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ON MIXED JOINT DISCRETE UNIVERSALITY FOR A CLASS OF ZETA-FUNCTIONS

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ABSTRACT

We prove a mixed joint discrete universality theorem for the Matsumoto zeta-function $\varphi(s)$ (belonging to the Steuding subclass) and the periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{B})$. For this purpose, certain independence condition for the parameter α and the minimal step of discrete shifts of these functions is assumed. This paper is a continuation of authors' works [12] and [13].

1. INTRODUCTION

In analytic number theory, the problem of the so-called mixed joint universality in Voronin's sense is a very interesting problem since it solves a problem on simultaneous approximation of certain tuples of analytic functions by shifts of tuples consisting of zeta-functions having an Euler product expansion over the set of primes and other zeta-functions without such a product. For such a type of universality, a very important role is played by the parameters that occur in the definitions of zeta-functions.

The first result on mixed joint universality was obtained by Mishou [21]. He proved that the Riemann zeta-function $\zeta(s)$ and the Hurwitz zeta-function $\zeta(s,\alpha)$ with transcendental parameter α are jointly universal.

Let $\mathbb{P}$, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ be the sets of all primes, positive integers, nonnegative integers, integers, rational numbers, real numbers, and complex numbers, respectively. Denote by $s=\sigma+it$ a complex variable. Recall that the functions $\zeta(s)$ and $\zeta(s,\alpha)$, $0<\alpha\leqslant 1$, for $\sigma>1$, are defined by

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s}=\prod_{p\in\mathbb{P}}\left(1-\frac{1}{p^s}\right)^{-1} \quad \text{and} \quad \zeta(s,\alpha)=\sum_{m=0}^{\infty}\frac{1}{(m+\alpha)^s},$$

respectively. Both of them are analytically continued to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. Note that the Riemann zeta-function has the Euler product expansion, whereas in general the Hurwitz zeta-function does not have (except the cases $\alpha = \frac{1}{2}$, 1).

For Mishou's result and further statements, we introduce some notation. Let $D(a, b) = \{s \in \mathbb{C} : a < \sigma < b\}$ for any $a < b$. For every compact set $K \subset \mathbb{C}$, denote by $H^c(K)$ the set of all $\mathbb{C}$-valued continuous functions defined on K and holomorphic in the interior of K. By $H_0^c(K)$ we denote the subset of $H^c(K)$ consisting of all elements that are nonvanishing on K.

THEOREM 1 ([21]). *Suppose that α is a transcendental number. Let K_1 and K_2 be compact subsets of the strip $D(\frac{1}{2}, 1)$ with connected complements. Suppose that $f_1(s) \in H_0^c(K_1)$ and $f_2(s) \in H^c(K_2)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0, T] \;:\; \sup_{s\in K_1} |\zeta(s + i\tau) - f_1(s)| < \varepsilon, \\ \sup_{s\in K_2} |\zeta(s + i\tau, \alpha) - f_2(s)| < \varepsilon\Big\} > 0.$$

Here, as usual, $\text{meas}\{A\}$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

Note that Sander and Steuding [23] proved the same type of universality for rational α by a quite different method.

In [12], we consider the mixed joint universality property for a wide class of zeta-functions consisting of Matsumoto zeta-functions $\varphi(s)$ belonging to the Steuding class $\widetilde{S}$ and periodic Hurwitz zeta-functions $\zeta(s, \alpha; \mathfrak{B})$.

Recall the definition of the polynomial Euler products $\widetilde{\varphi}(s)$ or so-called Matsumoto zeta-functions. (Remark: The function $\widetilde{\varphi}(s)$ was introduced by the second author in [18].) For $m \in \mathbb{N}$, let $g(m)$ be a positive integer, and p_m the mth prime number. Moreover, let $a_m^{(j)} \in \mathbb{C}$ and $f(j, m) \in \mathbb{N}$, $1 \leqslant j \leqslant g(m)$. The function $\widetilde{\varphi}(s)$ is defined by the polynomial Euler product

$$\widetilde{\varphi}(s) = \prod_{m=1}^{\infty} \prod_{j=1}^{g(m)} \left(1 - a_m^{(j)} p_m^{-sf(j,m)}\right)^{-1}. \tag{1}$$

We assume that

$$g(m) \leqslant C_1 p_m^{\alpha} \quad \text{and} \quad |a_m^{(j)}| \leqslant p_m^{\beta} \tag{2}$$

with a positive constant C_1 and nonnegative constants α and β. In view of (2), the function $\widetilde{\varphi}(s)$ converges absolutely for $\sigma > \alpha + \beta + 1$, and hence, in this region, it can be given by the absolutely convergent Dirichlet series

$$\widetilde{\varphi}(s) = \sum_{k=1}^{\infty} \frac{\widetilde{c}_k}{k^s}. \tag{3}$$

The shifted function $\varphi(s)$ is given by

$$\varphi(s) = \sum_{k=1}^{\infty} \frac{\widetilde{c}_k}{k^{s+\alpha+\beta}} = \sum_{k=1}^{\infty} \frac{c_k}{k^s} \tag{4}$$

with $c_k = k^{-\alpha-\beta}\widetilde{c}_k$. For $\sigma > 1$, the last series in (4) converges absolutely too.
Also, suppose that, for the function $\varphi(s)$, the following assumptions hold (for the details, see [18]):

(a) $\varphi(s)$ can be continued meromorphically to $\sigma \geqslant \sigma_0$, where $\frac{1}{2} \leqslant \sigma_0 < 1$, and all poles in this region are included in a compact set that has no intersection with the line $\sigma = \sigma_0$,

(b) $\varphi(\sigma + it) = O(|t|^{C_2})$ for $\sigma \geqslant \sigma_0$, where C_2 is a positive constant,

