dot{f})_{f \in B}$ satisfies

$$(4.44) \quad \|\mathbf{x}\| = \|A^{-1}\| \|(R_f - \operatorname{Re} \dot{f})_{f \in B}\| \leq \|A^{-1}\| \max_{f \in B} \sqrt{r} |R \pm \operatorname{Re} \dot{f}| \\ \leq R\sqrt{r} \|A^{-1}\| \max_{f \in B} \left| 1 \pm \frac{\operatorname{Re} \dot{f}}{R} \right| \leq 2\sqrt{r} \|A^{-1}\| R$$

for all sufficiently large $R > 0$, where $\|A\|$ denotes the matrix norm of A . Thus choosing

$$(4.45) \quad d = (2\sqrt{r} \|A^{-1}\|)^{-1}$$

we find that the vertex \mathbf{x} of $U_{dR}(B)$ satisfies $\|\mathbf{x}\| \leq R$, so $U_{dR}(B) \subset W_R$.

We show the second inclusion. Denote by $K(R)$ the ball of radius R whose center is the origin. The distance between the origin and the hyperplane $\{\mathbf{x} \in V \mid \langle \vec{f}, \mathbf{x} \rangle + \operatorname{Re} \dot{f} = \pm R\}$ is given by $\frac{|R \pm \operatorname{Re} \dot{f}|}{\|\vec{f}\|}$. Since

$$(4.46) \quad R \min_{f \in B} \frac{1}{2\|\vec{f}\|} \leq \min_{f \in B} \frac{|R \pm \operatorname{Re} \dot{f}|}{\|\vec{f}\|}$$

holds for all sufficiently large $R > 0$, we find that $K(R \min(2\|\vec{f}\|)^{-1}) \subset U_R(B)$. Therefore, choosing

$$(4.47) \quad c^{-1} = \min_{f \in B} \frac{1}{2\sqrt{r}\|\vec{f}\|},$$

we obtain

$$(4.48) \quad W_R \subset K(\sqrt{r}R) = K(cR \min(2\|\vec{f}\|)^{-1}) \subset U_{cR}(B).$$

□

Lemma 4.10. *Assume $\Lambda = B = \{f_1, \dots, f_r\}$ with $\mathcal{B} = \{B\}$. Let c, d be as in Lemma 4.9. For $\mu > 0$ and $\mathbf{k} = (k_f)_{f \in \Lambda} \in \mathbb{N}_0^r$ there exists $K > 0$ such that for all $\mathbf{y} \in V \setminus \bigcup_{f \in \Lambda_1} (\mathfrak{H}_{\Lambda \setminus \{f\}} + \mathbb{Z}^r)$ and all sufficiently large $N \in \mathbb{N}$,*

$$(4.49) \quad \left| \sum_{\substack{\mathbf{v} \in \mathbb{Z}^r \cap (W_N \setminus U_{dN}(B)) \\ f(\mathbf{v}) \neq 0 \quad (f \in \Lambda_+) \\ f(\mathbf{v}) = 0 \quad (f \in \Lambda_0)}} e^{2\pi\sqrt{-1}\langle \mathbf{y}, \mathbf{v} \rangle} \prod_{f \in \Lambda_+} \frac{1}{f(\mathbf{v})^{k_f}} \right| \\ \leq KN^{-\frac{1}{\mu+1}} (\log N)^r \left(1 + \sum_{\mathbf{w} \in \mathbb{Z}^r / \langle \vec{B} \rangle} \sum_{f \in \Lambda_1} \left((1 - \langle \mathbf{y} + \mathbf{w}, \vec{f}^B \rangle)^{-\frac{1}{\mu+1}} + \langle \mathbf{y} + \mathbf{w}, \vec{f}^B \rangle^{-\frac{1}{\mu+1}} \right) \right).$$

Proof. For brevity, we put

$$(4.50) \quad G(\mathbf{y}, \mathbf{v}) = e^{2\pi\sqrt{-1}\langle \mathbf{y}, \mathbf{v} \rangle} \prod_{f \in \Lambda_+} \frac{1}{(\langle \vec{f}, \mathbf{v} \rangle + \dot{f})^{k_f}}.$$

We rearrange $\{f_1, \dots, f_l\} = \Lambda_+$ and $\{f_{l+1}, \dots, f_r\} = \Lambda_0$ and decompose

$$(4.51) \quad \left\{ \mathbf{v} \in \mathbb{Z}^r \cap (W_N \setminus U_{dN}(B)) \left| \begin{array}{l} f(\mathbf{v}) \neq 0 \quad (f \in \Lambda_+), \\ f(\mathbf{v}) = 0 \quad (f \in \Lambda_0) \end{array} \right. \right\} = \bigcup_{j=1}^l X_j(N)$$

with

$$(4.52) \quad X_j(N) = \left\{ \mathbf{v} \in \mathbb{Z}^r \left| \begin{array}{l} |\operatorname{Re} f_1(\mathbf{v})|, \dots, |\operatorname{Re} f_{j-1}(\mathbf{v})| \leq dN, |\operatorname{Re} f_j(\mathbf{v})| > dN, \\ f_i(\mathbf{v}) \neq 0 \quad (1 \leq i \leq l), f_i(\mathbf{v}) = 0 \quad (l+1 \leq i \leq r), \\ |v_k| \leq N \quad (1 \leq k \leq r) \end{array} \right. \right\}$$

for $1 \leq j \leq l$. Let $A = {}^t(\vec{f})_{f \in B}$ with \vec{f} regarded as column vectors. We rewrite the series in terms of $\mathbf{u} = A\mathbf{v}$. Let

$$(4.53) \quad Y_j(N) = \left\{ \mathbf{u} \in \mathbb{Z}^r \left| \begin{array}{l} |u_1 + \operatorname{Re} \dot{f}_1|, \dots, |u_{j-1} + \operatorname{Re} \dot{f}_{j-1}| \leq dN, |u_j + \operatorname{Re} \dot{f}_j| > dN, \\ u_i + \dot{f}_i \neq 0 \quad (1 \leq i \leq l), u_i + \dot{f}_i = 0 \quad (l+1 \leq i \leq r), \\ |\text{each entry of } A^{-1}\mathbf{u}| \leq N \end{array} \right. \right\}$$

for $1 \leq j \leq l$. In $Y_j(N)$ for a fixed $(u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_r)$ with sufficiently large $N \in \mathbb{N}$, we see that u_j runs over all integers such that $dN - \operatorname{Re} \dot{f}_j < u_j < H_j^+$ and $-H_j^- < u_j < -dN - \operatorname{Re} \dot{f}_j$ for some $H_j^\pm = H_j^\pm(u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_r) \geq 0$, where H_j^\pm are determined by the intersection point of the half line $\{(u_1, \dots, u_{j-1}, x, u_{j+1}, \dots, u_r) \mid \pm x > 0\}$ and the boundaries $\partial W_N = \bigcup_{1 \leq l \leq r} \{A\mathbf{v} \mid |v_k| \leq N \quad (k \neq l), v_l = \pm N\}$ of W_N . If there is no intersection point, then we put $H_j^\pm = 0$ accordingly. See Figure 1 for these sets and parameters.

From the proof of Lemma 4.7 we evaluate

$$(4.54) \quad \left| \sum_{\mathbf{v} \in X_j(N)} G(\mathbf{y}, \mathbf{v}) \right| \leq \frac{1}{\#\langle \mathbb{Z}^r / \langle \vec{B} \rangle \rangle} \sum_{\mathbf{w} \in \mathbb{Z}^r / \langle \vec{B} \rangle} \left| \sum_{\mathbf{u} \in Y_j(N)} \left(\prod_{i=1}^l \frac{e^{2\pi\sqrt{-1}\langle \mathbf{y} + \mathbf{w}, \vec{f}_i^B \rangle u_i}}{(u_i + \dot{f}_i)^{k_i}} \right) \right|.$$

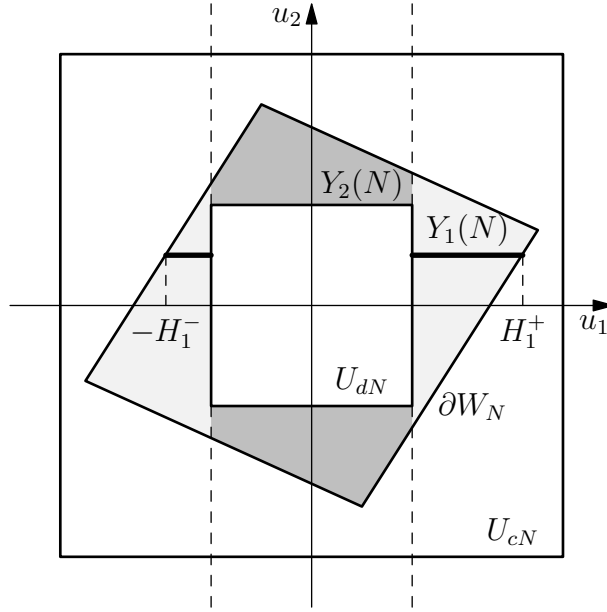


FIGURE 1

Further for each j and \mathbf{w} , we have

$$\begin{aligned}
& \left| \sum_{\mathbf{u} \in Y_j(N)} \left(\prod_{i=1}^l \frac{e^{2\pi\sqrt{-1}\langle \mathbf{y} + \mathbf{w}, \bar{f}_i^B \rangle u_i}}{(u_i + \dot{f}_i)^{k_i}} \right) \right| \\
&= \left| \sum_{\substack{u_i \in \mathbb{Z}, u_i + \dot{f}_i \neq 0 \quad (1 \leq i \leq l) \\ |u_i + \operatorname{Re} \dot{f}_i| \leq dN \quad (1 \leq i \leq j-1) \\ dN - \operatorname{Re} \dot{f}_j < u_j < H_j^+ \text{ or} \\ -H_j^- < u_j < -dN - \operatorname{Re} \dot{f}_j}} \left(\prod_{i=1}^l \frac{e^{2\pi\sqrt{-1}\langle \mathbf{y} + \mathbf{w}, \bar{f}_i^B \rangle u_i}}{(u_i + \dot{f}_i)^{k_i}} \right) \right| \\
(4.55) \quad &= \left| \sum_{\substack{u_i \in \mathbb{Z}, u_i + \dot{f}_i \neq 0 \quad (1 \leq i \leq l, i \neq j) \\ |u_i + \operatorname{Re} \dot{f}_i| \leq dN \quad (1 \leq i \leq j-1)}} \left(\prod_{\substack{i=1 \\ i \neq j}}^l \frac{e^{2\pi\sqrt{-1}\langle \mathbf{y} + \mathbf{w}, \bar{f}_i^B \rangle u_i}}{(u_i + \dot{f}_i)^{k_i}} \right) \sum_{\substack{dN - \operatorname{Re} \dot{f}_j < u_j < H_j^+ \text{ or} \\ -H_j^- < u_j < -dN - \operatorname{Re} \dot{f}_j}} \frac{e^{2\pi\sqrt{-1}\langle \mathbf{y} + \mathbf{w}, \bar{f}_j^B \rangle u_j}}{(u_j + \dot{f}_j)^{k_j}} \right| \\
&\leq \sum_{\substack{u_i \in \mathbb{Z}, u_i + \dot{f}_i \neq 0 \quad (1 \leq i \leq l, i \neq j) \\ |u_i + \operatorname{Re} \dot{f}_i| \leq dN \quad (1 \leq i \leq j-1) \\ |u_i + \operatorname{Re} \dot{f}_i| \leq cN \quad (j+1 \leq i \leq l)}} \left(\prod_{\substack{i=1 \\ i \neq j}}^l \frac{1}{|u_i + \dot{f}_i|^{k_i}} \right) \left| \sum_{\substack{dN - \operatorname{Re} \dot{f}_j < u_j < H_j^+ \text{ or} \\ -H_j^- < u_j < -dN - \operatorname{Re} \dot{f}_j}} \frac{e^{2\pi\sqrt{-1}\langle \mathbf{y} + \mathbf{w}, \bar{f}_j^B \rangle u_j}}{(u_j + \dot{f}_j)^{k_j}} \right|,
\end{aligned}$$

where in the last member, we added the extra conditions $|u_i + \operatorname{Re} \dot{f}_i| \leq cN$ for $j+1 \leq i \leq l$, which comes from Lemma 4.9. If $k_j = 1$, then by Lemma 4.4, and if $k_j \geq 2$, then directly we obtain

$$\begin{aligned}
(4.56) \quad & \left| \sum_{\substack{dN - \operatorname{Re} \dot{f}_j < u_j < H_j^+ \\ -H_j^- < u_j < -dN - \operatorname{Re} \dot{f}_j}} \frac{e^{2\pi\sqrt{-1}\langle \mathbf{y} + \mathbf{w}, \bar{f}_j^B \rangle u_j}}{(u_j + \dot{f}_j)^{k_j}} \right| \\
&\leq \begin{cases} KN^{-\frac{1}{\mu+1}} ((1 - \{\langle \mathbf{y} + \mathbf{w}, \bar{f}_j^B \rangle\})^{-\frac{1}{\mu+1}} + \{\langle \mathbf{y} + \mathbf{w}, \bar{f}_j^B \rangle\}^{-\frac{1}{\mu+1}}) & (k_j = 1) \\ KN^{-1} \leq KN^{-\frac{1}{\mu+1}} & (k_j \geq 2) \end{cases} \\
&=: Q_j(N, \mathbf{y}, \mathbf{w})
\end{aligned}$$

for some $K > 0$. Hence

$$(4.57) \quad \left| \sum_{\mathbf{u} \in Y_j(N)} \left(\prod_{i=1}^l \frac{e^{2\pi\sqrt{-1}\langle \mathbf{y} + \mathbf{w}, \vec{f}_i^{B_0} \rangle u_i}}{(u_i + \dot{f}_i)^{k_i}} \right) \right| \leq Q_j(N, \mathbf{y}, \mathbf{w}) \sum_{\substack{u_i \in \mathbb{Z}, u_i + \dot{f}_i \neq 0 \\ |u_i + \text{Re } \dot{f}_i| \leq dN \\ |u_i + \text{Re } \dot{f}_i| \leq cN}} \left(\prod_{\substack{i=1 \\ i \neq j}}^l \frac{1}{|u_i + \dot{f}_i|^{k_i}} \right) \\ \leq K' Q_j(N, \mathbf{y}, \mathbf{w}) (\log dN)^{j-1} (\log cN)^{l-j} \\ \leq K'' Q_j(N, \mathbf{y}, \mathbf{w}) (\log N)^r$$

for some $K', K'' > 0$. Substituting (4.57) into (4.54), we complete the proof. \square

The third step. Lastly we consider the general case. First we prove several preparatory lemmas.

Lemma 4.11. *Fix a decomposition $\Lambda = B_0 \cup L_0$ with $B_0 \in \mathcal{B}$. Let $\mathbf{y} \in V$ and $f \in B_0$. If $f \notin \tilde{\Lambda}$, then there exists $g \in L_0$ such that $\langle \vec{g}, \vec{f}^{B_0} \rangle \neq 0$. If $f \in \tilde{\Lambda}$ and $\mathbf{y} \notin \mathfrak{H}_{B_0 \setminus \{f\}} + \mathbb{Z}^r$, then there exists $c \in \mathbb{R} \setminus \mathbb{Z}$ such that*

$$(4.58) \quad \langle \mathbf{y} - \sum_{g \in L_0} x_g \vec{g}, \vec{f}^{B_0} \rangle = c$$

for all $\mathbf{x} = (x_g)_{g \in L_0} \in \mathbb{R}^{\sharp L_0}$.

Proof. The first assertion directly follows from the definition. Assume that $f \in \tilde{\Lambda}$. Then $\langle \vec{g}, \vec{f}^{B_0} \rangle = 0$ for all $g \in L_0$ and we have

$$(4.59) \quad \langle \mathbf{y} - \sum_{g \in L_0} x_g \vec{g}, \vec{f}^{B_0} \rangle = \langle \mathbf{y}, \vec{f}^{B_0} \rangle,$$

which is a constant function in \mathbf{x} . By Lemma 4.6, we find $\langle \mathbf{y}, \vec{f}^{B_0} \rangle \notin \mathbb{Z}$. This implies the second assertion. \square

For $\mathbf{y} \in V$, a decomposition $\Lambda = B_0 \cup L_0$ with $B_0 \in \mathcal{B}$ and $f \in B_0$, let

$$(4.60) \quad H(f, \mathbf{y}) = \left\{ \mathbf{x} = (x_g)_{g \in L_0} \in \mathbb{R}^{\sharp L_0} \mid \langle \mathbf{y} - \sum_{g \in L_0} x_g \vec{g}, \vec{f}^{B_0} \rangle \in \mathbb{Z} \right\}.$$

Lemma 4.12. *Let $\mathbf{y} \in V$, $f \in B_0$, and assume that $\mathbf{y} \notin \mathfrak{H}_{B_0 \setminus \{f\}} + \mathbb{Z}^r$ if $f \in \tilde{\Lambda}$. Then the set $H(f, \mathbf{y})$ is empty, or a collection of equally spaced parallel hyperplanes.*

Proof. Let $U = (\vec{g})_{g \in L_0}$ be an $r \times \sharp L_0$ matrix and $\mathbf{x} = (x_g)_{g \in L_0}$ be a column vector. Consider the equation

$$(4.61) \quad \langle \mathbf{y} - \sum_{g \in L_0} x_g \vec{g}, \vec{f}^{B_0} \rangle = n$$

for $n \in \mathbb{Z}$, which is equivalent to

$$(4.62) \quad \langle U\mathbf{x}, \vec{f}^{B_0} \rangle = \langle \mathbf{y}, \vec{f}^{B_0} \rangle - n.$$

Assume that $f \notin \tilde{\Lambda}$. Then there exists $g \in L_0$ such that $\langle \vec{g}, \vec{f}^{B_0} \rangle \neq 0$. We see that (4.62) has a solution $\mathbf{x} = \mathbf{x}_0 - n\mathbf{a}$ with

$$(4.63) \quad (\mathbf{x}_0)_h = \begin{cases} \langle \mathbf{y}, \vec{f}^{B_0} \rangle / \langle \vec{g}, \vec{f}^{B_0} \rangle & (h = g) \\ 0 & (h \neq g) \end{cases}, \quad (\mathbf{a})_h = \begin{cases} 1 / \langle \vec{g}, \vec{f}^{B_0} \rangle & (h = g) \\ 0 & (h \neq g) \end{cases},$$

and so the equation (4.62) is rewritten as

$$(4.64) \quad \langle U(\mathbf{x} - \mathbf{x}_0 + n\mathbf{a}), \vec{f}^{B_0} \rangle = 0.$$

The condition $\langle \vec{g}, \vec{f}^{B_0} \rangle \neq 0$ also implies that $\dim \ker \mathcal{U} = \sharp L_0 - 1$ for the linear functional $\mathcal{U}(\mathbf{v}) = \langle U\mathbf{v}, \vec{f}^{B_0} \rangle$, and

$$(4.65) \quad H(f, \mathbf{y}) = (\ker \mathcal{U} + \mathbf{x}_0) + \mathbb{Z}\mathbf{a}$$

is a collection of equally spaced parallel hyperplanes.

