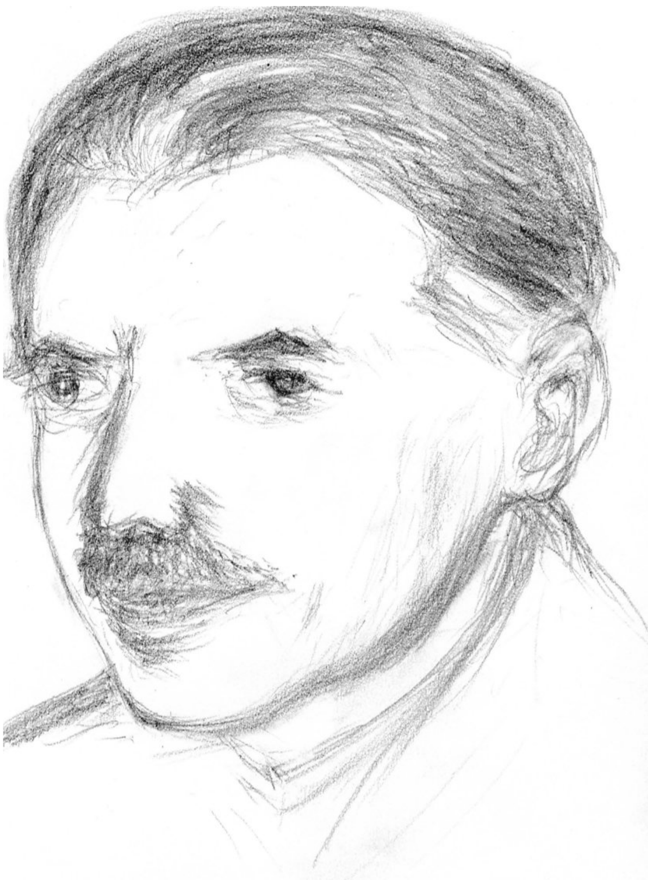

Dedicated to Schinzel on the occasion of his 70th birthday.
Sto lat, sto lat. Niech zyje nam.

PREFACE

The present volume is the proceedings of the 5th Japan-China Seminar on Number Theory "Dreaming in dreams" held during August 27-31, 2008 at Kinki University, Higashi-osaka, Japan, the organizers being Shigeru Kanemitsu and Jianya Liu with Professor Takashi Aoki as the local organizer.

The title sounded somewhat romantic or exotic and one of the participants, Professor Tim. D. Browning, a relative of the world famous poet R. Browning, expressed a poetic view that the title suggested that one could dream of proving the RH or whatsoever of the hardest nuts to crack in dreams. But we chose this title in view of the following due reason. Osaka is most well-known for its world famous Osaka Castle. The builder of Osaka Castle, Hideyoshi Toyotomi, a hero in the 16th century made a poem at his deathbed. "Like a dew drop was I born and into a dew drop am I fading, all that prevails in Naniwa (the present world) is like dreams in a dream." Here Naniwa sounds the same as the old name of Osaka. Also many of us went to the center of the city (for empty orchestra, perhaps), Namba, which is the modern name of Naniwa. This gives a good reason to entitle the seminar. This may not sound poetic but associatively logical. Indeed, at the end we have a poem composed by Professor Chaohua Jia. Thus we now have at least four poets among participants, including Professors Tianxin Cai (a professional), Chaohua Jia, Jianya Liu (who composed a poem in the proceedings of the 4th China-Japan Seminar), and Tim D. Browning.

The atmosphere was enjoyable as usual and we believe everyone enjoyed the 5 rich days. We organized various social activities including reception party at Sheraton-Miyako Hotel to which we thank for their great hospitality and generosity of providing us with champagne bottles. We made a tour to Kyoto (the bus was arranged through Sheraton-Miyako) and visited a few musts there. Some of the foreign participants paid multiple visits to Kinkakuji Temple and Kiyomizu Temple. Evening events were also entertaining which included empty orchestra activities at various places. We found that not only Chinese participants who were known to be good and enthusiastic singers, but most of the Western participants shared the same spirit. We found Professors Trevor Wooley, Winfried Kohnen, Katsuya Miyake and Yumiko Hironaka most entertaining. Professor Jörg Brüdern, though didn't sing, promised to play the guitar at the next occasion. S. Kanemitsu, to his regret, missed the chance of attending the ever-night show including Professor Tim. D. Browning and Professor Koichi Kawada.