(c) the mean-value estimate

$$\int_0^T |\varphi(\sigma_0 + it)|^2 dt = O(T). \tag{5}$$

It is possible to discuss functional limit theorems for Matsumoto zeta-functions (see Section 2), but this framework is too wide to consider the universality property. To investigate the universality, we introduce the Steuding subclass $\widetilde{S}$, for which the following slightly more restrictive conditions are required. We say that the function $\varphi(s)$ belongs to the class $\widetilde{S}$ if the following conditions are fulfilled:

(i) there exists a Dirichlet series expansion

$$\varphi(s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s}$$

with $a(m) = O(m^{\varepsilon})$ for every $\varepsilon > 0$;

(ii) there exists $\sigma_\varphi < 1$ such that $\varphi(s)$ can be meromorphically continued to the half-plane $\sigma > \sigma_\varphi$ and is holomorphic except for at most a pole at $s = 1$;

(iii) there exists a constant $c \geqslant 0$ such that

$$\varphi(\sigma + it) = O(|t|^{c+\varepsilon})$$

for every fixed $\sigma > \sigma_\varphi$ and $\varepsilon > 0$;

(iv) there exists the Euler product expansion over prime numbers, that is,

$$\varphi(s) = \prod_{p \in \mathbb{P}} \prod_{j=1}^{l} \left(1 - \frac{a_j(p)}{p^s}\right)^{-1};$$

(v) there exists a constant $\kappa > 0$ such that

$$\lim_{x \to \infty} \frac{1}{\pi(x)} \sum_{p \leqslant x} |a(p)|^2 = \kappa,$$

where $\pi(x)$ denotes the number of primes up to x, that is, $p \leqslant x$.

For $\varphi \in \widetilde{S}$, let σ^* be the infimum of all σ_1 such that

$$\frac{1}{2T}\int_{-T}^{T}|\varphi(\sigma+it)|^2 dt \sim \sum_{m=1}^{\infty}\frac{|a(m)|^2}{m^{2\sigma}}$$

for every $\sigma \geqslant \sigma_1$. Then it is known that $\frac{1}{2} \leqslant \sigma^* < 1$. (Remark: The class $\widetilde{S}$ was introduced by Steuding [24].)

Now we recall the definition of the periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{B})$ with a fixed parameter α, $0 < \alpha \leqslant 1$. (Remark: The function $\zeta(s,\alpha;\mathfrak{B})$ was introduced by Javtokas and Laurinčikas [8].) Let $\mathfrak{B} = \{b_m : m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers (not all zero) with minimal period $k \in \mathbb{N}$. For $\sigma > 1$, the function $\zeta(s,\alpha;\mathfrak{B})$ is defined by

$$\zeta(s,\alpha;\mathfrak{B}) = \sum_{m=0}^{\infty}\frac{b_m}{(m+\alpha)^s}.$$

It is known that

$$\zeta(s,\alpha;\mathfrak{B}) = \frac{1}{k^s}\sum_{l=0}^{k-1} b_l \zeta\left(s, \frac{l+\alpha}{k}\right), \quad \sigma > 1. \tag{6}$$

The last equality gives an analytic continuation of the function $\zeta(s,\alpha;\mathfrak{B})$ to the whole complex plane, except for a possible simple pole at the point $s = 1$ with residue

$$b := \frac{1}{k}\sum_{l=0}^{k-1} b_l.$$

If $b = 0$, then $\zeta(s,\alpha;\mathfrak{B})$ is an entire function.

In [12], we prove the mixed joint universality property of the functions $\varphi(s)$ and $\zeta(s,\alpha;\mathfrak{B})$.

THEOREM 2 ([12]). *Suppose that $\varphi(s) \in \widetilde{S}$, and α is a transcendental number. Let K_1 be a compact subset of $D(\sigma^*,1)$, and K_2 be a compact subset of $D(\frac{1}{2},1)$, both with connected complements. Suppose that $f_1 \in H_0^c(K_1)$ and $f_2 \in H^c(K_2)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\mathrm{meas}\Big\{\tau \in [0,T] : \sup_{s\in K_1}|\varphi(s+i\tau) - f_1(s)| < \varepsilon, \sup_{s\in K_2}|\zeta(s+i\tau,\alpha;\mathfrak{B}) - f_2(s)| < \varepsilon\Big\} > 0.$$

In [13], we obtain a generalization of Theorem 2, in which several periodic Hurwitz zeta-functions are involved.

More interesting and convenient in practical applications is the so-called discrete universality of zeta-functions (e.g., see [2]). This pushes us to extend our investigations of mixed joint universality for a class of zeta-functions to the discrete case. Recall that, in this case,

the pair of analytic functions is approximated by discrete shifts of tuple $\big(\varphi(s+ikh), \zeta(s+ikh,\alpha;\mathfrak{B})\big)$, $k \in \mathbb{N}_0$, where $h>0$ is the minimal step of given arithmetical progression.
The aim of this paper is to prove a mixed joint discrete universality theorem for the collection of the functions $(\varphi(s), \zeta(s,\alpha;\mathfrak{B}))$, that is, a discrete version of Theorem 2.
For $h>0$, let

$$L(\mathbb{P},\alpha,h)=\left\{(\log p : p\in\mathbb{P}), (\log(m+\alpha) : m\in\mathbb{N}_0), \frac{2\pi}{h}\right\}.$$

THEOREM 3. *Let $\varphi(s) \in \widetilde{S}$, K_1, K_2, $f_1(s)$, and $f_2(s)$ satisfy the conditions of Theorem 2. Suppose that the set $L(\mathbb{P},\alpha,h)$ is linearly independent over $\mathbb{Q}$. Then, for every $\varepsilon>0$,*

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\left\{0\leqslant k\leqslant N : \quad \sup_{s\in K_1}|\varphi(s+ikh)-f_1(s)|<\varepsilon, \right.$$
$$\left. \sup_{s\in K_2}|\zeta(s+ikh,\alpha;\mathfrak{B})-f_2(s)|<\varepsilon\right\}>0.$$

Remark 1. A typical situation when $L(\mathbb{P},\alpha,h)$ is linearly independent is the case where α and $\exp\left\{\frac{2\pi}{h}\right\}$ are algebraically independent over $\mathbb{Q}$. The proof of this fact is given in [3].