Assume that $f \in \tilde{\Lambda}$. Then by Lemma 4.11,

$$(4.66) \quad \langle \mathbf{y} - \sum_{g \in L_0} x_g \vec{g}, \vec{f}^{B_0} \rangle \in \mathbb{R} \setminus \mathbb{Z},$$

and hence $H(f, \mathbf{y}) = \emptyset$. □

Lemma 4.13. *Fix a decomposition $\Lambda = B_0 \cup L_0$ with $B_0 \in \mathcal{B}$. Assume $D \subset B_0$. For a fixed $\mathbf{y} \in V \setminus \bigcup_{f \in \tilde{\Lambda} \cap D} (\mathfrak{H}_{\Lambda \setminus \{f\}} + \mathbb{Z}^r)$, the measure of*

$$(4.67) \quad M(\mathbf{y}) = \left\{ (x_g)_{g \in L_0} \in \mathbb{R}^{\#L_0} \mid \mathbf{y} - \sum_{g \in L_0} x_g \vec{g} \in \bigcup_{f \in D} (\mathfrak{H}_{B_0 \setminus \{f\}} + \mathbb{Z}^r) \right\}$$

is zero.

Proof. By Lemma 4.6, we have $M(\mathbf{y}) = \bigcup_{f \in D} \bigcup_{\mathbf{w} \in \mathbb{Z}^r} H(f, \mathbf{y} + \mathbf{w})$. Further by (2.3) we have

$$(4.68) \quad \bigcup_{f \in \tilde{\Lambda} \cap D} (\mathfrak{H}_{\Lambda \setminus \{f\}} + \mathbb{Z}^r) = \bigcup_{f \in \tilde{\Lambda} \cap D} (\mathfrak{H}_{B_0 \setminus \{f\}} + \mathbb{Z}^r),$$

so from the assumption $\mathbf{y} \in V \setminus \bigcup_{f \in \tilde{\Lambda} \cap D} (\mathfrak{H}_{\Lambda \setminus \{f\}} + \mathbb{Z}^r)$ we see that $\mathbf{y} + \mathbf{w} \notin \bigcup_{f \in \tilde{\Lambda} \cap D} (\mathfrak{H}_{B_0 \setminus \{f\}} + \mathbb{Z}^r)$ for any $\mathbf{w} \in \mathbb{Z}^r$. Therefore we can apply Lemma 4.12 to find that for each $f \in D$ and $\mathbf{w} \in \mathbb{Z}^r$ the measure of $H(f, \mathbf{y} + \mathbf{w})$ is zero. □

Lemma 4.14. *Let $n \in \mathbb{N}$ and $P, Q \in \mathbb{N}_0$ with $P \geq Q$. Let $a_{ki} \in \mathbb{R}$ for $1 \leq k \leq P$ and $0 \leq i \leq n$ such that for each $k = 1, \dots, P$ there exists $i \geq 1$ such that $a_{ki} \neq 0$. If $\mu \geq P$, then*

$$(4.69) \quad \int_0^1 dx_1 \cdots \int_0^1 dx_n \left(\prod_{k=1}^Q (1 - \{a_{k0} + \sum_{i=1}^n a_{ki} x_i\})^{-\frac{1}{\mu+1}} \right) \left(\prod_{k=Q+1}^P \{a_{k0} + \sum_{i=1}^n a_{ki} x_i\}^{-\frac{1}{\mu+1}} \right) < \infty.$$

Proof. For $1 \leq k \leq P$ put

$$(4.70) \quad L_k(\mathbf{x}) = \sum_{i=1}^n a_{ki} x_i.$$

Since $[0, 1]^n$ is compact, by considering a neighborhood of each point \mathbf{x}_0 in $[0, 1]^n$ and shifting the point \mathbf{x}_0 to the origin, we see that it is sufficient to show that

$$(4.71) \quad I = \int_{[-\epsilon, \epsilon]^n} dx_1 \cdots dx_n \left(\prod_{k=1}^q (1 - \{L_k(\mathbf{x})\})^{-\frac{1}{\mu+1}} \right) \left(\prod_{k=q+1}^p \{L_k(\mathbf{x})\}^{-\frac{1}{\mu+1}} \right)$$

is finite for a sufficiently small $\epsilon > 0$, where $0 \leq q \leq Q$ and $q \leq p \leq P$. This is estimated as

$$(4.72) \quad I \leq \int_{[-\epsilon, \epsilon]^n} dx_1 \cdots dx_n \left(\prod_{k=1}^p |L_k(\mathbf{x})|^{-\frac{1}{\mu+1}} \right).$$

For this integral, we decompose the region

$$(4.73) \quad [-\epsilon, \epsilon]^n = \bigcup_{k=1}^p U_k,$$

where

$$(4.74) \quad U_k = \{\mathbf{x} \in [-\epsilon, \epsilon]^n \mid |L_k(\mathbf{x})| \leq |L_m(\mathbf{x})| \text{ for any } m \neq k\}.$$

We show that the integral on each U_k is finite. Since on U_k

$$(4.75) \quad |L_m(\mathbf{x})|^{-\frac{1}{\mu+1}} \leq |L_k(\mathbf{x})|^{-\frac{1}{\mu+1}}$$

for any $m \neq k$, we have

$$(4.76) \quad \left(\prod_{m=1}^p |L_m(\mathbf{x})|^{-\frac{1}{\mu+1}} \right) \leq |L_k(\mathbf{x})|^{-\frac{p}{\mu+1}}.$$

Fix i such that $a_{ki} \neq 0$. Then by changing variables as $y_i = L_k(\mathbf{x})$ and $y_j = x_j$ for $j \neq i$, we obtain

$$(4.77) \quad \int_{U_k} dx_1 \cdots dx_n \left(\prod_{m=1}^p |L_m(\mathbf{x})|^{-\frac{1}{\mu+1}} \right) \leq \int_{U_k} dx_1 \cdots dx_n |L_k(\mathbf{x})|^{-\frac{p}{\mu+1}} \\ \leq |a_{ki}| \int_{-r}^r |y_i|^{-\frac{p}{\mu+1}} dy_i \int_{[-\epsilon, \epsilon]^{n-1}} \prod_{j \neq i} dy_j$$

for some $r > 0$. By the assumption $\mu \geq P$, the right-hand side is finite because

$$(4.78) \quad -\frac{p}{\mu+1} > -\frac{P}{P+1} > -1.$$

□

Lemma 4.15. For $k \in \mathbb{N}_0$, $m \in \mathbb{Z}$ and $b \in \mathbb{C}$,

$$(4.79) \quad -\frac{(2\pi\sqrt{-1})^k}{k!} \int_0^1 C(k, x; b) e^{-2\pi\sqrt{-1}mx} dx = \begin{cases} -1 & (m+b=0, k=0), \\ 0 & (m+b \neq 0, k=0), \\ 0 & (m+b=0, k \neq 0), \\ \frac{1}{(m+b)^k} & (m+b \neq 0, k \neq 0). \end{cases}$$

Proof. By definition (4.1), for $0 \leq x < 1$, we have

$$(4.80) \quad \frac{1}{k!} C(k, x; b) e^{-2\pi\sqrt{-1}mx} = \frac{1}{2\pi\sqrt{-1}} \int_{|t|=\epsilon} \frac{t e^{(t-2\pi\sqrt{-1}(m+b))x}}{e^{t-2\pi\sqrt{-1}b} - 1} t^{-k-1} dt$$

for sufficiently small $\epsilon > 0$. By integrating the both sides in the region $0 \leq x < 1$, we obtain

$$(4.81) \quad \frac{1}{k!} \int_0^1 C(k, x; b) e^{-2\pi\sqrt{-1}mx} dx = \frac{1}{2\pi\sqrt{-1}} \int_{|t|=\epsilon} \frac{t}{t - 2\pi\sqrt{-1}(m+b)} t^{-k-1} dt.$$

Since

$$(4.82) \quad \frac{t}{t - 2\pi\sqrt{-1}(m+b)} = \begin{cases} 1 & (m+b=0), \\ -\sum_{l=1}^{\infty} \frac{1}{(2\pi\sqrt{-1}(m+b))^l} t^l & (m+b \neq 0), \end{cases}$$

we obtain the assertion. □

Proof of Proposition 4.1. Applying Lemma 4.15 with $m = \langle \vec{g}, \mathbf{v} \rangle$ and $b = \dot{g}$ (for $g \in L_0$) to (4.4), for $N > 0$ we have

$$\begin{aligned}
(4.83) \quad Z_1(N; \mathbf{k}, \mathbf{y}; \Lambda; B_0) &= (-1)^{\sharp \Lambda_0} \sum_{\substack{\mathbf{v} \in \mathbb{Z}^r \cap U_N(B_0) \\ f(\mathbf{v}) \neq 0 \quad (f \in \Lambda_+ \cap B_0) \\ f(\mathbf{v}) = 0 \quad (f \in \Lambda_0 \cap B_0)}} e^{2\pi\sqrt{-1}\langle \mathbf{y}, \mathbf{v} \rangle} \prod_{f \in \Lambda_+ \cap B_0} \frac{1}{f(\mathbf{v})^{k_f}} \\
&\quad \times \prod_{g \in L_0} (-1)^{\sharp(\Lambda_0 \cap L_0)} \left(-\frac{(2\pi\sqrt{-1})^{k_g}}{k_g!} \int_0^1 C(k_g, x_g; \dot{g}) e^{-2\pi\sqrt{-1}\langle \vec{g}, \mathbf{v} \rangle x_g} dx_g \right) \\
&= \prod_{g \in L_0} \left(-\frac{(2\pi\sqrt{-1})^{k_g}}{k_g!} \right) \int_0^1 \cdots \int_0^1 \left(\prod_{g \in L_0} C(k_g, x_g; \dot{g}) dx_g \right) \\
&\quad \times (-1)^{\sharp(\Lambda_0 \cap B_0)} \sum_{\substack{\mathbf{v} \in \mathbb{Z}^r \cap U_N(B_0) \\ f(\mathbf{v}) \neq 0 \quad (f \in \Lambda_+ \cap B_0) \\ f(\mathbf{v}) = 0 \quad (f \in \Lambda_0 \cap B_0)}} e^{2\pi\sqrt{-1}\langle \mathbf{y} - \sum_{g \in L_0} x_g \vec{g}, \mathbf{v} \rangle} \left(\prod_{f \in \Lambda_+ \cap B_0} \frac{1}{f(\mathbf{v})^{k_f}} \right) \\
&= \prod_{g \in L_0} \left(-\frac{(2\pi\sqrt{-1})^{k_g}}{k_g!} \right) \int_0^1 \cdots \int_0^1 \left(\prod_{g \in L_0} C(k_g, x_g; \dot{g}) dx_g \right) \\
&\quad \times Z_1(N; \mathbf{k}(B_0), \mathbf{y} - \sum_{g \in L_0} x_g \vec{g}; B_0; B_0),
\end{aligned}$$

where $\mathbf{k}(B_0) = (k_f)_{f \in B_0}$.

We want to take the limit $N \rightarrow \infty$. First we claim that it is possible to exchange the limit and the integrals. By Lemma 4.8 with $\mu = \sharp B_0 = r$, we have

$$\begin{aligned}
(4.84) \quad & \left| \left(\prod_{g \in L_0} C(k_g, x_g; \dot{g}) \right) \sum_{\substack{\mathbf{v} \in \mathbb{Z}^r \cap U_N(B_0) \\ f(\mathbf{v}) \neq 0 \quad (f \in \Lambda_+ \cap B_0) \\ f(\mathbf{v}) = 0 \quad (f \in \Lambda_0 \cap B_0)}} e^{2\pi\sqrt{-1}\langle \mathbf{y} - \sum_{g \in L_0} x_g \vec{g}, \mathbf{v} \rangle} \left(\prod_{f \in \Lambda_+ \cap B_0} \frac{1}{f(\mathbf{v})^{k_f}} \right) \right| \\
& \leq K \sum_{\mathbf{w} \in \mathbb{Z}^r / \langle \vec{B}_0 \rangle} \prod_{f \in \Lambda_1 \cap B_0} (1 + (1 - \{ \langle \mathbf{y} + \mathbf{w} - \sum_{g \in L_0} x_g \vec{g}, \vec{f}^{B_0} \rangle \})^{-\frac{1}{r+1}} + \{ \langle \mathbf{y} + \mathbf{w} - \sum_{g \in L_0} x_g \vec{g}, \vec{f}^{B_0} \rangle \}^{-\frac{1}{r+1}}) \\
& = K \sum_{\mathbf{w} \in \mathbb{Z}^r / \langle \vec{B}_0 \rangle} \prod_{f \in \Lambda_1 \cap B_0} X_f,
\end{aligned}$$

say. When $f \in \tilde{\Lambda}$, then under the condition $\mathbf{y} \notin \bigcup_{f \in \tilde{\Lambda} \cap \Lambda_1 \cap B_0} (\mathfrak{H}_{B_0 \setminus \{f\}} + \mathbb{Z}^r)$, we see that X_f is just a constant because of the second assertion of Lemma 4.11. When $f \notin \tilde{\Lambda}$, by the first assertion of Lemma 4.11 we see that X_f fulfills the assumption of Lemma 4.14, and hence by the lemma it is integrable since $r \geq \sharp(\Lambda_1 \cap B_0)$. Therefore our claim follows from Lebesgue's dominated convergence theorem. Note that $\tilde{\Lambda} \cap \Lambda_1 \cap B_0 = \tilde{\Lambda} \cap \Lambda_1$ because of (2.3).

Therefore from (4.83) we now obtain

$$\begin{aligned}
(4.85) \quad & \lim_{N \rightarrow \infty} Z_1(N; \mathbf{k}, \mathbf{y}; \Lambda; B_0) \\
& = \prod_{g \in L_0} \left(-\frac{(2\pi\sqrt{-1})^{k_g}}{k_g!} \right) \int_0^1 \cdots \int_0^1 \left(\prod_{g \in L_0} C(k_g, x_g; \dot{g}) dx_g \right) \\
& \quad \times \lim_{N \rightarrow \infty} Z_1(N; \mathbf{k}(B_0), \mathbf{y} - \sum_{g \in L_0} x_g \vec{g}; B_0; B_0).
\end{aligned}$$

By Lemma 4.13 with $D = \Lambda_1 \cap B_0$ the measure of $M(\mathbf{y})$ is 0, and if $(x_g)_{g \in L_0} \notin M(\mathbf{y})$, then by Lemma 4.7 with $B = B_0$ we see that $Z_1(N; \mathbf{k}(B_0), \mathbf{y} - \sum_{g \in L_0} x_g \vec{g}; B_0; B_0)$ converges as $N \rightarrow \infty$. That is, the integrand

on the right-hand side of (4.85) converges almost everywhere, and (4.31) of Lemma 4.7 implies

$$\begin{aligned}
(4.86) \quad S_1(\mathbf{k}, \mathbf{y}; \Lambda; B_0) &= \lim_{N \rightarrow \infty} Z_1(N; \mathbf{k}, \mathbf{y}; \Lambda; B_0) \\
&= \prod_{g \in L_0} \left(-\frac{(2\pi\sqrt{-1})^{k_g}}{k_g!} \right) \int_0^1 \cdots \int_0^1 \left(\prod_{g \in L_0} C(k_g, x_g; \dot{g}) dx_g \right) \\
&\quad \times \frac{1}{|\mathbb{Z}^r / \langle \vec{B}_0 \rangle|} \sum_{\mathbf{w} \in \mathbb{Z}^r / \langle \vec{B}_0 \rangle} \prod_{f \in B_0} \left(-\frac{(2\pi\sqrt{-1})^{k_f}}{k_f!} C(k_f, \{\mathbf{y} + \mathbf{w} - \sum_{g \in L_0} x_g \vec{g}\}_{B_0, f}; \dot{f}) \right) \\
&= \frac{1}{\sharp(\mathbb{Z}^r / \langle \vec{B}_0 \rangle)} \prod_{f \in \Lambda} \left(-\frac{(2\pi\sqrt{-1})^{k_f}}{k_f!} \right) \sum_{\mathbf{w} \in \mathbb{Z}^r / \langle \vec{B}_0 \rangle} \int_0^1 \cdots \int_0^1 \left(\prod_{g \in L_0} C(k_g, x_g; \dot{g}) dx_g \right) \\
&\quad \times \prod_{f \in B_0} \left(C(k_f, \{\mathbf{y} + \mathbf{w} - \sum_{g \in L_0} x_g \vec{g}\}_{B_0, f}; \dot{f}) \right).
\end{aligned}$$

The right-hand side of this equation coincides with that of (4.2). Therefore, to complete the proof of the proposition, the only remaining task is to show that $S_1(\mathbf{k}, \mathbf{y}; \Lambda; B_0) = S(\mathbf{k}, \mathbf{y}; \Lambda)$.