Now about the contents of the seminar and the present proceedings. The talks ranged over a wide spectrum of contemporary number theory. As can be seen from the papers themselves as well as in the following brief descriptions in this volume, we succeeded in assembling topics from Analytic Number Theory (Classical and Modern with emphasis on additive number theory), Theory of Modular Forms, Algebraic Groups and Algebraic Number Theory.

In the proceedings we collected not only papers from the participants but from those invitees who could not attend the seminar, including Professors Andrzej Schinzel (who was about to come), Igor Shparlinski and Ken Yamamura.

In [Browning] a new direction of research in analytic number theory is exhibited, i. e. a quantitative study on the distribution of rational points of some variety—specifically, a del Pezzo surface of degree 4, $V \subset \mathbb{P}^4$ which is defined over the rationals and is assumed to have a conic bundle structure. The main result is the asymptotic formula for the counting function

$$N_{U_0,H}(B) = \#\{x \in U_0(\mathbb{Q}) : \, H(x) \le B\},$$

where U_0 is a certain Zariski open subset and $H(x)$ is a certain norm function. The formula reads

$$N_{U_0,H}(B) = c_{V_0,H} B(\log B)^4 + \text{error term},$$

establishing the Manin conjecture in this case, where $c_{V_0,H}$ is the constant conjectured by Peyre.

The proof involves various ingredients, the geometric Picard group $\mathrm{Pic}(V_0) \simeq \mathbb{Z}^5$, analysis of conic sections and classical techniques including lattice points counting and divisor problems for binary forms.

[Brüdern-Kawada-Wooley] is the 8th of their series of papers "Additive representation in thin sequences" I-VII and is a timely summary which looks over their recent results in an enlightening way.

Their main concern is the diagonal form

$$\lambda_1 x_1^k + \cdots + \lambda_s x_s^k,$$

as $x_1, \cdots, x_s$ ranges over $\mathbb{Z}$ or a subset thereof, where $s \geq 2$, $k \geq 1$ and $\lambda_1, \cdots, \lambda_s$ are non-zero real numbers.

The purpose of the paper is two-fold; on one hand, it centers around Diophantine inequalities and on the other on the potentials of the methods developed in the series, including the Davenport-Heilbronn Fourier transform method, a counterpart of the Hardy-Littlewood circle method for Diophantine inequalities. Starting from the case of additive cubic forms, the authors give a very clear survey on their hitherto contributions, giving proofs of some of the important theorems, which makes the paper more instructive and readable.

In the paper [Hoshi-Miyake] the authors are concerned with the FIP (Fixed Isomorphism Problem) on k-generic polynomial $f_{\mathbf{t}}^G(X) \in k(\mathbf{t})[X]$, where k is a field of arbitrary characteristic, G is a finite group and $k(\mathbf{t})$ is the rational function field over k with n indeterminates $\mathbf{t} = (t_1, t_2, \cdots, t_n)$ and where a monic separable polynomial $f_{\mathbf{t}}^G(X)$ is called a k-generic polynomial for G if

- (G_1) the Galois group of $f_{\mathbf{t}}^G(X)$ over $k(\mathbf{t})$ is isomorphic to G and

- (G_1) every G-extension $L/K, K \supset k$ may be obtained as $L = \mathrm{Spl}_K f_{\mathbf{a}}^G$, the splitting field of $f_{\mathbf{a}}^G$ over K, for some $\mathbf{a} = (a_1, a_2, \cdots, a_n) \in K^n$.

The FIP reads: For a field $K \supset k$ and $\mathbf{a}, \mathbf{b} \in K^n$, determine whether $L = \mathrm{Spl}_K f_{\mathbf{a}}^G$ and $L = \mathrm{Spl}_K f_{\mathbf{b}}^G$ are isomorphic over K or not.

Two more related problems are stated without details: Subfield Problem and Field Intersection Problem for generic polynomials and numerical examples are given in the cases $G = D_4, D_5$ (dihedral) and $G = C_4$ (cyclic).