Now recall some known facts of discrete universality, which are directly related to the objects under our interests.
The discrete universality property for the Matsumoto zeta-function under the condition that $\exp\{\frac{2\pi k}{h}\}$ is irrational for every nonzero integer k was obtained by the first author in [9], whereas the discrete universality of the periodic Hurwitz zeta-functions was proved by Laurinčikas and Macaitienė [17].
Also, some results on discrete analogue of mixed universality are known. The first attempt in this direction was done by the first author in [10] under the assumption that α is transcendental and $\exp\left\{\frac{2\pi}{h}\right\}$ is rational. Unfortunately, the proof in [10] is incomplete, as mentioned by Laurinčikas [4] in 2014. However, the argument in [10] gives a correct proof for the modified L-functions where all Euler factors corresponding to primes in the set of all prime numbers appearing as a prime factor of a or b such that $\frac{a}{b}=\exp\left\{\frac{2\pi}{h}\right\}\in\mathbb{Q}$, $a,b\in\mathbb{Z}$, $(a,b)=1$, are removed; see Section 5.
Buivydas and Laurinčikas [3, 4] proved the joint mixed discrete universality for the Riemann zeta-function $\zeta(s)$ and Hurwitz zeta-function $\zeta(s,\alpha)$. The first result [3] deals with the case where the minimal steps of arithmetical progressions h for both functions are common, whereas in the second paper [4], for $\zeta(s)$ and $\zeta(s,\alpha)$, the minimal steps h_1 and h_2 are different from each other.
The purpose of the present paper is to give the proof of the joint mixed discrete universality theorem (Theorem 3) for $(\varphi(s),\zeta(s,\alpha;\mathfrak{B}))$, which generalizes the result from [3], and to clarify the situation in [10].

2. A JOINT MIXED DISCRETE LIMIT THEOREM

The proof of Theorem 3 is based on a joint mixed discrete limit theorem in the sense of weakly convergent probability measures in the space of analytic functions for the Matsumoto zeta-functions $\varphi(s)$ and the periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{B})$ (for the detailed expositions of this method, see [15], [24]), which we prove in this section using the linear independence of the set $L(\mathbb{P},\alpha,h)$. In this section, $\varphi(s)$ denotes any general Matsumoto zeta-function.

For further statements, we start with some notation and definitions.

For a set S, denote by $\mathcal{B}(S)$ the set of all Borel subset of S. Let $\gamma=\{s\in\mathbb{C}:|s|=1\}$. Define

$$\Omega_1=\prod_{p\in\mathbb{P}}\gamma_p \quad \text{and} \quad \Omega_2=\prod_{m=0}^{\infty}\gamma_m,$$

where $\gamma_p=\gamma$ for all $p\in\mathbb{P}$, and $\gamma_m=\gamma$ for all $m\in\mathbb{N}_0$. By the Tikhonov theorem (see [14]), the tori Ω_1 and Ω_2 with the product topology and the pointwise multiplication are compact topological groups. Then

$$\Omega:=\Omega_1\times\Omega_2$$

is a compact topological Abelian group too, and we obtain the probability space $(\Omega,\mathcal{B}(\Omega), m_H)$. Here $m_H=m_{1H}\times m_{2H}$ with the probability Haar measures m_{1H} and m_{2H} defined on the spaces $(\Omega_1,\mathcal{B}(\Omega_1))$ and $(\Omega_2,\mathcal{B}(\Omega_2))$, respectively.

Let $\omega_1(p)$ stand for the projection of $\omega_1\in\Omega_1$ to the coordinate space γ_p, $p\in\mathbb{P}$, and, for every $m\in\mathbb{N}$, we put

$$\omega_1(m)=\prod_{j=1}^{r}\omega_1(p_j)^{l_j},$$

where, by factorizing of m into the primes, $m=p_1^{l_1}\cdots p_r^{l_r}$. Let $\omega_2(m)$ denotes the projection of $\omega_2\in\Omega_2$ to the coordinate space γ_m, $m\in\mathbb{N}_0$. Define $\omega=(\omega_1,\omega_2)$ for elements of Ω.

For any open subregion G in the complex plane, let $H(G)$ be the space of analytic functions on G equipped with the topology of uniform convergence in compacta.

The function $\varphi(s)$ has only finitely many poles by condition (a). Denote those poles by $s_1(\varphi),\dots,s_l(\varphi)$ and define

$$D_\varphi=\{s:\ \sigma>\sigma_0,\ \sigma\neq\Re s_j(\varphi),\ 1\leqslant j\leqslant l\}.$$

Then $\varphi(s)$ and its vertical shift $\varphi(s+ikh)$ are holomorphic in D_φ. The function $\zeta(s,\alpha;\mathfrak{B})$ can be written as a linear combination of Hurwitz zeta-functions (6), and therefore it is entire or has a simple pole at $s=1$. Therefore, $\zeta(s,\alpha;\mathfrak{B})$ and its vertical shift $\zeta(s+ikh,\alpha;\mathfrak{B})$ are holomorphic in

$$D_\zeta=\begin{cases}\{s\in\mathbb{C}:\ \sigma>\frac{1}{2}\} & \text{if} \quad \zeta(s,\alpha;\mathfrak{B}) \ \text{ is entire},\\ \{s:\ \sigma>\frac{1}{2},\sigma\neq 1\} & \text{if} \quad s=1 \ \text{ is a pole of } \ \zeta(s,\alpha;\mathfrak{B}).\end{cases}$$