The sum $Z(N; \mathbf{k}, \mathbf{y}; \Lambda)$ has the expression which is almost the same as (4.83), only the condition $\mathbf{v} \in \mathbb{Z}^r \cap U_N(B_0)$ is replaced by $\mathbf{v} \in \mathbb{Z}^r \cap W_N$. Therefore, by using Lemmas 4.9 and 4.10, we see that the difference $Z(N; \mathbf{k}, \mathbf{y}; \Lambda) - Z_1(dN; \mathbf{k}, \mathbf{y}; \Lambda; B_0)$ is evaluated as

$$\begin{aligned}
(4.87) \quad &|Z(N; \mathbf{k}, \mathbf{y}; \Lambda) - Z_1(dN; \mathbf{k}, \mathbf{y}; \Lambda; B_0)| \\
&= \left| (-1)^{\sharp(\Lambda_0 \cap B_0)} \prod_{g \in L_0} \left(-\frac{(2\pi\sqrt{-1})^{k_g}}{k_g!} \right) \int_0^1 \cdots \int_0^1 \left(\prod_{g \in L_0} C(k_g, x_g; \dot{g}) dx_g \right) \right. \\
&\quad \times \sum_{\substack{\mathbf{v} \in \mathbb{Z}^r \cap (W_N \setminus U_{dN}(B_0)) \\ f(\mathbf{v}) \neq 0 \quad (f \in \Lambda_+ \cap B_0) \\ f(\mathbf{v}) = 0 \quad (f \in \Lambda_0 \cap B_0)}} e^{2\pi\sqrt{-1}\langle \mathbf{y} - \sum_{g \in L_0} x_g \vec{g}, \mathbf{v} \rangle} \left(\prod_{f \in \Lambda_+ \cap B_0} \frac{1}{f(\mathbf{v})^{k_f}} \right) \Big| \\
&\leq K \int_0^1 \cdots \int_0^1 \prod_{g \in L_0} dx_g \left| \sum_{\substack{\mathbf{v} \in \mathbb{Z}^r \cap (W_N \setminus U_{dN}(B_0)) \\ f(\mathbf{v}) \neq 0 \quad (f \in \Lambda_+ \cap B_0) \\ f(\mathbf{v}) = 0 \quad (f \in \Lambda_0 \cap B_0)}} e^{2\pi\sqrt{-1}\langle \mathbf{y} - \sum_{g \in L_0} x_g \vec{g}, \mathbf{v} \rangle} \left(\prod_{f \in \Lambda_+ \cap B_0} \frac{1}{f(\mathbf{v})^{k_f}} \right) \right| \\
&\leq K' N^{-\frac{1}{r+1}} (\log N)^r \int_0^1 \cdots \int_0^1 \prod_{g \in L_0} dx_g \\
&\quad \left(1 + \sum_{\mathbf{w} \in \mathbb{Z}^r / \langle \vec{B}_0 \rangle} \sum_{f \in \Lambda_1 \cap B_0} \left((1 - \{\langle \mathbf{y} + \mathbf{w} - \sum_{g \in L_0} x_g \vec{g}, \vec{f}^{B_0} \rangle\})^{-\frac{1}{r+1}} + \{\langle \mathbf{y} + \mathbf{w} - \sum_{g \in L_0} x_g \vec{g}, \vec{f}^{B_0} \rangle\}^{-\frac{1}{r+1}} \right) \right)
\end{aligned}$$

for some $K, K' > 0$. Again by Lemmas 4.11 and 4.14, we obtain

$$(4.88) \quad |Z(N; \mathbf{k}, \mathbf{y}; \Lambda) - Z_1(dN; \mathbf{k}, \mathbf{y}; \Lambda; B_0)| \leq K'' N^{-\frac{1}{r+1}} (\log N)^r$$

for some $K'' > 0$. Hence we have

$$(4.89) \quad S(\mathbf{k}, \mathbf{y}; \Lambda) = \lim_{N \rightarrow \infty} Z(N; \mathbf{k}, \mathbf{y}; \Lambda) = \lim_{N \rightarrow \infty} Z_1(N; \mathbf{k}, \mathbf{y}; \Lambda; B_0) = S_1(\mathbf{k}, \mathbf{y}; \Lambda; B_0).$$

□

We have shown the convergence of $S(\mathbf{k}, \mathbf{y}; \Lambda)$ in Proposition 4.1. Therefore to complete the proof of Theorem 2.2, we have only to show the continuity of $S(\mathbf{k}, \mathbf{y}; \Lambda)$ in \mathbf{y} on $V \setminus \bigcup_{f \in \tilde{\Lambda} \cap \Lambda_1} (\mathfrak{H}_{\Lambda} \setminus \{f\} + \mathbb{Z}^r)$.

Let $\mathbf{y}_0 \in V \setminus \bigcup_{f \in \tilde{\Lambda} \cap \Lambda_1} (\mathfrak{H}_{\Lambda \setminus \{f\}} + \mathbb{Z}^r)$, and let $G(\mathbf{y}, (x_g))$ be the integrand of (4.2). Since $G(\mathbf{y}, (x_g))$ is bounded, we have

$$(4.90) \quad \lim_{\mathbf{y} \rightarrow \mathbf{y}_0} \int G(\mathbf{y}, (x_g)) \prod_{g \in L_0} dx_g = \int \lim_{\mathbf{y} \rightarrow \mathbf{y}_0} G(\mathbf{y}, (x_g)) \prod_{g \in L_0} dx_g.$$

Thus it is sufficient to show that

$$(4.91) \quad \lim_{\mathbf{y} \rightarrow \mathbf{y}_0} G(\mathbf{y}, (x_g)) = G(\mathbf{y}_0, (x_g))$$

almost everywhere in (x_g) . By Lemmas 4.2 and 4.4, we see that $C(k, \{y\}; b)$ is continuous in y on \mathbb{R} if $k \neq 1$, and on $\mathbb{R} \setminus \mathbb{Z}$ if $k = 1$. Hence if (x_g) satisfies

$$(4.92) \quad \langle \mathbf{y}_0 + \mathbf{w} - \sum_{g \in L_0} x_g \vec{g}, \vec{f}^{B_0} \rangle \notin \mathbb{Z}$$

for all $f \in \Lambda_1 \cap B_0$, then (4.91) holds. Therefore it is sufficient to show that (x_g) satisfies (4.92) almost everywhere. Since $\mathbf{y}_0 + \mathbf{w} \in V \setminus \bigcup_{f \in \tilde{\Lambda} \cap \Lambda_1} (\mathfrak{H}_{\Lambda \setminus \{f\}} + \mathbb{Z}^r)$, we see that the measure of $M(\mathbf{y}_0 + \mathbf{w})$ is zero by Lemma 4.13 with $D = \Lambda_1 \cap B_0$.

The proof of Theorem 2.2 is thus complete.

5. THE STRUCTURE OF THE PROOF OF THEOREM 2.4 AND THEOREM 2.5

Now we start the proof of Theorem 2.4 and Theorem 2.5. We first prove the assertion (i) of Theorem 2.4. Let

$$(5.1) \quad \mathfrak{H}_{\mathcal{R}} := \bigcup_{R \in \mathcal{R}} (\mathfrak{H}_R + \mathbb{Z}^r).$$

Lemma 5.1. *The set $\mathfrak{H}_{\mathcal{R}}$ is a locally finite collection of hyperplanes, that is, for any $\mathbf{y} \in V$ there exists a neighborhood U of \mathbf{y} such that U intersects only finitely many hyperplanes.*

Proof. Let \mathbf{n}_R be a normal vector of \mathfrak{H}_R . We may assume that $\mathbf{n}_R \in \mathbb{Z}^r$, because $\vec{g}_1, \dots, \vec{g}_{r-1} \in \mathbb{Z}^r$. Then the hyperplane

$$(5.2) \quad \mathfrak{H}_R + \mathbf{v} = \{\mathbf{y} + \mathbf{v} \mid \langle \mathbf{y}, \mathbf{n}_R \rangle = 0\}$$

with $\mathbf{v} \in \mathbb{Z}^r$ can be rewritten as

$$(5.3) \quad \{\mathbf{y} \mid \langle \mathbf{y}, \mathbf{n}_R \rangle - \langle \mathbf{v}, \mathbf{n}_R \rangle = 0\} = \{\mathbf{y} \mid \langle \mathbf{y} - m\mathbf{e}_R, \mathbf{n}_R \rangle = 0\},$$

where $m = \langle \mathbf{v}, \mathbf{n}_R \rangle \in \mathbb{Z}$ and $\mathbf{e}_R = \mathbf{n}_R / \langle \mathbf{n}_R, \mathbf{n}_R \rangle$. Therefore

$$(5.4) \quad \mathfrak{H}_R + \mathbf{v} = \mathfrak{H}_R + m\mathbf{e}_R,$$

and so

$$(5.5) \quad \mathfrak{H}_{\mathcal{R}} \subset \bigcup_{R \in \mathcal{R}} (\mathfrak{H}_R + \mathbb{Z}\mathbf{e}_R).$$

Hence the assertion follows from this expression and $\#\mathcal{R} < \infty$. \square

Lemma 5.2.

$$(5.6) \quad \lim_{c \rightarrow 0^+} \{\mathbf{y} + c\phi\}_{B,f} = \{\mathbf{y}\}_{B,f}$$

for $\mathbf{y} \in V$.

Proof. By Lemma 5.1, for any $\mathbf{y} \in V$, we see that $\mathbf{y} + c\phi \notin \mathfrak{H}_{\mathcal{R}}$ and so $\langle \mathbf{y} + c\phi, \vec{f}^B \rangle \notin \mathbb{Z}$ for all sufficiently small $c > 0$. Therefore, if $\langle \mathbf{y}, \vec{f}^B \rangle \notin \mathbb{Z}$, then

$$(5.7) \quad \lim_{c \rightarrow 0^+} \{\mathbf{y} + c\phi\}_{B,f} = \lim_{c \rightarrow 0^+} \{\langle \mathbf{y} + c\phi, \vec{f}^B \rangle\} = \{\langle \mathbf{y}, \vec{f}^B \rangle\} = \{\mathbf{y}\}_{B,f}$$

by (2.6). If $\langle \mathbf{y}, \vec{f}^B \rangle \in \mathbb{Z}$, then

$$(5.8) \quad \lim_{c \rightarrow 0^+} \{\mathbf{y} + c\phi\}_{B,f} = \begin{cases} \lim_{c \rightarrow 0^+} \{\langle \mathbf{y} + c\phi, \vec{f}^B \rangle\} = \lim_{c \rightarrow 0^+} \{c\langle \phi, \vec{f}^B \rangle\} = 0 = \{\langle \mathbf{y}, \vec{f}^B \rangle\} & (\langle \phi, \vec{f}^B \rangle > 0), \\ \lim_{c \rightarrow 0^+} 1 - \{-\langle \mathbf{y} + c\phi, \vec{f}^B \rangle\} = \lim_{c \rightarrow 0^+} 1 - \{-c\langle \phi, \vec{f}^B \rangle\} = 1 = 1 - \{-\langle \mathbf{y}, \vec{f}^B \rangle\} & (\langle \phi, \vec{f}^B \rangle < 0). \end{cases}$$

Hence we have the assertion. \square

By this lemma we immediately obtain

$$(5.9) \quad \lim_{c \rightarrow 0^+} F(\mathbf{t}, \mathbf{y} + c\phi; \Lambda) = F(\mathbf{t}, \mathbf{y}; \Lambda).$$

This shows the assertion (i) of Theorem 2.4.

Next, observe that the right-hand side of (4.2) can be defined for any $\mathbf{y} \in V$ (though (4.2) itself is valid only under the assumption of Proposition 4.1). Therefore, we can define $\tilde{C}(\mathbf{k}, \mathbf{y}; \Lambda)$ for any $\mathbf{y} \in V$ as the $\left(\prod_{f \in \Lambda} \frac{k_f!}{(2\pi\sqrt{-1})^{k_f}}\right)$ multiple of the right-hand side of (4.2), and we introduce the generating function of $\tilde{C}(\mathbf{k}, \mathbf{y}; \Lambda)$ of the form

$$(5.10) \quad \tilde{F}(\mathbf{t}, \mathbf{y}; \Lambda) = \sum_{\mathbf{k} \in \mathbb{N}_0^{\#\Lambda}} \tilde{C}(\mathbf{k}, \mathbf{y}; \Lambda) \prod_{f \in \Lambda} \frac{t_f^{k_f}}{k_f!},$$

where $\mathbf{t} = (t_f)_{f \in \Lambda} \in \mathbb{C}^{\#\Lambda}$. A more explicit form of the generating function can be deduced by substituting the formula of Proposition 4.1 into (5.10). In fact,

Lemma 5.3. *For any $\mathbf{y} \in V$, the series on the right-hand side of (5.10) is absolutely and uniformly convergent in the neighborhood of the origin with respect to $\mathbf{t} \in \mathbb{C}^{\#\Lambda}$. Furthermore we have*

$$(5.11) \quad \tilde{F}(\mathbf{t}, \mathbf{y}; \Lambda) = \left(\prod_{f \in \Lambda} \frac{t_f}{\exp(t_f - 2\pi\sqrt{-1}\dot{f}) - 1} \right) \frac{1}{\#\langle \mathbb{Z}^r / \langle \vec{B}_0 \rangle \rangle} \sum_{\mathbf{w} \in \mathbb{Z}^r / \langle \vec{B}_0 \rangle} \int_0^1 \cdots \int_0^1 \prod_{g \in L_0} dx_g \\ \times \exp\left(\sum_{g \in L_0} (t_g - 2\pi\sqrt{-1}\dot{g})x_g + \sum_{f \in B_0} (t_f - 2\pi\sqrt{-1}\dot{f})\{\mathbf{y} + \mathbf{w} - \sum_{g \in L_0} x_g \vec{g}\}_{B_0, f} \right).$$

Proof. The following proof is similar to that of [7, Lemma 7]. By (4.80) we see that, for $b \in \mathbb{C}$, there exists a sufficiently small $R_b > 0$ such that

$$(5.12) \quad \frac{C(k, y; b)}{k!} = \frac{1}{2\pi\sqrt{-1}} \int_{|z|=R_b} \frac{ze^{(z-2\pi\sqrt{-1}b)y}}{e^{z-2\pi\sqrt{-1}b} - 1} \frac{dz}{z^{k+1}}$$

holds for $y \in \mathbb{R}$. Thus we have for $0 \leq y \leq 1$

$$(5.13) \quad \left| \frac{C(k, y; b)}{k!} \right| \leq \frac{1}{2\pi} \int_{|z|=R_b} \left| \frac{ze^{(z-2\pi\sqrt{-1}b)y}}{e^{z-2\pi\sqrt{-1}b} - 1} \right| \frac{|dz|}{R_b^{k+1}} \leq \frac{C_b}{R_b^k},$$

where

$$(5.14) \quad C_b = \max \left\{ \left| \frac{ze^{(z-2\pi\sqrt{-1}b)y}}{e^{z-2\pi\sqrt{-1}b} - 1} \right| \mid |z| = R_b, 0 \leq y \leq 1 \right\}.$$

Fix r such that $0 < r < \min_{f \in \Lambda} R_{\dot{f}}$. Then for $|t_f| < r$ ($f \in \Lambda$),

$$\begin{aligned}
(5.15) \quad \left| \tilde{C}(\mathbf{k}, \mathbf{y}; \Lambda) \prod_{f \in \Lambda} \frac{t_f^{k_f}}{k_f!} \right| &\leq \frac{1}{\#\langle \mathbb{Z}^r / \langle \vec{B}_0 \rangle \rangle} \sum_{\mathbf{w} \in \mathbb{Z}^r / \langle \vec{B}_0 \rangle} \left(\prod_{f \in \Lambda} |t_f|^{k_f} \right) \int_0^1 \cdots \int_0^1 \left| \left(\prod_{g \in L_0} \frac{C(k_g, x_g; \dot{g})}{k_g!} dx_g \right) \right. \\
&\quad \left. \times \prod_{f \in B_0} \left(\frac{C(k_f, \{\mathbf{y} + \mathbf{w} - \sum_{g \in L_0} x_g \vec{g}\}_{B_0, f}; \dot{f})}{k_f!} \right) \right| \\
&\leq \frac{1}{\#\langle \mathbb{Z}^r / \langle \vec{B}_0 \rangle \rangle} \sum_{\mathbf{w} \in \mathbb{Z}^r / \langle \vec{B}_0 \rangle} \left(\prod_{f \in \Lambda} r^{k_f} \right) \int_0^1 \cdots \int_0^1 \prod_{f \in \Lambda} \frac{C_{\dot{f}}}{R_{\dot{f}}^{k_f}} \prod_{g \in L_0} dx_g \\
&= \prod_{f \in \Lambda} C_{\dot{f}} \left(\frac{r}{R_{\dot{f}}} \right)^{k_f}.
\end{aligned}$$

Since

$$(5.16) \quad \sum_{\mathbf{k} \in \mathbb{N}_0^{\#\Lambda}} \prod_{f \in \Lambda} C_{\dot{f}} \left(\frac{r}{R_{\dot{f}}} \right)^{k_f} = \prod_{f \in \Lambda} \left(\frac{C_{\dot{f}}}{1 - r/R_{\dot{f}}} \right) < \infty,$$

we have the uniform and absolute convergence of $\tilde{F}(\mathbf{t}, \mathbf{y}; \Lambda)$, which implies the holomorphy of $\tilde{F}(\mathbf{t}, \mathbf{y}; \Lambda)$ in the neighborhood of the origin with respect to $\mathbf{t} \in \mathbb{C}^{\#\Lambda}$.