In his paper [Jia], C. -H. Jia gives some developments over the results on dynamics of the w-function introduced by W. -S. Goldring in 2006, where dynamics refers to the orbit of successive iterates of the function. For $n = p_1p_2p_3 \in S := C_3 \cup B_3$ (p_i's are prime, not all equal), the w-function is defined by

$$w(n) = P\left(p_1 + p_2\right) P\left(p_2 + p_3\right) P\left(p_3 + p_1\right),$$

where $P(n)$ signifies the largest prime factor of n. The objective is to classify the elements of S and there are many results obtained by Goldring, Y.-G. Chen et al.

The inverse problem of finding the inverse image of the w-function is also of interest. One of the conjectures of Goldring states that every element of C_3, which is the set of all $n = p_1p_2p_3$ with p_i all distinct primes, has infinitely many C_3-parents, i.e. there are infinitely many $m \in C_3$ such that $w(m) = n$. Jia proves, by making a novel use of the large sieve method, some quantitative results including Theorem 9 saying that there is an element of $r_2r_2q \in C_3$, where $r_1, r_2 \sim \sqrt{x}\log x$, $q \le 4x$ which has many distinct C_3-parents (where the data is quantitative).

In the paper [Kohnen], Kohnen surveys on the recent results his paper jointly with Mason on the generalized modular functions (GMF) (of weight zero) on Γ, where $\Gamma \subset SL_2(\mathbb{Z})$ is a subgroup of finite index. A GMF is a holomorphic function $f : \mathcal{H} \to \mathbf{C}$ satisfying the following two conditions.

i) $f(\gamma \circ z) = \chi(\gamma) f(z)$ $(\forall \gamma \in \Gamma)$, where $\chi : \Gamma \to \mathbf{C}^*$ is a (not necessarily unitary) character of Γ,

ii) f is meromorphic at the cusps of Γ.

In the paper [Komori-Matsumoto-Tsumura], the authors report on some recent developments on the second author's
Problem: Is it possible to find some functional relations for multiple zeta-functions which include some value-distribution for MZV's (multiple zeta values)? Here the multiple zeta-function of complex variables $\mathbf{s} = (s_1, s_2, \cdots .s_r)$ is defined by

$$\zeta(s_1, s_2, \ldots, s_r) = \sum_{m_1 > m_2 > \cdots > m_r \ge 1} \frac{1}{m_1^{s_1} m_2^{s_2} \cdots m_r^{s_r}}.$$

and the MZV of depth r is $\zeta(k_1, k_2, \ldots, k_r)$ with $k_1, k_2, \ldots, k_r \in \mathbb{N}, k_1 > 1$.

The authors are concerned with the multi-variable (version of the) Witten zeta-function defined by

$$\zeta_r(\mathbf{s}; \mathfrak{g}) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{\alpha \in \Delta_+} \langle \alpha^\vee, m_1\lambda_1 + \cdots + m_r\lambda_r \rangle^{-s_\alpha},$$

where $\mathfrak{g}$ is a complex semi-simple Lie algebra with rank r, $\mathbf{s} = (s_\alpha)_{\alpha \in \Delta_+} \in \mathbb{C}^n$ and the data appearing on the right-hand side are certain quantities associated with $\mathfrak{g}$.

Theorem 5.1 seems to be the culmination of the results which gives a general form of the functional relations for the multiple zeta-function $\zeta_r(\mathbf{s}, \mathbf{y}; \Delta)$ with an additive character of a root system, i.e. for

$$\sum_{w \in W^I} \Big(\prod_{\alpha \in \Delta_{w^{-1}}} (-1)^{-s_\alpha} \Big) \zeta_r(w^{-1}\mathbf{s}, w^{-1}\mathbf{y}; \Delta).$$

However, to deduce explicit functional relations for specific root systems from Theorem 5.1 seems rather cumbersome and the authors give some handy formulas in §6 which they apply in later sections to deduce explicit relations for $\mathfrak{g} = A_3$, B_2, B_3, C_3. There are given many concrete examples which make the paper readable.