Now, in view of the definitions of D_φ and D_ζ, let D_1 and D_2 be two open subsets of D_φ

and D_ζ, respectively. Let $\underline{H} = H(D_1) \times H(D_2)$. On $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $\underline{H}$-valued random element $\underline{Z}(\underline{s}, \omega)$ by the formula

$$\underline{Z}(\underline{s}, \omega) = \big(\varphi(s_1, \omega_1), \zeta(s_2, \alpha, \omega_2; \mathfrak{B})\big),$$

where $\underline{s} = (s_1, s_2) \in D_1 \times D_2$,

$$\varphi(s_1, \omega_1) = \sum_{k=1}^{\infty} \frac{c_k \omega_1(k)}{k^{s_1}}, \tag{7}$$

and

$$\zeta(s_2, \alpha, \omega_2; \mathfrak{B}) = \sum_{m=0}^{\infty} \frac{b_m \omega_2(m)}{(m+\alpha)^{s_2}}. \tag{8}$$

Denote by $P_{\underline{Z}}$ the distribution of $\underline{Z}(\underline{s}, \omega)$ as an $\underline{H}$-valued random element, that is,

$$P_{\underline{Z}}(A) = m_H\{\omega \in \Omega : \ \underline{Z}(\underline{s}, \omega) \in A\}, \quad A \in \mathcal{B}(\underline{H}).$$

Let $N > 0$. Define the probability measure P_N on $\underline{H}$ by the formula

$$P_N(A) = \frac{1}{N+1} \#\big\{0 \leqslant k \leqslant N : \ \underline{Z}(\underline{s} + ikh) \in A\big\}, \quad A \in \mathcal{B}(\underline{H}),$$

where $\underline{s} + ikh = (s_1 + ikh, s_2 + ikh)$ with $s_1 \in D_1$, $s_2 \in D_2$, and

$$\underline{Z}(\underline{s}) = (\varphi(s_1), \zeta(s_2, \alpha; \mathfrak{B})).$$

In the proof of Theorem 3, the first main goal is the following mixed joint discrete limit theorem.

THEOREM 4. *Suppose that the set $L(\mathbb{P}, \alpha, h)$ is linearly independent over $\mathbb{Q}$. Then the probability measure P_N converges weakly to $P_{\underline{Z}}$ as $N \to \infty$.*

We will omit some details of the proof because the proof follows the standard way (see, e.g., the proof of Theorem 7 of [3]). However, though the following lemma, a mixed joint discrete limit theorem on the torus Ω, is exactly the same as Lemma 1 of [3], we reproduce the detailed proof since this result plays a crucial role and from the proof we can see why the linear independence of $L(\mathbb{P}, \alpha, h)$ is necessary.
Define

$$Q_N(A) := \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N : \big(\big(p^{-ikh} : p \in \mathbb{P}\big), \big((m+\alpha)^{-ikh} : m \in \mathbb{N}_0\big)\big) \in A\Big\}, \quad A \in \mathcal{B}(\Omega).$$

LEMMA 1 ([3]). *Suppose that the set $L(\mathbb{P}, \alpha, h)$ satisfies the condition of Theorem 3. Then Q_N converges weakly to the Haar measure m_H as $N \to \infty$.*

Proof. For the proof of Lemma 1, we use the Fourier transformation method (for the details, see [15]). The dual group of Ω is isomorphic to the group

$$G := \Big(\bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p\Big) \bigoplus \Big(\bigoplus_{m\in\mathbb{N}_0} \mathbb{Z}_m\Big)$$

with $\mathbb{Z}_p = \mathbb{Z}$ for all $p \in \mathbb{P}$ and $\mathbb{Z}_m = \mathbb{Z}$ for all $m \in \mathbb{N}_0$. The element of G is written as $(\underline{k}, \underline{l}) = ((k_p : p \in \mathbb{P}), (l_m : m \in \mathbb{N}_0))$, where only a finite number of integers k_p and l_m are nonzero, and acts on Ω by

$$(\omega_1, \omega_2) \to (\omega_1^{\underline{k}}, \omega_2^{\underline{l}}) = \prod_{p\in\mathbb{P}} \omega_1^{k_p}(p) \prod_{m\in\mathbb{N}_0} \omega_2^{l_m}(m).$$

Let $g_N(\underline{k}, \underline{l})$, $(\underline{k}, \underline{l}) \in G$, be the Fourier transform of the measure $Q_N(A)$. Then we have

$$g_N(\underline{k}, \underline{l}) = \int_\Omega \Big(\prod_{p\in\mathbb{P}} \omega_1^{k_p}(p) \prod_{m\in\mathbb{N}_0} \omega_2^{l_m}(m)\Big) \mathrm{d}Q_N.$$

Thus, from the definition of $Q_N(A)$,

$$\begin{aligned} g_N(\underline{k}, \underline{l}) &= \frac{1}{N+1} \sum_{k=0}^{N} \prod_{p\in\mathbb{P}} p^{-ikk_p h} \prod_{m\in\mathbb{N}_0} (m+\alpha)^{-ikl_m h} \\ &= \frac{1}{N+1} \sum_{k=0}^{N} \exp\Big\{ -ikh\Big(\sum_{p\in\mathbb{P}} k_p \log p + \sum_{m\in\mathbb{N}_0} l_m \log(m+\alpha)\Big)\Big\}. \end{aligned} \tag{9}$$