Furthermore by exchanging the sum and the integral and using (4.1) we obtain

$$\begin{aligned}
(5.17) \quad \tilde{F}(\mathbf{t}, \mathbf{y}; \Lambda) &= \frac{1}{\#\langle \mathbb{Z}^r / \langle \vec{B}_0 \rangle \rangle} \sum_{\mathbf{w} \in \mathbb{Z}^r / \langle \vec{B}_0 \rangle} \int_0^1 \cdots \int_0^1 \left(\prod_{g \in L_0} dx_g \right) \left(\prod_{g \in L_0} \frac{t_g \exp((t_g - 2\pi\sqrt{-1}\dot{g})x_g)}{\exp(t_g - 2\pi\sqrt{-1}\dot{g}) - 1} \right) \\
&\quad \times \left(\prod_{f \in B_0} \frac{t_f \exp((t_f - 2\pi\sqrt{-1}\dot{f})\{\mathbf{y} + \mathbf{w} - \sum_{g \in L_0} x_g \vec{g}\}_{B_0, f})}{\exp(t_f - 2\pi\sqrt{-1}\dot{f}) - 1} \right),
\end{aligned}$$

which yields (5.11). \square

Lemma 5.4. $\tilde{F}(\mathbf{t}, \mathbf{y}; \Lambda)$ is continuous in \mathbf{y} on $V \setminus \bigcup_{f \in \tilde{\Lambda}} (\mathfrak{H}_{\Lambda \setminus \{f\}} + \mathbb{Z}^r)$ and has one-sided continuity in $\mathbf{y} \in V$ in the direction ϕ .

Proof. The proof is almost the same as that of the continuity of $S(\mathbf{k}, \mathbf{y}; \Lambda)$ in (4.90). Let $G(\mathbf{y}, (x_g))$ be the integrand of the last expression of (5.11). In this case, the continuity comes from (4.92) for all $f \in B_0$. Hence the first assertion follows from Lemma 4.13 with $D = B_0$. The second assertion immediately follows from Lemma 5.2. \square

We have obtained the assertions, corresponding to (i), (ii) and (iii) of Theorem 2.4, for $\tilde{F}(\mathbf{t}, \mathbf{y}; \Lambda)$. In the following sections, we will prove

$$(5.18) \quad \tilde{F}(\mathbf{t}, \mathbf{y}; \Lambda) = F(\mathbf{t}, \mathbf{y}; \Lambda)$$

for $\mathbf{y} \in V \setminus \mathfrak{H}_{\mathcal{R}}$. Then $\tilde{F}(\mathbf{t}, \mathbf{y}; \Lambda) = F(\mathbf{t}, \mathbf{y}; \Lambda)$ on the whole V by the one-sided continuity of $F(\mathbf{t}, \mathbf{y}; \Lambda)$ and $\tilde{F}(\mathbf{t}, \mathbf{y}; \Lambda)$ which we have already shown. Thus automatically the assertions (ii) and (iii) of Theorem 2.4 will follow.

Also, comparing (2.9) with (5.10), we find that

$$(5.19) \quad \tilde{C}(\mathbf{k}, \mathbf{y}; \Lambda) = C(\mathbf{k}, \mathbf{y}; \Lambda).$$

By the definition of $\tilde{C}(\mathbf{k}, \mathbf{y}; \Lambda)$ and Proposition 4.1, we have

$$\tilde{C}(\mathbf{k}, \mathbf{y}; \Lambda) = \left(\prod_{f \in \Lambda} -\frac{k_f!}{(2\pi\sqrt{-1})^{k_f}} \right) S(\mathbf{k}, \mathbf{y}; \Lambda)$$

for $\mathbf{y} \in V \setminus \bigcup_{f \in \tilde{\Lambda} \cap \Lambda_1} (\mathfrak{H}_{\Lambda \setminus \{f\}} + \mathbb{Z}^r)$. Combining this with (5.19), we obtain the assertion of Theorem 2.5. Therefore the only remaining task is to show (5.18) for $\mathbf{y} \in V \setminus \mathfrak{H}_{\mathcal{F}}$.

6. THE GENERATING FUNCTION AND CONVEX POLYTOPES

The aim of the following three sections is to prove (5.18), which will be shown in Section 8. The present and the next sections are devoted to the preparations for the proof of (5.18), which are connected with the theory of convex polytopes.

First, we summarize some definitions and facts about convex polytopes (see [6, 7, 13]). For a subset $X \subset \mathbb{R}^N$, we denote by $\text{Conv}(X)$ the convex hull of X . A subset $\mathcal{P} \subset \mathbb{R}^N$ is called a convex polytope if $\mathcal{P} = \text{Conv}(X)$ for some finite subset $X \subset \mathbb{R}^N$. Let \mathcal{P} be a d -dimensional polytope. Let \mathcal{H} be a hyperplane in \mathbb{R}^N . Then \mathcal{H} divides \mathbb{R}^N into two half-spaces. If \mathcal{P} is entirely contained in one of the two closed half-spaces and $\mathcal{P} \cap \mathcal{H} \neq \emptyset$, then \mathcal{H} is called a supporting hyperplane of \mathcal{P} . For a supporting hyperplane \mathcal{H} and a subset $\mathcal{F} = \mathcal{P} \cap \mathcal{H} \neq \emptyset$, the subset \mathcal{F} is called a face of the polytope \mathcal{P} and \mathcal{H} a supporting hyperplane associated with \mathcal{F} . If the dimension of a face \mathcal{F} is j , then we call it a j -face \mathcal{F} . A 0-face is called a vertex, a 1-face an edge and a $(d-1)$ -face a facet. For convenience, we regard \mathcal{P} itself as its unique d -face. Let $\text{Vert}(\mathcal{P})$ be the set of all vertices of \mathcal{P} . Then

$$(6.1) \quad \mathcal{F} = \text{Conv}(\text{Vert}(\mathcal{P}) \cap \mathcal{F}),$$

for a face \mathcal{F} . A d -dimensional simple polytope is a polytope whose vertices are adjacent to exactly d edges.

For $\mathbf{a} = {}^t(a_1, \dots, a_N)$, $\mathbf{b} = {}^t(b_1, \dots, b_N) \in \mathbb{R}^N$ we define $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + \dots + a_N b_N$. The definition of polytopes above is that of ‘‘V-polytopes’’. We mainly deal with another representation of polytopes, ‘‘H-polytopes’’ instead, that is, a bounded subset of the form

$$(6.2) \quad \mathcal{P} = \bigcap_{i \in I} \mathcal{H}_i^+ \subset \mathbb{R}^N,$$

where $\sharp I < \infty$ and $\mathcal{H}_i^+ = \{\mathbf{x} \in \mathbb{R}^N \mid \mathbf{a}_i \cdot \mathbf{x} \geq h_i\}$ with $\mathbf{a}_i \in \mathbb{R}^N$ and $h_i \in \mathbb{R}$. It is known (Weyl–Minkowski) that H-polytopes are V-polytopes and vice versa.

We have an expression of k -faces in terms of hyperplanes $\mathcal{H}_i = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}_i \cdot \mathbf{x} = h_i\}$.

Proposition 6.1 ([7, Proposition 2.7]). *Let $J \subset I$. Assume that $\mathcal{F} = \mathcal{P} \cap \bigcap_{j \in J} \mathcal{H}_j \neq \emptyset$. Then \mathcal{F} is a face.*

Proposition 6.2 ([7, Proposition 2.8]). *Let \mathcal{H} be a supporting hyperplane and $\mathcal{F} = \mathcal{P} \cap \mathcal{H}$ is a k -face. Then there exists a set of indices $J \subset I$ such that $\sharp J = (\dim \mathcal{P}) - k$ and $\mathcal{F} = \mathcal{P} \cap \bigcap_{j \in J} \mathcal{H}_j$.*

Lemma 6.3 ([6, Lemma 6.5]). *Let \mathcal{P} be a simple polytope and $\{\mathbf{p}_0, \dots, \mathbf{p}_K\}$ be the vertices of \mathcal{P} . Let*

$$(6.3) \quad E_k = \{j \mid \text{Conv}(\{\mathbf{p}_k, \mathbf{p}_j\}) \text{ is an edge}\}.$$

Then we have

$$(6.4) \quad \int_{\mathcal{P}} e^{\mathbf{a} \cdot \mathbf{x}} d\mathbf{x} = \sum_{k=0}^K |\det(\mathbf{p}_k - \mathbf{p}_j)_{j \in E_k}| \frac{e^{\mathbf{a} \cdot \mathbf{p}_k}}{\prod_{j \in E_k} \mathbf{a} \cdot (\mathbf{p}_k - \mathbf{p}_j)}.$$

Now we present a fundamental proposition, which gives an expression of $\tilde{F}(\mathbf{t}, \mathbf{y}; \Lambda)$ involving integrals over certain convex polytopes. This proposition is a generalization of [7, Theorem 7].

Proposition 6.4.

$$(6.5) \quad \begin{aligned} \tilde{F}(\mathbf{t}, \mathbf{y}; \Lambda) &= \left(\prod_{f \in \Lambda} \frac{t_f}{\exp(t_f - 2\pi\sqrt{-1}f) - 1} \right) \frac{1}{\sharp(\mathbb{Z}^r / \langle \vec{B}_0 \rangle)} \\ &\times \sum_{\mathbf{m} \in \mathbb{Z}^r} \exp\left(\sum_{f \in B_0} (t_f - 2\pi\sqrt{-1}f) \langle \mathbf{y} + \mathbf{m}, \vec{f}^{B_0} \rangle \right) \\ &\times \int_{\mathcal{P}(\mathbf{m}; \mathbf{y})} \exp\left(\sum_{g \in L_0} t_g^* x_g \right) \prod_{g \in L_0} dx_g, \end{aligned}$$

where

$$(6.6) \quad t_g^* = (t_g - 2\pi\sqrt{-1}\dot{g}) - \sum_{f \in B_0} (t_f - 2\pi\sqrt{-1}\dot{f}) \langle \vec{g}, \vec{f}^{B_0} \rangle$$

and

$$(6.7) \quad \mathcal{P}(\mathbf{m}; \mathbf{y}) = \left\{ \mathbf{x} = (x_g)_{g \in L_0} \left| \begin{array}{l} 0 \leq x_g \leq 1 \quad (g \in L_0) \\ \langle \mathbf{y} + \mathbf{m} - \vec{f}, \vec{f}^{B_0} \rangle \leq \sum_{g \in L_0} x_g \langle \vec{g}, \vec{f}^{B_0} \rangle \leq \langle \mathbf{y} + \mathbf{m}, \vec{f}^{B_0} \rangle \quad (f \in B_0) \end{array} \right. \right\}$$

is a convex polytope or an empty set.

Proof. We fix a representative of each $\mathbf{w} \in \mathbb{Z}^r / \langle \vec{B}_0 \rangle$ in \mathbb{Z}^r . Let $\mathbf{m} = (m_f)_{f \in B_0} \in \mathbb{Z}^r$, and denote by $\mathcal{Q}(\mathbf{w}, \mathbf{m})$ the set of all $\mathbf{x} = (x_g)_{g \in L_0}$ satisfying the conditions $0 \leq x_g \leq 1$ ($g \in L_0$) and

$$(6.8) \quad -m_f \leq \left\langle \mathbf{y} + \mathbf{w} - \sum_{g \in L_0} x_g \vec{g}, \vec{f}^{B_0} \right\rangle < -m_f + 1.$$

This condition is equivalent to

$$(6.9) \quad \left\langle \mathbf{y} + \mathbf{w} + \sum_{h \in B_0} m_h \vec{h} - \vec{f}, \vec{f}^{B_0} \right\rangle < \sum_{g \in L_0} x_g \langle \vec{g}, \vec{f}^{B_0} \rangle \leq \left\langle \mathbf{y} + \mathbf{w} + \sum_{h \in B_0} m_h \vec{h}, \vec{f}^{B_0} \right\rangle,$$

because $\left\langle \sum_{h \in B_0} m_h \vec{h}, \vec{f}^{B_0} \right\rangle = m_f$. Also we have

$$(6.10) \quad \begin{aligned} \left\{ \left\langle \mathbf{y} + \mathbf{w} - \sum_{g \in L_0} x_g \vec{g}, \vec{f}^{B_0} \right\rangle \right\} &= \left\langle \mathbf{y} + \mathbf{w} - \sum_{g \in L_0} x_g \vec{g}, \vec{f}^{B_0} \right\rangle + m_f \\ &= \left\langle \mathbf{y} + \mathbf{w} + \sum_{h \in B_0} m_h \vec{h}, \vec{f}^{B_0} \right\rangle - \sum_{g \in L_0} x_g \langle \vec{g}, \vec{f}^{B_0} \rangle. \end{aligned}$$

Denote the multiple integral on the right-hand side of (5.11) by $I(\mathbf{w})$, and divide it as

$$I(\mathbf{w}) = \sum_{\mathbf{m} \in \mathbb{Z}^r} \int_{\mathcal{Q}(\mathbf{w}, \mathbf{m})}.$$

Applying (6.10), we obtain

$$(6.11) \quad \begin{aligned} I(\mathbf{w}) &= \sum_{\mathbf{m} \in \mathbb{Z}^r} \exp \left(\sum_{f \in B_0} (t_f - 2\pi\sqrt{-1}\dot{f}) \langle \mathbf{y} + \mathbf{w} + \sum_{h \in B_0} m_h \vec{h}, \vec{f}^{B_0} \rangle \right) \\ &\quad \times \int_{\mathcal{Q}(\mathbf{w}, \mathbf{m})} \exp \left(\sum_{g \in L_0} t_g^* x_g \right) \prod_{g \in L_0} dx_g. \end{aligned}$$

Note that $\mathbf{w} + \sum_{h \in B_0} m_h \vec{h}$ runs over \mathbb{Z}^r , when $\mathbf{w} \in \mathbb{Z}^r / \langle \vec{B}_0 \rangle$ and $\mathbf{m} \in \mathbb{Z}^r$ run. Therefore, rewriting $\mathbf{w} + \sum_{h \in B_0} m_h \vec{h}$ as \mathbf{m} , we obtain the assertion of the proposition. \square

Remark 6.5. For readers' convenience, we give typical pictures of $\mathcal{P}(\mathbf{m}; \mathbf{y})$ in the cases $\mathbf{y} \in \mathfrak{H}_{\mathcal{A}}$ and $\mathbf{y} \notin \mathfrak{H}_{\mathcal{A}}$, which will be treated below in Lemmas 7.1, 7.2, 7.3 and 7.4.

Let $V = \mathbb{R}^2$ ($r = 2$). Let $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and

$$(6.12) \quad \Lambda = \{f_1 = (\mathbf{e}_1, \alpha_1), f_2 = (\mathbf{e}_2, \alpha_2), g = (a\mathbf{e}_1 + b\mathbf{e}_2, \alpha_g), h = (c\mathbf{e}_1 + d\mathbf{e}_2, \alpha_h)\} = B_0 \cup L_0,$$

$$(6.13) \quad B_0 = \{(\mathbf{e}_1, \alpha_1), (\mathbf{e}_2, \alpha_2)\},$$

$$(6.14) \quad L_0 = \{(a\mathbf{e}_1 + b\mathbf{e}_2, \alpha_g), (c\mathbf{e}_1 + d\mathbf{e}_2, \alpha_h)\},$$

$$(6.15) \quad \mathfrak{H}_{\mathcal{A}} = (\mathbb{R}\mathbf{e}_1 + \mathbb{Z}^2) \cup (\mathbb{R}\mathbf{e}_2 + \mathbb{Z}^2) \cup (\mathbb{R}(a\mathbf{e}_1 + b\mathbf{e}_2) + \mathbb{Z}^2) \cup (\mathbb{R}(c\mathbf{e}_1 + d\mathbf{e}_2) + \mathbb{Z}^2),$$

where a, b, c, d are positive integers, which corresponds to the series

$$(6.16) \quad S(\mathbf{k}, \mathbf{y}; \Lambda) = \sum_{n_1, n_2} \frac{e^{2\pi\sqrt{-1}(n_1 y_1 + n_2 y_2)}}{(n_1 + \alpha_1)^{k_1} (n_2 + \alpha_2)^{k_2} (a n_1 + b n_2 + \alpha_g)^{k_g} (c n_1 + d n_2 + \alpha_h)^{k_h}},$$

where $\alpha_1, \alpha_2, \alpha_g, \alpha_h \in \mathbb{C}$, $k_1, k_2, k_g, k_h \geq 2$ and n_1, n_2 run over all integers such that the denominator does not vanishes. Then we have

$$(6.17) \quad \mathcal{P}(\mathbf{m}; \mathbf{y}) = \left\{ \mathbf{x} = (x_g, x_h) \left| \begin{array}{l} 0 \leq x_g \leq 1, \\ 0 \leq x_h \leq 1, \\ y_1 + m_1 - 1 \leq a x_g + c x_h \leq y_1 + m_1, \\ y_2 + m_2 - 1 \leq b x_g + d x_h \leq y_2 + m_2 \end{array} \right. \right\}.$$

In the case $a = b = c = 1, d = 2$, the polytope $\mathcal{P}(\mathbf{m}; \mathbf{y})$ is drawn as in Figure 2 if $\mathbf{y} \in \mathfrak{H}_{\mathcal{R}}$ and in Figure 3 if $\mathbf{y} \notin \mathfrak{H}_{\mathcal{R}}$. In the former case, there are more than 2 ($= \#\Lambda - r$) hyperplanes at some vertices while in the latter case, there are only 2 hyperplanes at each vertex, which ensures that $\mathcal{P}(\mathbf{m}; \mathbf{y})$ is a simple polytope in higher dimensions.