[Liu] is a short introduction to Maass wave forms, which originated from his lectures at Postech in 2007. Restricting to the simplest setting of the Maass forms of weight 0 on the full modular group, he manages to give a quick introduction to its formidable theory and liquidates the situation more familiar to analytic number-theorists. From the contents one can see the flowchart of the paper. He starts from Fourier expansion of Maass forms, proving thereby the Chowla-Selberg formula for the Eisenstein series, goes on to the spectral decomposition of the Hilbert space of square-integrable automorphic functions with respect to the non-Euclidean Laplacian, establishing the facts about the Laplace eigenvalues. After introducing Hecke theory, he introduces the automorphic L-functions and develop analytic methods to study them. Towards the end of the paper, a Linnik-type problem for Maass forms is studied, exhibiting how analytic number theory can develop on such exotic stages.

There are two nice collections of problems by Andrzej and Igor'.

In [Schinzel] there is a collection of problems concerning the number $N(f)$ of non-zero coefficients of a polynomial $f \in K$, K being a field. f is called an $N(f)$-nomial, e. g. $x^n - a$ is a binomial while $4x^{20} + 7x^{18} + 64$ is a trinomial. Problems are about the estimate from above or below on $N(f)$. E. g. Problem 2 asks about the existence (and boundedness if it exists) of a constant $C(K)$ such that every trinomial over K has an irreducible factor f with $N(f) < C(K)$.

In [Shparlinski] many open problems are given on the estimate of exponential and character sums according to the author's taste and interest, with enlightening annotation. Problems fall into two categories; in the first category new improvements of the estimates are asked for, while the second is concerned with applications of these sums. We shall talk about the second category.

Problem **??** has a natural interpretation in the study of polynomially growing sequences on orbits of the dynamical system generated by the map $u \to gu$ in $\mathbb{Z}/g\mathbb{Z}$. Problem **??** gives the estimate for complexity of factorization algorithm for polynomials over $\mathbb{F}_p$. Problem **??** has applications to the study of distribution of Selmer groups of a certain family of elliptic curves. Problem **??** is concerned with the estimate of the exponential sum of a sparse polynomial and has applications to number theory, computer science and cryptography. Problem **??** is about the exponential sum of a non-linear recurrence sequence and has applications to pseudo-random number generation.

We invited Professor Yamamura to contribute his list of determinantal expressions for the class number of Abelian number fields. The theory started from the paper of Carlitz and Olson in 1955 and has been pursued constantly. The list is in the spirit of Dilcher-Skula-Slavutskii volume on Bernoulli numbers, is complete and will be useful for researchers in the relevant problems.

And we also need to add one sentence, Professor Haruo Tsukada added corrigendum to his paper published in the last proceedings.

Finally, vote of thanks is due. We would like to thank Kinki University for its generous permission of using its excellent facilities. The conference room was equipped with modern conveniences and was very useful in conducting the seminar. We would like to thank Professor Kohji Chinen for his help in preparing posters of the seminar as well as constant support in keeping the working conditions in the conference room pleasant. We would like to thank Dr. Hiromitsu Tanaka for technical help in manipulating modern devices and for his thoughtful arrangement of things. We would like to thank Sheraton-Miyako Hotel for its excellent service and hospitality; especially thanks are due to Messers H. Fujihara and K. Morimoto for their thoughtful support throughout.

As in the case of the last proceedings, Professor Jing Ma from Jilin University made a devoted help in editing and we record here our hearty thanks to her for her excellent and beautiful preparation of the manuscript of the proceedings. It was a pity that she could not attend the seminar, but she came to Japan in July, 2009 to complete the editing work.

Finally, we would like to express our hearty thanks to S. Kanemitsu's students, Mr. N. -L. Wang and Ms. X. -H. Wang for their devoted support in making the stay of foreign participants more comfortable and pleasant.

As usual we complete the preface by a poem. This time Professor Chaohua Jia composed it.