By the assumption of the lemma the set $L(\mathbb{P}, \alpha, h)$ is linearly independent over $\mathbb{Q}$. Then the set $\{(\log p : p \in \mathbb{P}), (\log(m+\alpha) : m \in \mathbb{N}_0)\}$ is linearly independent over $\mathbb{Q}$, and

$$\sum_{p\in\mathbb{P}} k_p \log p + \sum_{m\in\mathbb{N}_0} l_m \log(m+\alpha) = 0$$

if and only if $\underline{k} = \underline{0}$ and $\underline{l} = \underline{0}$. Moreover, if $(\underline{k}, \underline{l}) \neq (\underline{0}, \underline{0})$, then

$$\exp\Big\{ -ih\Big(\sum_{p\in\mathbb{P}} k_p \log p + \sum_{m\in\mathbb{N}_0} l_m \log(m+\alpha)\Big)\Big\} \neq 1. \tag{10}$$

In fact, if (10) were false, then

$$\sum_{p\in\mathbb{P}} k_p \log p + \sum_{m\in\mathbb{N}_0} l_m \log(m+\alpha) = \frac{2\pi a}{h} \tag{11}$$

with some $a \in \mathbb{Z} \setminus \{0\}$. But this contradicts to the linear independence of the set $L(\mathbb{P}, \alpha, h)$.

Therefore, from (9) and (10) we find that

$$g_N(\underline{k},\underline{l}) = \begin{cases} 1 & \text{if } (\underline{k},\underline{l}) = (\underline{0},\underline{0}), \\ \frac{1-\exp\left\{-i(N+1)h\left(\sum_{p\in\mathbb{P}} k_p \log p + \sum_{m\in\mathbb{N}_0} l_m \log(m+\alpha)\right)\right\}}{(N+1)\left(1-\exp\left\{-ih\left(\sum_{p\in\mathbb{P}} k_p \log p + \sum_{m\in\mathbb{N}_0} l_m \log(m+\alpha)\right)\right\}\right)} & \text{if } (\underline{k},\underline{l}) \neq (\underline{0},\underline{0}). \end{cases}$$

Hence,

$$\lim_{N\to\infty} g_N(\underline{k},\underline{l}) = \begin{cases} 1 & \text{if } (\underline{k},\underline{l}) = (\underline{0},\underline{0}), \\ 0 & \text{otherwise.} \end{cases}$$

By the continuity theorem for probability measures on compact groups (see [7]) we obtain the statement of the lemma, that is, $Q_N(A)$ converges weakly to m_H as $N \to \infty$.

Now, using Lemma 1, we may prove a joint mixed discrete limit theorem for absolutely convergent Dirichlet series.
Let, for fixed $\widehat{\sigma} > \frac{1}{2}$,

$$v_1(m,n) = \exp\left\{ -\left(\frac{m}{n}\right)^{\widehat{\sigma}} \right\}, \quad m,n \in \mathbb{N},$$

and

$$v_2(m,n,\alpha) = \exp\left\{ -\left(\frac{m+\alpha}{n+\alpha}\right)^{\widehat{\sigma}} \right\}, \quad m \in \mathbb{N}_0, \quad n \in \mathbb{N}.$$

Define the series

$$\begin{aligned} \varphi_n(s) &= \sum_{m=1}^{\infty} \frac{c_m v_1(m,n)}{m^s}, \\ \zeta_n(s,\alpha;\mathfrak{B}) &= \sum_{m=0}^{\infty} \frac{b_m v_2(m,n,\alpha)}{(m+\alpha)^s}, \end{aligned}$$

and, for $\widehat{\omega} := \left(\widehat{\omega}_1, \widehat{\omega}_2\right) \in \Omega$,

$$\begin{aligned} \varphi_n(s,\widehat{\omega}_1) &= \sum_{m=1}^{\infty} \frac{\widehat{\omega}_1(m) c_m v_1(m,n)}{m^s}, \\ \zeta_n(s,\alpha,\widehat{\omega}_2;\mathfrak{B}) &= \sum_{m=0}^{\infty} \frac{\widehat{\omega}_2(m) b_m v_2(m,n,\alpha)}{(m+\alpha)^s}. \end{aligned}$$

These series are absolutely convergent for $\sigma > \frac{1}{2}$.
For brevity, denote

$$\underline{Z}_n(\underline{s}) = (\varphi_n(s_1), \zeta_n(s_2,\alpha;\mathfrak{B}))$$

and

$$\underline{Z}_n(\underline{s},\widehat{\omega}) = (\varphi_n(s_1,\widehat{\omega}_1), \zeta_n(s_2,\alpha,\widehat{\omega}_2;\mathfrak{B})).$$

Now, on the space $(\underline{H}, \mathcal{B}(\underline{H}))$, we consider the weak convergence of the measures

$$P_{N,n}(A) = \frac{1}{N+1} \# \Big\{ 0 \leqslant k \leqslant N : \underline{Z}_n(\underline{s} + ikh) \in A \Big\}$$

and, for $\widehat{\omega} \in \Omega$,

$$\widehat{P}_{N,n}(A) = \frac{1}{N+1} \# \Big\{ 0 \leqslant k \leqslant N : \underline{Z}_n(\underline{s} + ikh, \widehat{\omega}) \in A \Big\}.$$

LEMMA 2. *Suppose that the set $L(\mathbb{P}, \alpha, h)$ is linearly independent over $\mathbb{Q}$. Then, on $(\underline{H}, \mathcal{B}(\underline{H}))$, there exists a probability measure P_n such that the measures $P_{N,n}$ and $\widehat{P}_{N,n}$ both converge weakly to P_n as $N \to \infty$.*

Proof. The proof of the lemma is analogous to that of Lemma 2 from [3].