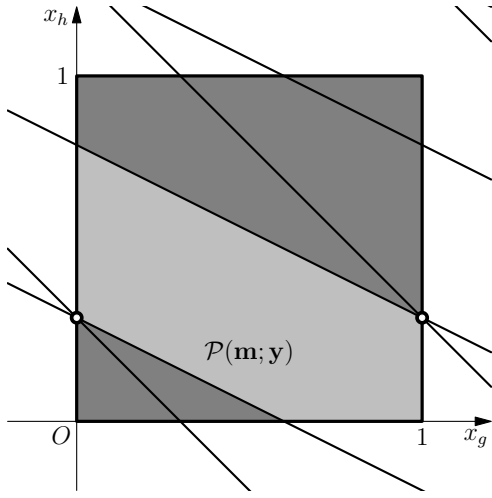


FIGURE 2. $\mathbf{y} \in \mathfrak{H}_{\mathcal{R}}$

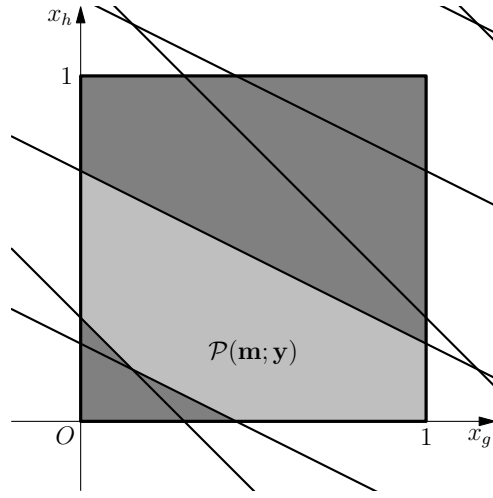


FIGURE 3. $\mathbf{y} \notin \mathfrak{H}_{\mathcal{R}}$

7. PROPERTIES OF THE POLYTOPES $\mathcal{P}(\mathbf{m}; \mathbf{y})$

The argument developed in this and the next sections is a generalization of that in [6, Section 6].

Let $\mathcal{A} = \{0, 1\}^{\#\Lambda - r}$. Let \mathcal{B}' be the set of all subsets of Λ which have r elements, and define $\mathcal{W}' = \mathcal{B}' \times \mathcal{A}$ and $\mathcal{W} = \mathcal{B} \times \mathcal{A}$. Obviously $\mathcal{W} \subset \mathcal{W}'$. For an element $W = (B, A) \in \mathcal{W}'$, we number $A = (a_g)_{g \in \Lambda \setminus B}$. We fix a decomposition $\Lambda = B_0 \cup L_0$ with $B_0 \in \mathcal{B}$, and for $f \in \Lambda$, $a \in \{0, 1\}$, $\mathbf{m} \in \mathbb{Z}^r$ and $\mathbf{y} \in V$, we define $\mathbf{u}(f, a) \in \mathbb{R}^{\#L_0}$ by

$$(7.1) \quad \mathbf{u}(f, a)_g = \begin{cases} (-1)^{1-a} \langle \vec{g}, \vec{f}^{B_0} \rangle & \text{if } f \in B_0, \\ (-1)^a \delta_{gf} & \text{if } f \notin B_0, \end{cases}$$

where g runs over L_0 , and define $v(f, a; \mathbf{m}; \mathbf{y}) \in \mathbb{R}$ by

$$(7.2) \quad v(f, a; \mathbf{m}; \mathbf{y}) = \begin{cases} (-1)^{1-a} \langle \mathbf{y} + \mathbf{m} - a\vec{f}, \vec{f}^{B_0} \rangle & \text{if } f \in B_0, \\ (-1)^a a = -a & \text{if } f \notin B_0. \end{cases}$$

Further we define the hyperplanes

$$(7.3) \quad \mathcal{H}(f, a; \mathbf{m}; \mathbf{y}) = \{ \mathbf{x} = (x_g)_{g \in L_0} \in \mathbb{R}^{\#L_0} \mid \mathbf{u}(f, a) \cdot \mathbf{x} = v(f, a; \mathbf{m}; \mathbf{y}) \},$$

and the half-spaces

$$(7.4) \quad \mathcal{H}^+(f, a; \mathbf{m}; \mathbf{y}) = \{ \mathbf{x} = (x_g)_{g \in L_0} \in \mathbb{R}^{\#L_0} \mid \mathbf{u}(f, a) \cdot \mathbf{x} \geq v(f, a; \mathbf{m}; \mathbf{y}) \},$$

where for $\mathbf{w} = (w_g)$, $\mathbf{x} = (x_g) \in \mathbb{C}^{\sharp L_0}$, we have set

$$(7.5) \quad \mathbf{w} \cdot \mathbf{x} = \sum_{g \in L_0} w_g x_g.$$

Then we have

$$(7.6) \quad \mathcal{P}(\mathbf{m}; \mathbf{y}) = \bigcap_{\substack{f \in \Lambda \\ a \in \{0,1\}}} \mathcal{H}^+(f, a; \mathbf{m}; \mathbf{y}).$$

Lemma 7.1. *All vertices of $\mathcal{P}(\mathbf{m}; \mathbf{y})$ are of the form*

$$(7.7) \quad \bigcap_{f \in L'} \mathcal{H}(f, a_f; \mathbf{m}; \mathbf{y}),$$

where $L' \subset \Lambda$ with $\sharp L' = \sharp L_0 = \sharp \Lambda - r$ and $(a_f)_{f \in L'} \in \mathcal{A}$.

Proof. By Proposition 6.2, we see that any vertex of $\mathcal{P}(\mathbf{m}; \mathbf{y})$ is obtained as the intersection of $(\sharp L_0)$ hyperplanes. Since for $f \in \Lambda$, two hyperplanes $\mathcal{H}(f, a; \mathbf{m}; \mathbf{y})$ ($a = 0, 1$) are parallel and hence their intersection is empty, we see that a vertex must be of the form (7.7). \square

Since $\sharp L' = \sharp \Lambda - r$, we have $B = \Lambda \setminus L' \in \mathcal{B}'$. Therefore Lemma 7.1 implies that any vertex of $\mathcal{P}(\mathbf{m}; \mathbf{y})$ determines an element $(B, A) \in \mathcal{W}'$. The next lemma is a kind of converse assertion.

Lemma 7.2. *Let $(B, A) \in \mathcal{W}'$ and $L' = \Lambda \setminus B$. The set*

$$(7.8) \quad \bigcap_{g \in L'} \mathcal{H}(g, a_g; \mathbf{m}; \mathbf{y})$$

consists of only one point, which we denote by $\mathbf{p}(\mathbf{m}; \mathbf{y}; W)$, if and only if $W = (B, A) \in \mathcal{W}$.

Proof. Let $B = \{f_1, \dots, f_r\} \in \mathcal{B}'$ and $a_f \in \{0, 1\}$ for $f \in L' = \Lambda \setminus B$. Consider the intersection of $(\sharp \Lambda - r)$ hyperplanes (7.8). Then this set consists of the solutions of the system of the $(\sharp L_0)$ linear equations

$$(7.9) \quad \begin{cases} \sum_{g \in L_0} x_g \langle \vec{g}, \vec{f}^{B_0} \rangle = \langle \mathbf{y} + \mathbf{m} - a_f \vec{f}, \vec{f}^{B_0} \rangle & \text{for } f \in B_0 \setminus B, \\ x_f = a_f & \text{for } f \in L_0 \setminus B. \end{cases}$$

The system of the linear equations (7.9) has a unique solution if and only if

$$(7.10) \quad \det(\langle \vec{g}, \vec{f}^{B_0} \rangle)_{\substack{g \in B \setminus B_0 \\ f \in B_0 \setminus B}} \neq 0,$$

and hence if and only if $B \in \mathcal{B}$, since

$$(7.11) \quad \left| \det \begin{pmatrix} \langle \vec{g}, \vec{f}^{B_0} \rangle_{\substack{g \in B \setminus B_0 \\ f \in B_0 \setminus B}} & \langle \vec{g}, \vec{f}^{B_0} \rangle_{\substack{g \in B \setminus B_0 \\ f \in B_0 \cap B}} \\ \langle \vec{g}, \vec{f}^{B_0} \rangle_{\substack{g \in B \cap B_0 \\ f \in B_0 \setminus B}} & \langle \vec{g}, \vec{f}^{B_0} \rangle_{\substack{g \in B \cap B_0 \\ f \in B_0 \cap B}} \end{pmatrix} \right| = \left| \det \begin{pmatrix} \langle \vec{g}, \vec{f}^{B_0} \rangle_{\substack{g \in B \setminus B_0 \\ f \in B_0 \setminus B}} & * \\ 0 & E_{\sharp(B \cap B_0)} \end{pmatrix} \right| \\ = \left| \det(\langle \vec{g}, \vec{f}^{B_0} \rangle)_{\substack{g \in B \setminus B_0 \\ f \in B_0 \setminus B}} \right|,$$

where E_p is the $p \times p$ identity matrix. \square

Lemma 7.3. *Let $W = (B, A) \in \mathcal{W}$. The point $\mathbf{p}(\mathbf{m}; \mathbf{y}; W)$ is a vertex of $\mathcal{P}(\mathbf{m}; \mathbf{y})$ if and only if*

$$(7.12) \quad \mathbf{p}(\mathbf{m}; \mathbf{y}; W) \in \bigcap_{\substack{f \in B \\ a \in \{0,1\}}} \mathcal{H}^+(f, a; \mathbf{m}; \mathbf{y}).$$

Proof. By (7.6), we see that $\mathcal{P}(\mathbf{m}; \mathbf{y})$ is defined by $(\sharp \Lambda)$ pairs of inequalities. By Proposition 6.1, the point $\mathbf{p}(\mathbf{m}; \mathbf{y}; W)$ is a vertex of $\mathcal{P}(\mathbf{m}; \mathbf{y})$ if and only if all of these inequalities hold. We see that $(\sharp L_0)$ pairs among them are automatically satisfied, because

$$(7.13) \quad \{\mathbf{p}(\mathbf{m}; \mathbf{y}; W)\} = \bigcap_{g \in \Lambda \setminus B} \mathcal{H}(g, a_g; \mathbf{m}; \mathbf{y}) \subset \bigcap_{\substack{g \in \Lambda \setminus B \\ a \in \{0,1\}}} \mathcal{H}^+(g, a; \mathbf{m}; \mathbf{y}).$$

Therefore $\mathbf{p}(\mathbf{m}; \mathbf{y}; W)$ is a vertex of $\mathcal{P}(\mathbf{m}; \mathbf{y})$ if and only if the remaining r pairs of inequalities are satisfied, which implies (7.12). \square

Lemma 7.4. *If $\mathbf{y} \in V \setminus \mathfrak{H}_{\mathcal{R}}$ and $\mathcal{P}(\mathbf{m}; \mathbf{y})$ is not empty, then $\mathcal{P}(\mathbf{m}; \mathbf{y})$ is a simple polytope.*

Proof. By Lemmas 7.1 and 7.2, it is sufficient to check the following claim: *If for $W = (B, A) \in \mathcal{W}$, the point $\mathbf{p}(\mathbf{m}; \mathbf{y}; W)$ lies on some other hyperplanes of the form (7.3) than the defining hyperplanes $\{\mathcal{H}(g, a_g; \mathbf{m}; \mathbf{y})\}_{g \in \Lambda \setminus B}$, then $\mathbf{y} \in \mathfrak{H}_{\mathcal{R}}$.* Because this claim implies that if $\mathbf{y} \in V \setminus \mathfrak{H}_{\mathcal{R}}$, we can uniquely determine the $(\sharp L_0)$ hyperplanes on which the point $\mathbf{p}(\mathbf{m}; \mathbf{y}; W)$ lies, and hence it implies the simplicity of the polytope $\mathcal{P}(\mathbf{m}; \mathbf{y})$.

Since

$$(7.14) \quad \mathbf{p}(\mathbf{m}; \mathbf{y}; W) \notin \bigcup_{g \in \Lambda \setminus B} \mathcal{H}(g, 1 - a_g; \mathbf{m}; \mathbf{y})$$

always holds, it is sufficient to check that

$$(7.15) \quad \mathbf{p}(\mathbf{m}; \mathbf{y}; W) \in \bigcup_{\substack{f \in B \\ a \in \{0,1\}}} \mathcal{H}(f, a; \mathbf{m}; \mathbf{y})$$

implies $\mathbf{y} \in \mathfrak{H}_{\mathcal{R}}$.

First we show the claim when

$$(7.16) \quad \mathbf{p}(\mathbf{m}; \mathbf{y}; W) \in \mathcal{H}(h, a_h; \mathbf{m}; \mathbf{y})$$

holds for some $h \in B \cap B_0$ and $a_h \in \{0, 1\}$. For $\mathbf{x} = \mathbf{p}(\mathbf{m}; \mathbf{y}; W)$, condition (7.16) is equivalent to

$$(7.17) \quad \sum_{g \in L_0} x_g \langle \vec{g}, \vec{h}^{B_0} \rangle = \langle \mathbf{y} + \mathbf{m} - a_h \vec{h}, \vec{h}^{B_0} \rangle.$$

Let $p = \sharp(B_0 \setminus B) = \sharp(B \setminus B_0)$. Divide the left-hand side of (the first formula of) (7.9) and (7.17) into two parts according to the conditions $g \in B \setminus B_0$ and $g \in L_0 \setminus B$ (with noting $L_0 = (B \setminus B_0) \cup (L_0 \setminus B)$). Then we obtain an overdetermined system with the p variables x_g for $g \in B \setminus B_0$ and the $(p + 1)$ equations

$$(7.18) \quad \sum_{g \in B \setminus B_0} x_g \langle \vec{g}, \vec{f}^{B_0} \rangle = \langle \mathbf{y} + \mathbf{m} - a_f \vec{f}, \vec{f}^{B_0} \rangle + c_f$$

for $f \in (B_0 \setminus B) \cup \{h\}$, where

$$(7.19) \quad c_f = - \sum_{g \in L_0 \setminus B} a_g \langle \vec{g}, \vec{f}^{B_0} \rangle.$$

Hence we have

$$(7.20) \quad ((x_g)_{g \in B \setminus B_0} \quad -1) M(\mathbf{y}) = (0 \quad \cdots \quad 0),$$

where $(x_g)_{g \in B \setminus B_0}$ is a row vector and $M(\mathbf{y})$ is a $(p + 1) \times (p + 1)$ matrix defined by

$$(7.21) \quad M(\mathbf{y}) = \begin{pmatrix} (\langle \vec{g}, \vec{f}^{B_0} \rangle)_{\substack{g \in B \setminus B_0 \\ f \in (B_0 \setminus B) \cup \{h\}}} \\ (\langle \mathbf{y} + \mathbf{m} - a_f \vec{f}, \vec{f}^{B_0} \rangle + c_f)_{f \in (B_0 \setminus B) \cup \{h\}} \end{pmatrix}.$$

As the consistency for these equations, we get $\det M(\mathbf{y}) = 0$. We may rewrite

$$(7.22) \quad \langle \mathbf{y} + \mathbf{m} - a_f \vec{f}, \vec{f}^{B_0} \rangle + c_f = \left\langle \mathbf{y} + \mathbf{m} - \sum_{g \in (\Lambda \setminus B) \cup \{h\}} a_g \vec{g}, \vec{f}^{B_0} \right\rangle,$$

because

$$\sum_{g \in (\Lambda \setminus B) \cup \{h\}} a_g \vec{g} - \left(\sum_{g \in L_0 \setminus B} a_g \vec{g} + a_f \vec{f} \right) = \sum_{\substack{g \in (B_0 \setminus B) \cup \{h\} \\ g \neq f}} a_g \vec{g}$$

and $\langle \vec{g}, \vec{f}^{B_0} \rangle = 0$ for all \vec{g} on the right-hand side.

Since the row vectors $(\langle \vec{g}, \vec{f}^{B_0} \rangle)_{f \in (B_0 \setminus B) \cup \{h\}}$ for $g \in B \setminus B_0$ are linearly independent, $\det M(\mathbf{y}) = 0$ implies that the last row vector is written as a linear combination of the other row vectors. That is, for $f \in (B_0 \setminus B) \cup \{h\}$ we have

$$(7.23) \quad \left\langle \mathbf{y} + \mathbf{m} - \sum_{g \in (\Lambda \setminus B) \cup \{h\}} a_g \vec{g}, \vec{f}^{B_0} \right\rangle = \sum_{g \in B \setminus B_0} q_g \langle \vec{g}, \vec{f}^{B_0} \rangle,$$

with some $q_g \in \mathbb{R}$, and so

$$(7.24) \quad \left\langle \mathbf{y} + \mathbf{m} - \sum_{g \in (\Lambda \setminus B) \cup \{h\}} a_g \vec{g} - \sum_{g \in B \setminus B_0} q_g \vec{g}, \vec{f}^{B_0} \right\rangle = 0.$$

Vectors which are orthogonal to all \vec{f}^{B_0} ($f \in (B_0 \setminus B) \cup \{h\}$) are spanned by \vec{g} ($g \in (B \cap B_0) \setminus \{h\}$). Therefore, since $\mathbf{m} - \sum_{g \in (\Lambda \setminus B) \cup \{h\}} a_g \vec{g} \in \mathbb{Z}^r$, we have

$$(7.25) \quad \mathbf{y} \in \sum_{g \in B \setminus B_0} \mathbb{R} \vec{g} + \mathbb{Z}^r + \sum_{g \in (B \cap B_0) \setminus \{h\}} \mathbb{R} \vec{g} = \sum_{g \in B \setminus \{h\}} \mathbb{R} \vec{g} + \mathbb{Z}^r \subset \mathfrak{H}_{\mathcal{B}},$$

which implies the desired claim.

Next we consider the condition $\mathbf{p}(\mathbf{m}; \mathbf{y}; W) \in \mathcal{H}(h, a_h; \mathbf{m}; \mathbf{y})$ for some $h \in B \setminus B_0$ and $a_h \in \{0, 1\}$. Then similarly as above, we see that \mathbf{y} lies in $\mathfrak{H}_{\mathcal{B}}$ because

$$(7.26) \quad \det \begin{pmatrix} (\langle \vec{g}, \vec{f}^{B_0} \rangle)_{f \in B_0 \setminus B}^{g \in B \setminus (B_0 \cup \{h\})} \\ (\langle \mathbf{y} + \mathbf{m} - a_f \vec{f}, \vec{f}^{B_0} \rangle + d_f)_{f \in B_0 \setminus B} \end{pmatrix} = 0,$$

where

$$(7.27) \quad d_f = - \sum_{g \in (L_0 \setminus B) \cup \{h\}} a_g \langle \vec{g}, \vec{f}^{B_0} \rangle.$$

This completes the proof of the lemma. \square

Remark 7.5. We draw the picture of a vertex of $\mathcal{P}(\mathbf{m}; \mathbf{y})$ in the same setting as in Figure 3. For example, for $W = (B, A) \in \mathcal{B}$ with $B = \{f_2, h\} \in \mathcal{B}$ and $A = (a_{f_1}, a_g) = (1, 0)$ ($\Lambda \setminus B = \{f_1, g\}$), the associated vertex is $\mathbf{p}(\mathbf{m}; \mathbf{y}; W) = \mathbf{p}(\mathbf{m}; \mathbf{y}; (\{f_2, h\}, (1, 0)))$, which is uniquely determined by the hyperplanes $\mathcal{H}(f_1, 1; \mathbf{m}; \mathbf{y})$ and $\mathcal{H}(g, 0; \mathbf{m}; \mathbf{y})$. See Figure 4.

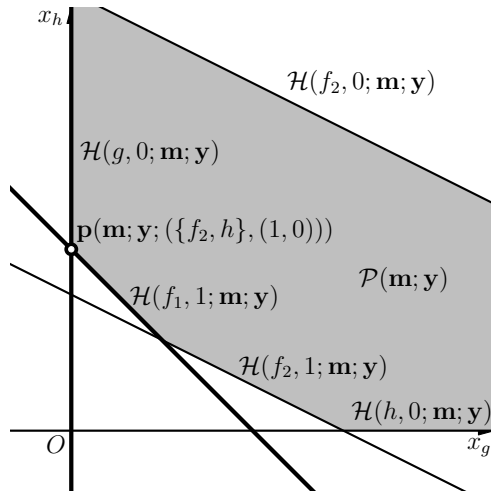


FIGURE 4. vertices and hyperplanes

Lemma 7.6. *Let $\mathbf{y} \in V \setminus \mathfrak{H}_{\mathcal{A}}$ and $W \in \mathcal{W}$. Then we have*

$$(7.28) \quad \sum_{f \in B_0} (t_f - 2\pi\sqrt{-1}\dot{f}) \langle \mathbf{y} + \mathbf{m}, \vec{f}^{B_0} \rangle + \mathbf{t}^* \cdot \mathbf{p}(\mathbf{m}; \mathbf{y}; W) \\ = \sum_{g \in \Lambda \setminus B} (t_g - 2\pi\sqrt{-1}\dot{g}) a_g + \sum_{f \in B} (t_f - 2\pi\sqrt{-1}\dot{f}) \left\langle \mathbf{y} + \mathbf{m} - \sum_{g \in \Lambda \setminus B} a_g \vec{g}, \vec{f}^B \right\rangle,$$

where $\mathbf{t}^* = (t_g^*)_{g \in L_0}$ with t_g^* defined by (6.6).

Proof. By (7.8), the point $\mathbf{p}(\mathbf{m}; \mathbf{y}; W) = (x_g)_{g \in L_0}$ satisfies

$$(7.29) \quad \begin{cases} \sum_{g \in L_0} x_g \langle \vec{g}, \vec{f}^{B_0} \rangle = \langle \mathbf{y} + \mathbf{m} - a_f \vec{f}, \vec{f}^{B_0} \rangle & \text{for } f \in B_0 \setminus B, \\ x_f = a_f & \text{for } f \in L_0 \setminus B. \end{cases}$$

By Lemma 7.2, the system of these equations has a unique solution.

In the case $f \in B_0 \setminus B$, we have

$$(7.30) \quad \sum_{h \in B \setminus B_0} x_h \langle \vec{h}, \vec{f}^{B_0} \rangle = \langle \mathbf{y} + \mathbf{m} - a_f \vec{f}, \vec{f}^{B_0} \rangle - \sum_{g \in L_0 \setminus B} a_g \langle \vec{g}, \vec{f}^{B_0} \rangle \\ = \left\langle \mathbf{y} + \mathbf{m} - \sum_{g \in \Lambda \setminus B} a_g \vec{g}, \vec{f}^{B_0} \right\rangle.$$

On the other hand, in the case $f \in B_0 \cap B$ we have

$$(7.31) \quad \vec{f}^B = \vec{f}^{B_0} - \sum_{h \in B \setminus B_0} \vec{h}^B \langle \vec{h}, \vec{f}^{B_0} \rangle.$$

In fact, since

$$(7.32) \quad \mathbf{z} = \sum_{h \in B} \vec{h}^B \langle \vec{h}, \mathbf{z} \rangle$$

holds for any $\mathbf{z} \in V$, we have

$$(7.33) \quad \vec{f}^{B_0} = \sum_{h \in B \cap B_0} \vec{h}^B \langle \vec{h}, \vec{f}^{B_0} \rangle + \sum_{h \in B \setminus B_0} \vec{h}^B \langle \vec{h}, \vec{f}^{B_0} \rangle = \vec{f}^B + \sum_{h \in B \setminus B_0} \vec{h}^B \langle \vec{h}, \vec{f}^{B_0} \rangle.$$

Here we note that for $h \in B \setminus B_0$,

$$(7.34) \quad x_h = \left\langle \mathbf{y} + \mathbf{m} - \sum_{g \in \Lambda \setminus B} a_g \vec{g}, \vec{h}^B \right\rangle$$

holds. Because, using (7.32), for $f \in B_0 \setminus B$ we obtain

$$(7.35) \quad \sum_{h \in B \setminus B_0} \left\langle \mathbf{y} + \mathbf{m} - \sum_{g \in \Lambda \setminus B} a_g \vec{g}, \vec{h}^B \right\rangle \langle \vec{h}, \vec{f}^{B_0} \rangle = \sum_{h \in B} \left\langle \mathbf{y} + \mathbf{m} - \sum_{g \in \Lambda \setminus B} a_g \vec{g}, \vec{h}^B \right\rangle \langle \vec{h}, \vec{f}^{B_0} \rangle \\ = \left\langle \mathbf{y} + \mathbf{m} - \sum_{g \in \Lambda \setminus B} a_g \vec{g}, \vec{f}^{B_0} \right\rangle.$$

Comparing (7.30) with (7.35), we obtain (7.34) due to the uniqueness of the solution.

Noting $L_0 = (L_0 \setminus B) \cup (B \setminus B_0)$, we have

$$\begin{aligned}
(7.36) \quad & \sum_{f \in B_0} (t_f - 2\pi\sqrt{-1}\dot{f}) \langle \mathbf{y} + \mathbf{m}, \vec{f}^{B_0} \rangle + \mathbf{t}^* \cdot \mathbf{p}(\mathbf{m}; \mathbf{y}; W) \\
&= \sum_{f \in B_0} (t_f - 2\pi\sqrt{-1}\dot{f}) \langle \mathbf{y} + \mathbf{m}, \vec{f}^{B_0} \rangle + \sum_{g \in L_0} \left((t_g - 2\pi\sqrt{-1}\dot{g}) - \sum_{f \in B_0} (t_f - 2\pi\sqrt{-1}\dot{f}) \langle \vec{g}, \vec{f}^{B_0} \rangle \right) x_g \\
&= \sum_{g \in L_0} (t_g - 2\pi\sqrt{-1}\dot{g}) x_g + \sum_{f \in B_0} (t_f - 2\pi\sqrt{-1}\dot{f}) \left(\langle \mathbf{y} + \mathbf{m}, \vec{f}^{B_0} \rangle - \sum_{g \in L_0} x_g \langle \vec{g}, \vec{f}^{B_0} \rangle \right) \\
&= \sum_{g \in L_0 \setminus B} (t_g - 2\pi\sqrt{-1}\dot{g}) a_g + \sum_{h \in B \setminus B_0} (t_h - 2\pi\sqrt{-1}\dot{h}) x_h \\
&\quad + \sum_{f \in B_0 \cap B} (t_f - 2\pi\sqrt{-1}\dot{f}) \left(\langle \mathbf{y} + \mathbf{m}, \vec{f}^{B_0} \rangle - \sum_{g \in L_0 \setminus B} a_g \langle \vec{g}, \vec{f}^{B_0} \rangle - \sum_{h \in B \setminus B_0} x_h \langle \vec{h}, \vec{f}^{B_0} \rangle \right) \\
&\quad + \sum_{f \in B_0 \setminus B} (t_f - 2\pi\sqrt{-1}\dot{f}) \left(\langle \mathbf{y} + \mathbf{m}, \vec{f}^{B_0} \rangle - \sum_{g \in L_0 \setminus B} a_g \langle \vec{g}, \vec{f}^{B_0} \rangle - \sum_{h \in B \setminus B_0} x_h \langle \vec{h}, \vec{f}^{B_0} \rangle \right).
\end{aligned}$$

By the first equality of (7.30), the last term on the last member of (7.36) is equal to

$$\begin{aligned}
(7.37) \quad & \sum_{f \in B_0 \setminus B} (t_f - 2\pi\sqrt{-1}\dot{f}) \left(\langle \mathbf{y} + \mathbf{m}, \vec{f}^{B_0} \rangle - \sum_{g \in L_0 \setminus B} a_g \langle \vec{g}, \vec{f}^{B_0} \rangle - \sum_{h \in B \setminus B_0} x_h \langle \vec{h}, \vec{f}^{B_0} \rangle \right) \\
&= \sum_{f \in B_0 \setminus B} (t_f - 2\pi\sqrt{-1}\dot{f}) \langle a_f \vec{f}, \vec{f}^{B_0} \rangle = \sum_{f \in B_0 \setminus B} (t_f - 2\pi\sqrt{-1}\dot{f}) a_f.
\end{aligned}$$

On the other hand, for $f \in B_0 \cap B$, we have

$$\begin{aligned}
(7.38) \quad & \langle \mathbf{y} + \mathbf{m}, \vec{f}^{B_0} \rangle - \sum_{g \in L_0 \setminus B} a_g \langle \vec{g}, \vec{f}^{B_0} \rangle - \sum_{h \in B \setminus B_0} x_h \langle \vec{h}, \vec{f}^{B_0} \rangle \\
&= \left\langle \mathbf{y} + \mathbf{m} - \sum_{g \in L_0 \setminus B} a_g \vec{g}, \vec{f}^{B_0} \right\rangle - \sum_{h \in B \setminus B_0} \left\langle \mathbf{y} + \mathbf{m} - \sum_{g \in \Lambda \setminus B} a_g \vec{g}, \vec{h}^B \right\rangle \langle \vec{h}, \vec{f}^{B_0} \rangle \\
&= \left\langle \mathbf{y} + \mathbf{m} - \sum_{g \in \Lambda \setminus B} a_g \vec{g}, \vec{f}^{B_0} - \sum_{h \in B \setminus B_0} \vec{h}^B \langle \vec{h}, \vec{f}^{B_0} \rangle \right\rangle \\
&= \left\langle \mathbf{y} + \mathbf{m} - \sum_{g \in \Lambda \setminus B} a_g \vec{g}, \vec{f}^B \right\rangle,
\end{aligned}$$

by (7.34) and (7.31). Therefore we finally obtain (7.28). \square

Lemma 7.7. *Let $\mathbf{y} \in V \setminus \mathfrak{H}_{\mathcal{A}}$ and $W \in \mathcal{W}$. Then the point $\mathbf{p}(\mathbf{m}; \mathbf{y}; W)$ is a vertex of $\mathcal{P}(\mathbf{m}; \mathbf{y})$ if and only if*

$$(7.39) \quad 0 \leq \left\langle \mathbf{y} + \mathbf{m} - \sum_{g \in \Lambda \setminus B} a_g \vec{g}, \vec{f}^B \right\rangle \leq 1$$

for all $f \in B$.

Proof. By Lemma 7.3, the point $\mathbf{p}(\mathbf{m}; \mathbf{y}; W) = (x_g)_{g \in L_0}$ is indeed a vertex if and only if

$$(7.40) \quad \begin{cases} \langle \mathbf{y} + \mathbf{m} - \vec{f}, \vec{f}^{B_0} \rangle \leq \sum_{g \in L_0} x_g \langle \vec{g}, \vec{f}^{B_0} \rangle \leq \langle \mathbf{y} + \mathbf{m}, \vec{f}^{B_0} \rangle, & \text{for } f \in B \cap B_0, \\ 0 \leq x_f \leq 1, & \text{for } f \in B \setminus B_0. \end{cases}$$

For $f \in B \cap B_0$, applying (7.29) (the second equality) and (7.34), we have

$$\begin{aligned}
(7.41) \quad & \sum_{g \in L_0} x_g \langle \vec{g}, \vec{f}^{B_0} \rangle = \sum_{g \in L_0 \setminus B} x_g \langle \vec{g}, \vec{f}^{B_0} \rangle + \sum_{h \in B \setminus B_0} x_h \langle \vec{h}, \vec{f}^{B_0} \rangle \\
&= \left\langle \sum_{g \in \Lambda \setminus B} a_g \vec{g}, \vec{f}^{B_0} \right\rangle + \left\langle \mathbf{y} + \mathbf{m} - \sum_{g \in \Lambda \setminus B} a_g \vec{g}, \sum_{h \in B \setminus B_0} \vec{h}^B \langle \vec{h}, \vec{f}^{B_0} \rangle \right\rangle.
\end{aligned}$$

Therefore, noting (7.31), we see that the first pair of inequalities of (7.40) is

$$\left\langle \mathbf{y} + \mathbf{m} - \vec{f} - \sum_{g \in \Lambda \setminus B} a_g \vec{g}, \vec{f}^{B_0} \right\rangle \leq \left\langle \mathbf{y} + \mathbf{m} - \sum_{g \in \Lambda \setminus B} a_g \vec{g}, \vec{f}^{B_0} - \vec{f}^B \right\rangle \leq \left\langle \mathbf{y} + \mathbf{m} - \sum_{g \in \Lambda \setminus B} a_g \vec{g}, \vec{f}^{B_0} \right\rangle,$$

which is equivalent to

$$(7.42) \quad 0 \leq \left\langle \mathbf{y} + \mathbf{m} - \sum_{g \in \Lambda \setminus B} a_g \vec{g}, \vec{f}^B \right\rangle \leq 1.$$

For $f \in B \setminus B_0$, noting (7.34) we see that the second pair of inequalities of (7.40) is again of the same form as (7.42). Therefore the desired assertion follows. \square

Fix $W = (B, A) \in \mathcal{W}$. Let U be the $(\sharp L_0) \times (\sharp L_0)$ matrix whose f -th column consists of $\mathbf{u}(f, a_f)$ for $f \in \Lambda \setminus B$ and $U(h, \mathbf{v})$ be the matrix U with only the h -th column replaced by \mathbf{v} . Then we have the following two lemmas.

Lemma 7.8.

$$(7.43) \quad |\det U| = \frac{\sharp(\mathbb{Z}^r / \langle \vec{B} \rangle)}{\sharp(\mathbb{Z}^r / \langle \vec{B}_0 \rangle)}.$$

Proof. By rearranging rows and columns we see that

$$(7.44) \quad \begin{aligned} |\det U| &= \left| \det(\mathbf{u}(f, a_f)_g)_{f \in \Lambda \setminus B}^{g \in L_0} \right| \\ &= \left| \det \begin{pmatrix} (\mathbf{u}(f, a_f)_g)_{f \in B_0 \setminus B}^{g \in B \setminus B_0} & (\mathbf{u}(f, a_f)_g)_{f \in L_0 \setminus B}^{g \in B \setminus B_0} \\ (\mathbf{u}(f, a_f)_g)_{f \in B_0 \setminus B}^{g \in L_0 \setminus B} & (\mathbf{u}(f, a_f)_g)_{f \in L_0 \setminus B}^{g \in L_0 \setminus B} \end{pmatrix} \right| \\ &= \left| \det \begin{pmatrix} (\langle \vec{g}, \vec{f}^{B_0} \rangle)_{f \in B_0 \setminus B}^{g \in B \setminus B_0} & 0 \\ * & E_{\sharp(L_0 \setminus B)} \end{pmatrix} \right| \\ &= \left| \det(\langle \vec{g}, \vec{f}^{B_0} \rangle)_{f \in B_0 \setminus B}^{g \in B \setminus B_0} \right| \\ &= \left| \det(\langle \vec{g}, \vec{f}^{B_0} \rangle)_{f \in B_0}^{g \in B} \right| \end{aligned}$$

by (7.11). Further

$$(7.45) \quad \left| \det(\langle \vec{g}, \vec{f}^{B_0} \rangle)_{f \in B_0}^{g \in B} \right| = \left| \det(\vec{g})_{g \in B} \det(\vec{f}^{B_0})_{f \in B_0} \right| = \frac{|\det(\vec{g})_{g \in B}|}{|\det(\vec{f})_{f \in B_0}|} = \frac{\sharp(\mathbb{Z}^r / \langle \vec{B} \rangle)}{\sharp(\mathbb{Z}^r / \langle \vec{B}_0 \rangle)}.$$

\square

Lemma 7.9. For $f \in \Lambda \setminus B$, we have

$$(7.46) \quad \frac{\det U(f, \mathbf{t}^*)}{\det U} = (-1)^{a_f} \left((t_f - 2\pi\sqrt{-1}\dot{f}) - \sum_{g \in B} (t_g - 2\pi\sqrt{-1}\dot{g}) \langle \vec{f}, \vec{g}^B \rangle \right).$$

Proof. We show that $\mathbf{x} = (x_f)_{f \in \Lambda \setminus B}$ defined by

$$(7.47) \quad x_f = (-1)^{a_f} \left((t_f - 2\pi\sqrt{-1}\dot{f}) - \sum_{g \in B} (t_g - 2\pi\sqrt{-1}\dot{g}) \langle \vec{f}, \vec{g}^B \rangle \right)$$

is a unique solution of the linear equation

$$(7.48) \quad U\mathbf{x} = \mathbf{t}^*.$$

Then the statement follows from Cramer's rule.