The next step of the proof is to approximate the tuple $(\underline{Z}(\underline{s}), \underline{Z}(\underline{s}, \widehat{\omega}))$ by the tuple $\big(\underline{Z}_n(\underline{s}), \underline{Z}_n(\underline{s}, \widehat{\omega})\big)$. For this purpose, we will use the metric on the space $\underline{H}$. For any open region G, it is known (see [5] or [15]) that there exists a sequence of compact sets $\{K_l : l \in \mathbb{N}\} \subset G$ satisfying conditions:

1. $G = \bigcup\limits_{l=1}^{\infty} K_l$,
2. $K_l \subset K_{l+1}$ for any $l \in \mathbb{N}$,
3. if K is a compact set, then $K \subset K_l$ for some $l \in \mathbb{N}$.

For functions $g_1, g_2 \in H(G)$, define the metric ϱ_G by the formula

$$\varrho_G(g_1, g_2) = \sum_{l=1}^{\infty} \frac{1}{2^l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|},$$

which induces the topology of uniform convergence on compacta. Put $\varrho_1 = \varrho_{D_1}$ and $\varrho_2 = \varrho_{D_2}$. Define, for $\underline{g}_1 = (g_{11}, g_{21})$ and $\underline{g}_2 = (g_{12}, g_{22})$ from $\underline{H}$,

$$\underline{\varrho}(\underline{g}_1, \underline{g}_2) = \max \big\{ \varrho_1(g_{11}, g_{12}), \varrho_2(g_{21}, g_{22}) \big\}.$$

In such a way, we obtain a metric on the space $\underline{H}$ inducing its topology.

LEMMA 3. *Suppose that the set $L(\mathbb{P}, \alpha, h)$ is linearly independent over $\mathbb{Q}$. Then*

$$\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \underline{\varrho}\big(\underline{Z}(\underline{s} + ikh), \underline{Z}_n(\underline{s} + ikh)\big) = 0 \tag{12}$$

and, for almost all $\omega \in \Omega$,

$$\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \underline{\varrho}\big(\underline{Z}(\underline{s} + ikh, \omega), \underline{Z}_n(\underline{s} + ikh, \omega)\big) = 0. \tag{13}$$

Proof. This can be shown in a way similar to the proofs of Lemmas 3 and 4 of [3], respectively. The main body of the argument in [3], based on an application of Gallagher's lemma, is going back to the proof of Theorem 4.1 of [17]. We just indicate some different points from the proof in [3] and [17].

The starting point of the proof of (12) is the integral expressions

$$\varphi_n(s) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\varphi(s+z)l_n(z)\frac{dz}{z} \tag{14}$$

and

$$\zeta_n(s,\alpha;\mathfrak{B}) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\zeta(s+z,\alpha;\mathfrak{B})l_n(z,\alpha)\frac{dz}{z}, \tag{15}$$

where $a > \frac{1}{2}$, and

$$l_n(z) = \frac{z}{a}\Gamma\left(\frac{z}{a}\right)n^z \quad\text{and}\quad l_n(z,\alpha) = \frac{z}{a}\Gamma\left(\frac{z}{a}\right)(n+\alpha)^z,$$

respectively. We shift the paths to the left and apply the residue calculus. The case (15) is discussed in [17], where the path is moved to $\Re z = b - \sigma$ with $\frac{1}{2} < b < 1$ and $\sigma > b$. In this case, the relevant poles are only $z = 0$ and $z = 1 - s$. As for (14), we shift the path to $\Re z = \sigma_0 + \delta_0 - \sigma$, where δ_0 is a small positive number such that $\varphi(s)$ is holomorphic in the strip $\sigma_0 \leqslant \Re s \leqslant \sigma_0 + \delta_0$. We encounter all the poles $z = s_j(\varphi) - s$, $1 \leqslant j \leqslant l$, so we have to consider all the residues coming from those poles. But they can be handled by the same method as described in the proof of Theorem 4.1 of [17].

To complete the proof of (12), it is also necessary to show the discrete mean square estimate

$$\sum_{k=0}^{N}|\varphi(\sigma_0+\delta_0+it+ikh)|^2 \ll N(1+|t|). \tag{16}$$

This is an analogue of Lemma 4.3 of [17] and can be obtained similarly from (5) and Gallagher's lemma (Lemma 1.4 of [22]).

As for the proof of (13), we need the "random" version of (5), that is,

$$\int_0^T |\varphi(\sigma+it,\omega_1)|^2 dt = O(T), \quad \sigma > \sigma_0, \tag{17}$$

for almost all $\omega_1 \in \Omega_1$. This is actually a special case of Lemma 10 of [16]. The corresponding mean value result for $\zeta(s,\alpha,\omega_2;\mathfrak{B})$ has been shown in [8]. Using those mean value results, we can show (13) in the same way as in the proof of Lemma 4 of [3].

Lemma 3, together with the weak convergence of of the measures $P_{N,n}$ and $\widehat{P}_{N,n}$ (Lemma 2), enables us to prove that the probability measure P_N and one more probability measure defined as

$$\widehat{P}_N(A) = \frac{1}{N+1}\#\big\{0 \leqslant k \leqslant N : \underline{Z}(\underline{s}+ikh,\omega) \in A\big\}, \quad A \in \mathcal{B}(\underline{H}),$$

both converge weakly to the same probability measure P, that is, the following statement holds.

LEMMA 4. *Suppose that the set $L(\mathbb{P}, \alpha, h)$ is linearly independent over $\mathbb{Q}$. Then, on $(\underline{H}, \mathcal{B}(\underline{H}))$, there exists a probability measure P such that the measures P_N and $\widehat{P}_N$ both converge weakly to P as $N \to \infty$.*

Proof. This lemma can be shown analogously to Lemma 5 from [3].