Now we show that (7.47) satisfies (7.48). First observe

$$\begin{aligned}
(U\mathbf{x})_h &= \sum_{f \in B_0 \setminus B} (-1)^{1-a_f} \langle \vec{h}, \vec{f}^{B_0} \rangle (-1)^{a_f} \left((t_f - 2\pi\sqrt{-1}\dot{f}) - \sum_{g \in B} (t_g - 2\pi\sqrt{-1}\dot{g}) \langle \vec{f}, \vec{g}^B \rangle \right) \\
&\quad + \sum_{f \in L_0 \setminus B} (-1)^{a_f} \delta_{hf} (-1)^{a_f} \left((t_f - 2\pi\sqrt{-1}\dot{f}) - \sum_{g \in B} (t_g - 2\pi\sqrt{-1}\dot{g}) \langle \vec{f}, \vec{g}^B \rangle \right) \\
(7.49) \quad &= - \sum_{f \in B_0 \setminus B} \langle \vec{h}, \vec{f}^{B_0} \rangle \left((t_f - 2\pi\sqrt{-1}\dot{f}) - \sum_{g \in B} (t_g - 2\pi\sqrt{-1}\dot{g}) \langle \vec{f}, \vec{g}^B \rangle \right) \\
&\quad + \sum_{f \in L_0 \setminus B} \delta_{hf} \left((t_h - 2\pi\sqrt{-1}\dot{h}) - \sum_{g \in B} (t_g - 2\pi\sqrt{-1}\dot{g}) \langle \vec{h}, \vec{g}^B \rangle \right).
\end{aligned}$$

Assume $h \in B \setminus B_0$. Then $\delta_{hf} = 0$ for all $f \in L_0 \setminus B$, and hence

$$\begin{aligned}
(U\mathbf{x})_h &= - \sum_{f \in B_0 \setminus B} \langle \vec{h}, \vec{f}^{B_0} \rangle \left((t_f - 2\pi\sqrt{-1}\dot{f}) - \sum_{g \in B} (t_g - 2\pi\sqrt{-1}\dot{g}) \langle \vec{f}, \vec{g}^B \rangle \right) \\
(7.50) \quad &= - \sum_{f \in B_0 \setminus B} \langle \vec{h}, \vec{f}^{B_0} \rangle (t_f - 2\pi\sqrt{-1}\dot{f}) + \sum_{f \in B_0 \setminus B} \sum_{g \in B} (t_g - 2\pi\sqrt{-1}\dot{g}) \langle \vec{h}, \vec{f}^{B_0} \rangle \langle \vec{f}, \vec{g}^B \rangle.
\end{aligned}$$

From (7.32) we have

$$\mathbf{z} = \sum_{f \in B_0 \setminus B} \vec{f}^{B_0} \langle \vec{f}, \mathbf{z} \rangle + \sum_{f \in B_0 \cap B} \vec{f}^{B_0} \langle \vec{f}, \mathbf{z} \rangle,$$

and hence

$$(7.51) \quad \langle \vec{h}, \mathbf{z} \rangle = \sum_{f \in B_0 \setminus B} \langle \vec{h}, \vec{f}^{B_0} \rangle \langle \vec{f}, \mathbf{z} \rangle + \sum_{f \in B_0 \cap B} \langle \vec{h}, \vec{f}^{B_0} \rangle \langle \vec{f}, \mathbf{z} \rangle.$$

Putting $\mathbf{z} = \sum_{g \in B} (t_g - 2\pi\sqrt{-1}\dot{g}) \vec{g}^B$ in (7.51), we obtain

$$\begin{aligned}
(7.52) \quad &\sum_{f \in B_0 \setminus B} \sum_{g \in B} (t_g - 2\pi\sqrt{-1}\dot{g}) \langle \vec{h}, \vec{f}^{B_0} \rangle \langle \vec{f}, \vec{g}^B \rangle \\
&= \sum_{g \in B} (t_g - 2\pi\sqrt{-1}\dot{g}) \delta_{hg} - \sum_{f \in B_0 \cap B} \sum_{g \in B} (t_g - 2\pi\sqrt{-1}\dot{g}) \langle \vec{h}, \vec{f}^{B_0} \rangle \langle \vec{f}, \vec{g}^B \rangle.
\end{aligned}$$

Substituting this into the right-hand side of (7.50), we obtain

$$\begin{aligned}
(7.53) \quad (U\mathbf{x})_h &= - \sum_{f \in B_0 \setminus B} \langle \vec{h}, \vec{f}^{B_0} \rangle (t_f - 2\pi\sqrt{-1}\dot{f}) + (t_h - 2\pi\sqrt{-1}\dot{h}) - \sum_{f \in B_0 \cap B} \sum_{g \in B} (t_g - 2\pi\sqrt{-1}\dot{g}) \langle \vec{h}, \vec{f}^{B_0} \rangle \langle \vec{f}, \vec{g}^B \rangle \\
&= (t_h - 2\pi\sqrt{-1}\dot{h}) - \sum_{f \in B_0 \setminus B} \langle \vec{h}, \vec{f}^{B_0} \rangle (t_f - 2\pi\sqrt{-1}\dot{f}) - \sum_{f \in B_0 \cap B} (t_f - 2\pi\sqrt{-1}\dot{f}) \langle \vec{h}, \vec{f}^{B_0} \rangle = t_h^*.
\end{aligned}$$

Assume $h \in L_0 \setminus B$. Then again using (7.52), we have

$$\begin{aligned}
(7.54) \quad (U\mathbf{x})_h &= - \sum_{f \in B_0 \setminus B} \langle \vec{h}, \vec{f}^{B_0} \rangle (t_f - 2\pi\sqrt{-1}\dot{f}) + \sum_{f \in B_0 \setminus B} \sum_{g \in B} (t_g - 2\pi\sqrt{-1}\dot{g}) \langle \vec{h}, \vec{f}^{B_0} \rangle \langle \vec{f}, \vec{g}^B \rangle \\
&\quad + (t_h - 2\pi\sqrt{-1}\dot{h}) - \sum_{g \in B} (t_g - 2\pi\sqrt{-1}\dot{g}) \langle \vec{h}, \vec{g}^B \rangle \\
&= - \sum_{f \in B_0 \setminus B} \langle \vec{h}, \vec{f}^{B_0} \rangle (t_f - 2\pi\sqrt{-1}\dot{f}) - \sum_{f \in B_0 \cap B} \sum_{g \in B} (t_g - 2\pi\sqrt{-1}\dot{g}) \langle \vec{h}, \vec{f}^{B_0} \rangle \langle \vec{f}, \vec{g}^B \rangle \\
&\quad + (t_h - 2\pi\sqrt{-1}\dot{h}) \\
&= t_h^*.
\end{aligned}$$

□

8. COMPLETION OF THE PROOF OF THEOREM 2.4 AND THEOREM 2.5

In this section we prove (5.18) to complete the proof of Theorems 2.4 and 2.5. First we show an elementary lemma.

Let $n \in \mathbb{N}$ and let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a standard basis of \mathbb{R}^n . Let $M = (m_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$ be an $n \times n$ matrix. For $1 \leq j \leq n$, and $\mathbf{v} \in \mathbb{R}^n$, let $M(j, \mathbf{v})$ be the matrix M with only the j -th column replaced by \mathbf{v} . Let

$$(8.1) \quad \mathbf{u}(j) = \det M(j, (\mathbf{e}_i)_{1 \leq i \leq n}) = \det \begin{pmatrix} m_{11} & \cdots & m_{1j-1} & \mathbf{e}_1 & m_{1j+1} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nj-1} & \mathbf{e}_n & m_{nj+1} & \cdots & m_{nn} \end{pmatrix} = \sum_{i=1}^n b_{ij} \mathbf{e}_i \in \mathbb{R}^n,$$

where b_{ij} is the cofactor of (i, j) -entry of M . The following lemma (properties of cofactor matrices) is shown by elementary linear algebra, and we omit the proof.

Lemma 8.1. (1) For $\mathbf{v} \in \mathbb{R}^n$, we have

$$(8.2) \quad \mathbf{u}(j) \cdot \mathbf{v} = \det M(j, \mathbf{v}).$$

(2) Let $U = (\mathbf{u}(j))_{1 \leq j \leq n}$ be the $n \times n$ matrix whose j -th column consists of $\mathbf{u}(j)$. Then

$$(8.3) \quad \det U = (\det M)^{n-1}.$$

Now we start the proof of (5.18), that is $F(\mathbf{t}, \mathbf{y}; \Lambda) = \tilde{F}(\mathbf{t}, \mathbf{y}; \Lambda)$ for $\mathbf{y} \in V \setminus \mathfrak{H}_{\mathcal{A}}$.

Since all the polytopes $\mathcal{P}(\mathbf{m}; \mathbf{y})$ for $\mathbf{y} \in V \setminus \mathfrak{H}_{\mathcal{A}}$ are empty or simple by Lemma 7.4, we may apply Lemma 6.3 to the right-hand side of Proposition 6.4. In the present case the vertices are of the form $\mathbf{p}(\mathbf{m}; \mathbf{y}; W)$, satisfying (7.39) by Lemma 7.7. Therefore we have

$$(8.4) \quad \begin{aligned} \tilde{F}(\mathbf{t}, \mathbf{y}; \Lambda) &= \left(\prod_{f \in \Lambda} \frac{t_f}{\exp(t_f - 2\pi\sqrt{-1}f) - 1} \right) \frac{1}{\sharp(\mathbb{Z}^r / \langle \vec{B}_0 \rangle)} \sum_{\mathbf{m} \in \mathbb{Z}^r} \sum_W \\ &\times \exp \left(\sum_{f \in B_0} (t_f - 2\pi\sqrt{-1}f) \langle \mathbf{y} + \mathbf{m}, \vec{f}^{B_0} \rangle + \mathbf{t}^* \cdot \mathbf{p}(\mathbf{m}; \mathbf{y}; W) \right) \\ &\times \frac{|\det(\mathbf{p}(\mathbf{m}; \mathbf{y}; W) - \mathbf{p}(\mathbf{m}; \mathbf{y}; W'))_{W' \in E(\mathbf{m}; W)}|}{\prod_{W' \in E(\mathbf{m}; W)} \mathbf{t}^* \cdot (\mathbf{p}(\mathbf{m}; \mathbf{y}; W) - \mathbf{p}(\mathbf{m}; \mathbf{y}; W'))}, \end{aligned}$$

where $E(\mathbf{m}; W)$ is the set of all indices W' such that $\text{Conv}(\{\mathbf{p}(\mathbf{m}; \mathbf{y}; W), \mathbf{p}(\mathbf{m}; \mathbf{y}; W')\})$ is an edge of $\mathcal{P}(\mathbf{m}; \mathbf{y})$, and for each $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r$, $W = (B, A) \in \mathcal{W}$ runs over those satisfying

$$(8.5) \quad 0 \leq \left\langle \mathbf{y} + \mathbf{m} - \sum_{g \in \Lambda \setminus B} a_g \vec{g}, \vec{f}^B \right\rangle \leq 1.$$

Recall that a vertex $\mathbf{p}(\mathbf{m}; \mathbf{y}; W)$ satisfies $(\sharp\Lambda - r)$ equations of the form

$$(8.6) \quad \mathbf{u}(f, a_f) \cdot \mathbf{p}(\mathbf{m}; \mathbf{y}; W) = v(f, a_f; \mathbf{m}; \mathbf{y})$$

for $f \in \Lambda \setminus B$ with $W = (B, A)$ (see (7.13)). For $W' = (B', A') \in E(\mathbf{m}; W)$, we see that the two distinct vertices $\mathbf{p}(\mathbf{m}; \mathbf{y}; W)$ and $\mathbf{p}(\mathbf{m}; \mathbf{y}; W')$ share common $(\sharp\Lambda - r - 1)$ hyperplanes, that is, there exists $h \in \Lambda \setminus B$ such that $\Lambda \setminus (B \cup \{h\}) \subset \Lambda \setminus B'$ and $a_f = a'_f$ for $f \in \Lambda \setminus (B \cup \{h\})$, which implies

$$(8.7) \quad \begin{aligned} \mathbf{u}(f, a_f) \cdot \mathbf{p}(\mathbf{m}; \mathbf{y}; W) &= v(f, a_f; \mathbf{m}; \mathbf{y}), \\ \mathbf{u}(f, a_f) \cdot \mathbf{p}(\mathbf{m}; \mathbf{y}; W') &= v(f, a_f; \mathbf{m}; \mathbf{y}) \end{aligned}$$

for $f \in \Lambda \setminus (B \cup \{h\})$ and

$$(8.8) \quad \begin{aligned} \mathbf{u}(h, a_h) \cdot \mathbf{p}(\mathbf{m}; \mathbf{y}; W) &= v(h, a_h; \mathbf{m}; \mathbf{y}), \\ \mathbf{u}(h, a_h) \cdot \mathbf{p}(\mathbf{m}; \mathbf{y}; W') &> v(h, a_h; \mathbf{m}; \mathbf{y}). \end{aligned}$$

This h is unique because otherwise we have $\mathbf{p}(\mathbf{m}; \mathbf{y}; W) = \mathbf{p}(\mathbf{m}; \mathbf{y}; W')$. Since $\sharp E(\mathbf{m}; W) = \sharp(\Lambda \setminus B) = \sharp\Lambda - r$ (because $\mathcal{P}(\mathbf{m}; \mathbf{y})$ is simple), we find that there is a one-to-one corresponding between $E(\mathbf{m}; W)$ and $\Lambda \setminus B$.

By (8.7), we see that the set of the equations and the inequality with respect to \mathbf{v}

$$(8.9) \quad \begin{cases} \mathbf{u}(f, a_f) \cdot \mathbf{v} = 0 & (f \in \Lambda \setminus (B \cup \{h\})), \\ \mathbf{u}(h, a_h) \cdot \mathbf{v} < 0 \end{cases}$$

has a solution $\mathbf{v} = \mathbf{p}(\mathbf{m}; \mathbf{y}; W) - \mathbf{p}(\mathbf{m}; \mathbf{y}; W')$.

We construct another vector $\mathbf{e}(\mathbf{m}; W, W')$ satisfying (8.9) so that

$$(8.10) \quad \mathbf{p}(\mathbf{m}; \mathbf{y}; W) - \mathbf{p}(\mathbf{m}; \mathbf{y}; W') = c(\mathbf{m}; \mathbf{y}; W, W') \mathbf{e}(\mathbf{m}; W, W'),$$

where $c(\mathbf{m}; \mathbf{y}; W, W') > 0$. Then it follows that

$$(8.11) \quad \frac{|\det(\mathbf{p}(\mathbf{m}; \mathbf{y}; W) - \mathbf{p}(\mathbf{m}; \mathbf{y}; W'))_{W' \in E(\mathbf{m}; W)}|}{\prod_{W' \in E(\mathbf{m}; W)} \mathbf{t}^* \cdot (\mathbf{p}(\mathbf{m}; \mathbf{y}; W) - \mathbf{p}(\mathbf{m}; \mathbf{y}; W'))} \\ = \frac{|\prod_{W' \in E(\mathbf{m}; W)} c(W, W'; \mathbf{m}; \mathbf{y})| |\det(\mathbf{e}(\mathbf{m}; W, W'))_{W' \in E(\mathbf{m}; W)}|}{\prod_{W' \in E(\mathbf{m}; W)} c(\mathbf{m}; \mathbf{y}; W, W') \mathbf{t}^* \cdot \mathbf{e}(\mathbf{m}; W, W')} \\ = \frac{|\det(\mathbf{e}(\mathbf{m}; W, W'))_{W' \in E(\mathbf{m}; W)}|}{\prod_{W' \in E(\mathbf{m}; W)} \mathbf{t}^* \cdot \mathbf{e}(\mathbf{m}; W, W')}.$$

The construction of $\mathbf{e}(\mathbf{m}; W, W')$ is as follows. Let \mathbf{e}_g for $g \in L_0$ be the standard orthonormal basis of $\mathbb{R}^{\#\Lambda}$. Let U be the $(\#\Lambda - r) \times (\#\Lambda - r)$ matrix whose f -th column consists of $\mathbf{u}(f, a_f)$ for $f \in \Lambda \setminus B$ with $W = (B, A)$. For $h \in \Lambda \setminus B$, let $U(h, \mathbf{v})$ be the matrix U with only the h -th column replaced by \mathbf{v} . Note that $\det U \neq 0$ by Lemma 7.8. Define

$$(8.12) \quad \mathbf{e}(\mathbf{m}; W, W') = -(\operatorname{sgn} \det U) \det U(h, (\mathbf{e}_g)_{g \in L_0}) = -(\operatorname{sgn} \det U) \sum_{g \in L_0} b_{gh} \mathbf{e}_g$$

(the second equality is due to (8.1)), where $W' = (B', A')$ such that $\Lambda \setminus (B \cup \{h\}) \subset \Lambda \setminus B'$ and b_{gh} is the cofactor of (g, h) -entry of U . Then by Lemma 8.1(1), we have

$$(8.13) \quad \begin{aligned} \mathbf{e}(\mathbf{m}; W, W') \cdot \mathbf{u}(f, a_f) &= -(\operatorname{sgn} \det U) \det U(h, \mathbf{u}(f, a_f)) = 0, \\ \mathbf{e}(\mathbf{m}; W, W') \cdot \mathbf{u}(h, a_h) &= -(\operatorname{sgn} \det U) \det U(h, \mathbf{u}(h, a_h)) = -(\operatorname{sgn} \det U) \det U = -|\det U| < 0 \end{aligned}$$

for $f \in \Lambda \setminus (B \cup \{h\})$ as required.

We observed that h runs over $\Lambda \setminus B$ when W' runs over $E(\mathbf{m}; W)$. Therefore by Lemma 8.1(2) we see that (8.12) implies $|\det(\mathbf{e}(\mathbf{m}; W, W'))_{W' \in E(\mathbf{m}; W)}| = |\det U|^{\#\Lambda - r - 1}$.

Also, from (8.12) we have

$$\mathbf{t}^* \cdot \mathbf{e}(\mathbf{m}; W, W') = -(\operatorname{sgn} \det U) \sum_{g \in L_0} b_{gh} t_g^* = -(\operatorname{sgn} \det U) \det U(h, \mathbf{t}^*).$$

Therefore

$$(8.14) \quad \frac{|\det(\mathbf{e}(\mathbf{m}; W, W'))_{W' \in E(\mathbf{m}; W)}|}{\prod_{W' \in E(\mathbf{m}; W)} \mathbf{t}^* \cdot \mathbf{e}(\mathbf{m}; W, W')} = (-1)^{\#\Lambda - r} (\operatorname{sgn} \det U)^{\#\Lambda - r} \frac{|\det U|^{\#\Lambda - r - 1}}{\prod_{h \in \Lambda \setminus B} \det U(h, \mathbf{t}^*)} \\ = (-1)^{\#\Lambda - r} \frac{1}{|\det U| \prod_{h \in \Lambda \setminus B} \det U(h, \mathbf{t}^*)}.$$

Substituting this into the right-hand side of (8.4) and using Lemmas 7.6, 7.8 and 7.9, we have

$$(8.15) \quad \begin{aligned} \tilde{F}(\mathbf{t}, \mathbf{y}; \Lambda) &= (-1)^{\#\Lambda - r} \left(\prod_{f \in \Lambda} \frac{t_f}{\exp(t_f - 2\pi\sqrt{-1}\dot{f}) - 1} \right) \sum_{\mathbf{m} \in \mathbb{Z}^r} \sum_W \frac{1}{\#\langle \mathbb{Z}^r / \langle \vec{B} \rangle \rangle} \\ &\quad \times \exp \left(\sum_{g \in \Lambda \setminus B} (t_g - 2\pi\sqrt{-1}\dot{g}) a_g + \sum_{f \in B} (t_f - 2\pi\sqrt{-1}\dot{f}) \left\langle \mathbf{y} + \mathbf{m} - \sum_{g \in \Lambda \setminus B} a_g \vec{g}, \vec{f}^B \right\rangle \right) \\ &\quad \times \prod_{h \in \Lambda \setminus B} \frac{(-1)^{a_h}}{(t_h - 2\pi\sqrt{-1}\dot{h}) - \sum_{g \in B} (t_g - 2\pi\sqrt{-1}\dot{g}) \langle \vec{h}, \vec{g}^B \rangle}. \end{aligned}$$

We rewrite the double sum on the first line of the above so as to exchange the order of the sums with respect to $W \in \mathscr{W}$ and $\mathbf{m} \in \mathbb{Z}^r$. For each $W \in \mathscr{W}$, we see that

$$(8.16) \quad \mathbf{m} - \sum_{g \in \Lambda \setminus B} a_g \vec{g}$$

runs over \mathbb{Z}^r when \mathbf{m} runs over \mathbb{Z}^r . Thus (8.5) can be rewritten in terms of $\mathbf{v} \in \mathbb{Z}^r$, that is, \mathbf{v} runs over those satisfying

$$(8.17) \quad 0 \leq \langle \mathbf{y} + \mathbf{v}, \vec{f}^B \rangle \leq 1$$

for all $f \in B$. If there exist $f \in B$ and $\mathbf{v} \in \mathbb{Z}^r$ such that $c = \langle \mathbf{y} + \mathbf{v}, \vec{f}^B \rangle \in \mathbb{Z}$, then we can write $\mathbf{y} + \mathbf{v} = \sum_{g \in R} c_g \vec{g} + c \vec{f}$, where $R = B \setminus \{f\} \in \mathscr{R}$ and $c_g \in \mathbb{R}$. Therefore we have $\mathbf{y} + \mathbf{v} - c \vec{f} \in \mathfrak{H}_R$, hence $\mathbf{y} \in \mathfrak{H}_{\mathscr{R}}$, which contradicts with the assumption. Thus condition (8.17) can be replaced by

$$(8.18) \quad 0 < \langle \mathbf{y} + \mathbf{v}, \vec{f}^B \rangle < 1.$$

Let $G = \{\mathbf{v} \in \mathbb{Z}^r \mid 0 < \langle \mathbf{y} + \mathbf{v}, \vec{f}^B \rangle < 1 \text{ for all } f \in B\}$. We show that the natural projection $g : G \rightarrow \mathbb{Z}^r / \langle \vec{B} \rangle$ is bijective. If $g(\mathbf{v}) = g(\mathbf{v}')$ for $\mathbf{v}, \mathbf{v}' \in G$, then $\mathbf{v} = \mathbf{v}' + \mathbf{x}$ for some $\mathbf{x} \in \langle \vec{B} \rangle$. Since $\langle \mathbf{x}, \vec{f}^B \rangle = \langle \mathbf{y} + \mathbf{v}, \vec{f}^B \rangle - \langle \mathbf{y} + \mathbf{v}', \vec{f}^B \rangle$, we have $-1 < \langle \mathbf{x}, \vec{f}^B \rangle < 1$ for all $f \in B$, which implies $\mathbf{x} = 0$. Conversely, for any $\mathbf{v} \in \mathbb{Z}^r$, putting $\mathbf{x} = \sum_{g \in B} c_g \vec{g} \in \langle \vec{B} \rangle$ with $c_g = -[\langle \mathbf{y} + \mathbf{v}, \vec{g}^B \rangle] \in \mathbb{Z}$, we have

$$\langle \mathbf{y} + \mathbf{v} + \mathbf{x}, \vec{f}^B \rangle = \langle \mathbf{y} + \mathbf{v}, \vec{f}^B \rangle - c_f = \{\langle \mathbf{y} + \mathbf{v}, \vec{f}^B \rangle\}$$

and so $0 < \langle \mathbf{y} + \mathbf{v} + \mathbf{x}, \vec{f}^B \rangle < 1$ because $\mathbf{y} \notin \mathfrak{H}_{\mathscr{R}}$. This implies the assertion. Hence replacing $\langle \mathbf{y} + \mathbf{v}, \vec{f}^B \rangle$ by $\{\langle \mathbf{y} + \mathbf{v}, \vec{f}^B \rangle\}$, we see that \mathbf{v} runs over all representatives of $\mathbb{Z}^r / \langle \vec{B} \rangle$.

Therefore by exchanging the order of the sums with respect to $W = (B, A) \in \mathscr{W} = \mathscr{B} \times \mathscr{A}$ and $\mathbf{m} \in \mathbb{Z}^r$, and summing with respect to \mathbf{v} , we have

$$(8.19) \quad \begin{aligned} \tilde{F}(\mathbf{t}, \mathbf{y}; \Lambda) &= (-1)^{\#\Lambda-r} \left(\prod_{f \in \Lambda} \frac{t_f}{\exp(t_f - 2\pi\sqrt{-1}\dot{f}) - 1} \right) \sum_{W \in \mathscr{W}} \frac{1}{\#(\mathbb{Z}^r / \langle \vec{B} \rangle)} \\ &\times \sum_{\mathbf{v} \in \mathbb{Z}^r / \langle \vec{B} \rangle} \exp\left(\sum_{g \in \Lambda \setminus B} (t_g - 2\pi\sqrt{-1}\dot{g})a_g + \sum_{f \in B} (t_f - 2\pi\sqrt{-1}\dot{f})\{\langle \mathbf{y} + \mathbf{v}, \vec{f}^B \rangle\} \right) \\ &\times \prod_{h \in \Lambda \setminus B} \frac{(-1)^{a_h}}{(t_h - 2\pi\sqrt{-1}\dot{h}) - \sum_{g \in B} (t_g - 2\pi\sqrt{-1}\dot{g})\langle \vec{h}, \vec{g}^B \rangle} \\ &= (-1)^{\#\Lambda-r} \left(\prod_{f \in \Lambda} \frac{t_f}{\exp(t_f - 2\pi\sqrt{-1}\dot{f}) - 1} \right) \sum_{B \in \mathscr{B}} \sum_{A \in \mathscr{A}} \left(\prod_{h \in \Lambda \setminus B} (-1)^{a_h} \exp((t_h - 2\pi\sqrt{-1}\dot{h})a_h) \right) \\ &\times \frac{1}{\#(\mathbb{Z}^r / \langle \vec{B} \rangle)} \sum_{\mathbf{v} \in \mathbb{Z}^r / \langle \vec{B} \rangle} \exp\left(\sum_{f \in B} (t_f - 2\pi\sqrt{-1}\dot{f})\{\langle \mathbf{y} + \mathbf{v}, \vec{f}^B \rangle\} \right) \\ &\times \prod_{h \in \Lambda \setminus B} \frac{1}{(t_h - 2\pi\sqrt{-1}\dot{h}) - \sum_{g \in B} (t_g - 2\pi\sqrt{-1}\dot{g})\langle \vec{h}, \vec{g}^B \rangle}. \end{aligned}$$

Since

$$(8.20) \quad (-1)^{a_h} \exp((t_h - 2\pi\sqrt{-1}\dot{h})a_h) = \begin{cases} -\exp(t_h - 2\pi\sqrt{-1}\dot{h}) & \text{if } a_h = 1, \\ 1 & \text{if } a_h = 0, \end{cases}$$

we have

$$(8.21) \quad \begin{aligned} \sum_{A \in \mathscr{A}} \left(\prod_{h \in \Lambda \setminus B} (-1)^{a_h} \exp((t_h - 2\pi\sqrt{-1}\dot{h})a_h) \right) &= \prod_{h \in \Lambda \setminus B} (1 - \exp(t_h - 2\pi\sqrt{-1}\dot{h})) \\ &= (-1)^{\#\Lambda-r} \prod_{h \in \Lambda \setminus B} (\exp(t_h - 2\pi\sqrt{-1}\dot{h}) - 1). \end{aligned}$$

Therefore the rightmost side of (8.19) is finally equal to

$$(8.22) \quad \begin{aligned} & \left(\prod_{f \in \Lambda} \frac{t_f}{\exp(t_f - 2\pi\sqrt{-1}\dot{f}) - 1} \right) \sum_{B \in \mathcal{B}} \left(\prod_{h \in \Lambda \setminus B} (\exp(t_h - 2\pi\sqrt{-1}\dot{h}) - 1) \right) \\ & \times \frac{1}{\#(\mathbb{Z}^r / \langle \vec{B} \rangle)} \sum_{\mathbf{v} \in \mathbb{Z}^r / \langle \vec{B} \rangle} \exp \left(\sum_{f \in B} (t_f - 2\pi\sqrt{-1}\dot{f}) \langle \mathbf{y} + \mathbf{v}, \vec{f}^B \rangle \right) \\ & \times \prod_{h \in \Lambda \setminus B} \frac{1}{(t_h - 2\pi\sqrt{-1}\dot{h}) - \sum_{g \in B} (t_g - 2\pi\sqrt{-1}\dot{g}) \langle \vec{h}, \vec{g}^B \rangle}, \end{aligned}$$

and coincides with (2.7) for $\mathbf{y} \in V \setminus \mathfrak{H}_{\mathcal{B}}$. This completes the proof of (5.18), and hence the proof of Theorems 2.4 and 2.5.

9. A HIERARCHY AND DIFFERENTIAL EQUATIONS

We conclude this paper with a theorem which asserts that the family of our generating functions has a hierarchy. Let

$$(9.1) \quad \mathbf{\Lambda}_r = \{ \Lambda \subset (\mathbb{Z}^r \setminus \{\vec{0}\}) \times \mathbb{C} \mid \#\Lambda < \infty, \text{rank}(\vec{\Lambda}) = r \}.$$

For $\mathbf{v} = (v_1, \dots, v_r) \in V$, let

$$(9.2) \quad \partial_{\mathbf{v}} = v_1 \partial_{y_1} + \dots + v_r \partial_{y_r},$$

where ∂_{y_j} is the j -th partial differential operator acting on $\mathbf{y} = (y_j)_{1 \leq j \leq r} \in V$. For $g = (\vec{g}, \dot{g}) \in (\mathbb{Z}^r \setminus \{\vec{0}\}) \times \mathbb{C}$, define

$$(9.3) \quad D_g = \frac{t_g - 2\pi\sqrt{-1}\dot{g}}{t_g} - \frac{1}{t_g} \partial_{\vec{g}}.$$

Theorem 9.1. *Let $\Lambda, \Lambda' \in \mathbf{\Lambda}_r$ with $\Lambda' \subset \Lambda$, and $\mathbf{t} = (t_g)_{g \in \Lambda}$, $\mathbf{t}' = (t_g)_{g \in \Lambda'}$. We have*

$$(9.4) \quad \left(\prod_{g \in \Lambda \setminus \Lambda'} D_g \right) F(\mathbf{t}, \mathbf{y}; \Lambda) = F(\mathbf{t}', \mathbf{y}; \Lambda'),$$

where on the left-hand side D_g is understood to act at $\mathbf{y} \in V \setminus \mathfrak{H}_{\mathcal{B}(\Lambda)}$ and the resulting function, to be continuously extended by the one-sided limit along ϕ .

Proof. It is sufficient to show the assertion in the case $\Lambda \setminus \Lambda' = \{g\}$ and $\mathbf{y} \in V \setminus \mathfrak{H}_{\mathcal{B}(\Lambda)}$. For $B \in \mathcal{B}(\Lambda)$ and $\mathbf{w} \in \mathbb{Z}^r / \langle \vec{B} \rangle$, we put

$$(9.5) \quad F_{B, \mathbf{w}}(\mathbf{t}, \mathbf{y}; \Lambda) := \left(\prod_{h \in \Lambda \setminus B} K(\mathbf{t}, h)^{-1} \right) \left(\prod_{f \in B} \frac{t_f \exp((t_f - 2\pi\sqrt{-1}\dot{f}) \langle \mathbf{y} + \mathbf{w} \rangle_{B, f})}{\exp(t_f - 2\pi\sqrt{-1}\dot{f}) - 1} \right),$$

where for $h \in \Lambda$,

$$(9.6) \quad K(\mathbf{t}, h) = \frac{t_h - 2\pi\sqrt{-1}\dot{h} - \sum_{f \in B} (t_f - 2\pi\sqrt{-1}\dot{f}) \langle \vec{h}, \vec{f}^B \rangle}{t_h},$$

so that

$$(9.7) \quad F(\mathbf{t}, \mathbf{y}; \Lambda) = \sum_{B \in \mathcal{B}(\Lambda)} \frac{1}{\#(\mathbb{Z}^r / \langle \vec{B} \rangle)} \sum_{\mathbf{w} \in \mathbb{Z}^r / \langle \vec{B} \rangle} F_{B, \mathbf{w}}(\mathbf{t}, \mathbf{y}; \Lambda).$$

By simple computations we obtain

$$(9.8) \quad D_g F_{B, \mathbf{w}}(\mathbf{t}, \mathbf{y}; \Lambda) = K(\mathbf{t}, g) F_{B, \mathbf{w}}(\mathbf{t}, \mathbf{y}; \Lambda).$$

If $g \in \Lambda \setminus B$, then the factor $K(\mathbf{t}, g)$ cancels with the factor $K(\mathbf{t}, g)^{-1}$ appearing on the right-hand side of (9.5), and so the variable t_g disappears. Thus we have

$$(9.9) \quad D_g F_{B, \mathbf{w}}(\mathbf{t}, \mathbf{y}; \Lambda) = F_{B, \mathbf{w}}(\mathbf{t}', \mathbf{y}; \Lambda \setminus \{g\}).$$

In this case $B \in \mathcal{B}(\Lambda')$.

If $g \in B$, then

$$(9.10) \quad \sum_{f \in B} (t_f - 2\pi\sqrt{-1}f) \langle \vec{g}, \vec{f}^B \rangle = t_g - 2\pi\sqrt{-1}g$$

and hence $K(\mathbf{t}, g) = 0$ and

$$(9.11) \quad D_g F_{B, \mathbf{w}}(\mathbf{t}, \mathbf{y}; \Lambda) = 0.$$

Thus the sum runs over all $\mathcal{B}(\Lambda')$ and

$$(9.12) \quad D_g F(\mathbf{t}, \mathbf{y}; \Lambda) = F(\mathbf{t}', \mathbf{y}; \Lambda \setminus \{g\}).$$

□

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