Proof of Theorem 4. As usual, in the last step of the proof of the functional discrete limit theorem, we show that the limit measure P in Lemma 4 coincides with $P_{\underline{Z}}$.
Define the measurable measure-preserving transformation $\Phi_h : \Omega \to \Omega$ on the group Ω by $\Phi_h(\omega) = f_h\omega$, $\omega \in \Omega$, where $f_h = \{(p^{-ih} : p \in \mathbb{P}), ((m+\alpha)^{-ih} : m \in \mathbb{N}_0)\}$. Again using (10), we see that $\{\Phi_h(s)\}$ is an ergodic one-parameter group. This, together with the well-known Birkhoff–Khintchine theorem (see [6]) and the weak convergence of $\widehat{P}_N(A)$, gives that $P(A) = P_{\underline{Z}}(A)$ for all $A \in \mathcal{B}(\underline{H})$. For the details, consult the proof of Theorem 7 of [3] or Theorem 6.1 of [17].

3. THE SUPPORT OF THE MEASURE $P_{\underline{Z}}$

To introduce the support of $P_{\underline{Z}}$, we repeat the arguments of Section 4 from [12].
Let $\varphi \in \widetilde{S}$, and let K_1, K_2, f_1, and f_2 be as in the statement of Theorem 3. Then we can find a real number σ_0 with $\sigma^* < \sigma_0 < 1$ and a positive number $M > 0$ such that K_1 is included in the open rectangle

$$D_M = \{s : \ \sigma_0 < \sigma < 1, \ |t| < M\}.$$

Since $\varphi(s) \in \widetilde{S}$, the pole of φ is at most at $s = 1$. Then, in this case, we find that

$$D_\varphi = \{s : \ \sigma > \sigma_0, \ \sigma \neq 1\}.$$

Therefore, D_M is an open subset of D_φ. Also, we can find $T > 0$ such that K_2 belongs to the open rectangle

$$D_T = \left\{s : \ \frac{1}{2} < \sigma < 1, \ |t| < T\right\}.$$

To obtain the support of the measure $P_{\underline{Z}}$, we will use Theorem 4 with $D_1 = D_M$ and $D_2 = D_T$. Let S_φ be the set of all $f \in H(D_M)$ that are nonvanishing on D_M or constantly equivalent to 0 on D_M.

THEOREM 5. *Suppose that the set $L(\mathbb{P}, \alpha, h)$ is linearly independent over $\mathbb{Q}$. The support of the measure $P_{\underline{Z}}$ is the set $S = S_\varphi \times H(D_T)$.*

Proof. This is an analogue to Lemma 4.3 of [12] or Theorem 8 from [3]. The fact that $\varphi \in \widetilde{S}$ is essentially used here.

4. PROOF OF THE MIXED JOINT DISCRETE UNIVERSALITY THEOREM

The proof of Theorem 3 follows from Theorems 4 and 5 and the Mergelyan theorem (see [20]), which we state as a lemma.

LEMMA 5 (Mergelyan). *Let $K \subset \mathbb{C}$ be a compact subset with connected complement, and $f(s)$ be a continuous function on K that is analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s\in K} |f(s) - p(s)| < \varepsilon.$$

Proof of Theorem 3. By Lemma 5 there exist polynomials $p_1(s)$ and $p_2(s)$ such that

$$\sup_{s\in K_1} \left|f_1(s) - e^{p_1(s)}\right| < \frac{\varepsilon}{2} \tag{18}$$

and

$$\sup_{s\in K_2} \left|f_2(s) - p_2(s)\right| < \frac{\varepsilon}{2}. \tag{19}$$

We introduce the set

$$G = \left\{(g_1, g_2) \in \underline{H} : \sup_{s\in K_1} |g_1(s) - e^{p_1(s)}| < \frac{\varepsilon}{2}, \sup_{s\in K_2} |g_2(s) - p_2(s)| < \frac{\varepsilon}{2}\right\}.$$

Then G is an open set of the space $\underline{H}$. By Theorem 5 it is an open neighborhood of the element $(e^{p_1(s)}, p_2(s))$ of the support of $P_{\underline{Z}}$. Thus, $P_{\underline{Z}}(G) > 0$. Using Theorem 4 and an equivalent statement of the weak convergence in terms of open sets (see [1]), we obtain

$$\liminf_{N\to\infty} P_N(G) \geqslant P_{\underline{Z}}(G) > 0.$$

This and the definitions of P_N and G show that

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \Big\{0 \leqslant k \leqslant N : \quad \sup_{s\in K_1} \left|\varphi(s + ikh) - e^{p_1(s)}\right| < \frac{\varepsilon}{2}, \quad \sup_{s\in K_2} \left|\zeta(s + ikh, \alpha; \mathfrak{B}) - p_2(s)\right| < \frac{\varepsilon}{2}\Big\} > 0. \tag{20}$$

From (18) and (19) we deduce that

$$\left\{0 \leqslant k \leqslant N : \sup_{s\in K_1} \left|\varphi(s + ikh) - f_1(s)\right| < \varepsilon, \sup_{s\in K_2} \left|\zeta(s + ikh, \alpha; \mathfrak{B}) - f_2(s)\right| < \varepsilon\right\}$$

$$\supset \left\{0 \leqslant k \leqslant N : \sup_{s\in K_1} \left|\varphi(s + ikh) - e^{p_1(s)}\right| < \frac{\varepsilon}{2}, \sup_{s\in K_2} \left|\zeta(s + ikh, \alpha; \mathfrak{B}) - p_2(s)\right| < \frac{\varepsilon}{2}\right\}.$$

This, together with inequality (20), gives the assertion of the theorem.

5. THE CASE OF MODIFIED ZETA-FUNCTIONS

In Section 1, we mentioned an incomplete point in [10]. An inaccuracy is actually included in a former paper [11], whose result is applied to [10]. On p. 103 of [11], inequality (10) for $(\underline{k}, \underline{l}) \neq (\underline{0}, \underline{0})$ is claimed under the assumption that α is transcendental and $\exp\{\frac{2\pi}{h}\}$ is rational. The same reasoning as in the case of (10) is valid if there is some $l_m \neq 0$ because from (11) we have

$$\prod_{p \in \mathbb{P}} p^{k_p} \prod_{m \in \mathbb{N}_0} (m+\alpha)^{l_m} = \left(\exp\left\{ \frac{2\pi}{h} \right\} \right)^a, \tag{21}$$

which contradicts the assumption. But if all $l_m = 0$, then (21) does not produce a contradiction. Therefore, the results in [11], and hence in [10], are to be amended.
Write $\exp\left\{\frac{2\pi}{h}\right\} = \frac{a}{b}$, $a, b \in \mathbb{Z}$, $(a,b) = 1$, and denote by $\mathbb{P}_h$ the set of all primes appearing as prime divisors of a or b. Instead of $Q_N(A)$ defined in Section 2, we define $Q_{N,h}(A)$ by replacing $\mathbb{P}$ in the definition of $Q_N(A)$ by $\mathbb{P} \setminus \mathbb{P}_h$. Let

$$\Omega_{1h} = \prod_{p \in \mathbb{P} \setminus \mathbb{P}_h} \gamma_p$$

and denote the probability Haar measure on $(\Omega_{1h}, \mathcal{B}(\Omega_{1h}))$ by m_{1hH}.

LEMMA 6. *Let α be transcendental, and $\exp\left\{\frac{2\pi}{h}\right\}$ be rational. Then $Q_{N,h}$ converges weakly to the Haar measure $m_{hH} = m_{1hH} \times m_{2H}$ on the space $\Omega_h = \Omega_{1h} \times \Omega_2$ as $N \to \infty$.*

Proof. If we replace $\mathbb{P}$ by $\mathbb{P} \setminus \mathbb{P}_h$ in (21), then the resulting equality is impossible even if all $l_m = 0$. Therefore, (10) is valid for any $(\underline{k}, \underline{l}) \neq (\underline{0}, \underline{0})$, and so we can mimic the proof of Lemma 1.

This lemma is a corrected version of Lemma 2.1 of [11]. Let χ be a Dirichlet character. Define the modified Dirichlet L-function by

$$L_h(s, \chi) = \prod_{p \in \mathbb{P} \setminus \mathbb{P}_h} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Then, using Lemma 6, we can show a mixed joint discrete universality theorem for $L_h(s, \chi)$ and a periodic Hurwitz zeta-function by the argument in [10]. This is the corrected version of Theorem 1.7 of [10], which was already mentioned in [19].
It is possible to generalize the above arguments to the class of Matsumoto zeta-functions. We conclude the present paper with the statement of such results.
Define the modified Matsumoto zeta-function by

$$\widetilde{\varphi}_h(s) = \prod_{m \in \mathbb{N} \setminus \mathbb{N}_h} \prod_{j=1}^{g(m)} \left(1 - a_m^{(j)} p_m^{-sf(j,m)}\right)^{-1}, \tag{22}$$

where $\mathbb{N}_h$ is the set of all $m \in \mathbb{N}$ such that $p_m \in \mathbb{P}_h$, and $\varphi_h(s) = \widetilde{\varphi}_h(s+\alpha+\beta)$. The difference between $\varphi_h(s)$ and $\varphi(s)$ is only finitely many Euler factors, so their analytic properties are not so different. In particular, if $\varphi(s)$ satisfies properties (a), (b), and (c), then so does $\varphi_h(s)$. Therefore, the method developed in the previous sections of the present paper can be applied to $\varphi_h(s)$. Let

$$\underline{Z}_h(\underline{s}) = (\varphi_h(s_1), \zeta(s_2, \alpha; \mathfrak{B})),$$
$$\underline{Z}_h(\underline{s}, \omega_h) = (\varphi_h(s_1, \omega_{1h}), \zeta(s_2, \alpha, \omega_2; \mathfrak{B})),$$

where $\omega_{1h} \in \Omega_{1h}$ and $\omega_h = (\omega_{1h}, \omega_2) \in \Omega_h$. Define $P_{\underline{Z},h}$ and $P_{N,h}$ analogously to $P_{\underline{Z}}$ and P_N, just replacing $\underline{Z}(\underline{s}, \omega)$ and $\underline{Z}(\underline{s}+ikh)$ by $\underline{Z}_h(\underline{s}, \omega_h)$ and $\underline{Z}_h(\underline{s}+ikh)$, respectively. Then, using Lemma 6, we obtain the following:

THEOREM 6. *Let α be transcendental, and $\exp\left\{\frac{2\pi}{h}\right\}$ be rational. Then $P_{N,h}$ converges weakly to $P_{\underline{Z},h}$ as $N \to \infty$.*

THEOREM 7. *Let $\varphi(s) \in \widetilde{S}$, K_1, K_2, $f_1(s)$, and $f_2(s)$ satisfy the conditions of Theorem 2. Suppose that α is transcendental, and $\exp\left\{\frac{2\pi}{h}\right\}$ is rational. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \quad \sup_{s\in K_1} |\varphi_h(s+ikh) - f_1(s)| < \varepsilon, \right.$$
$$\left. \sup_{s\in K_2} |\zeta(s+ikh, \alpha; \mathfrak{B}) - f_2(s)| < \varepsilon \right\} > 0.$$

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