大阪行

賈朝華

迷離難波夜
清澄澱河川
論道花叢裡
何須上深山

CONTENTS

Preface 1

Functional relations for zeta-functions of root systems 1
Yasushi Komori, Kohji Matsumoto and Hirofumi Tsumura

Index 51

FUNCTIONAL RELATIONS FOR ZETA-FUNCTIONS OF ROOT SYSTEMS

YASUSHI KOMORI

Graduate School of Mathematics, Nagoya University,
Chikusa-ku, Nagoya 464-8602, Japan
E-mail: komori@math.nagoya-u.ac.jp

KOHJI MATSUMOTO

Graduate School of Mathematics, Nagoya University,
Chikusa-ku, Nagoya 464-8602, Japan
E-mail: kohjimat@math.nagoya-u.ac.jp

HIROFUMI TSUMURA

Department of Mathematics and Information Sciences, Tokyo Metropolitan University,
1-1, Minami-Ohsawa, Hachioji, Tokyo 192-0397 Japan
E-mail: tsumura@tmu.ac.jp

We report on the theory of functional relations among zeta-functions of root systems, including known formulas for their special values. In the first part of this paper, we present known results on value-relations and functional relations for zeta-functions of root systems of A_2 type. Also, in view of the symmetry of underlying Weyl groups, we discuss a general framework of functional relations. In the second part of this paper, we prove several new results; we give a method for constructing functional relations systematically, and prove new functional relations among zeta-functions of root systems of types A_3, $C_2(\simeq B_2)$, B_3 and C_3, which include Witten's volume formulas as value-relations with explicit values of coefficients.

1. Introduction

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0$ the set of non-negative integers, $\mathbb{Z}$ the ring of rational integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{R}$ the field of real numbers and $\mathbb{C}$ the field of complex numbers.

The multiple zeta value (MZV) of depth r is defined by

$$\zeta(k_1, k_2, \ldots, k_r) = \sum_{m_1 > m_2 > \cdots > m_r \geq 1} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}} \qquad (1.1)$$

for $k_1, k_2, \dots, k_r \in \mathbb{N}$ with $k_1 > 1$ (see Zagier [51] and Hoffman [12]). It was Euler who first studied the double zeta values and gave some relation formulas among them such as

$$\sum_{j=2}^{k-1} \zeta(j, k-j) = \zeta(k) \tag{1.2}$$

for $k \in \mathbb{N}$ with $k \geq 3$, where $\zeta(s)$ is the Riemann zeta-function. Equation (1.2) is called the sum formula for double zeta values (see [9]). Research on MZVs has been conducted intensively in this decade (see the survey, [4,13,15]). A recent feature of studies on MZVs is to investigate the structure of the $\mathbb{Q}$-algebra generated by MZVs.

On the other hand, in the late 1990's, it was established that the multiple zeta-function $\zeta(s_1, s_2, ..., s_r)$ of complex variables can be continued meromorphically to the whole complex space $\mathbb{C}^r$ by, for example, Essouabri ([7,8]), Akiyama-Egami-Tanigawa ([1]), Arakawa-Kaneko ([3]), Zhao ([52]) and the second-named author ([23,25]).

Based on these researches, the second-named author raised the following problem several years ago (see, for example, [27]).

Problem. *Are the known relation formulas for multiple zeta values valid only at positive integers, or valid continuously also at other values?*

In other words, is it possible to find certain functional relations for multiple zeta-functions, which include some value-relations for MZVs? A classical example is the following formula which is often called the harmonic product relation:

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2).$$

As a related result, Bradley showed a certain class of functional relations called partition identities (see [5]). However, there are many kinds of relations among MZVs, so it is natural to expect that there will be many other classes of functional relations. For example, it seems interesting to prove certain functional relations which include sum formulas for MZVs. In order to give an "answer" to some specific cases of this Problem (the specification being clear from the context), we consider a wider class of multiple zeta-functions as follows.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with rank r. The Witten zeta-function associated with $\mathfrak{g}$ is defined by

$$\zeta_W(s; \mathfrak{g}) = \sum_{\varphi} (\dim \varphi)^{-s}, \tag{1.3}$$

where the summation runs over all finite dimensional irreducible representations φ of $\mathfrak{g}$.

Witten's motivation [50] for introducing the above zeta-function is to express the volumes of certain moduli spaces in terms of special values of (1.3). The expression is called Witten's volume formula, which especially implies that

$$\zeta_W(2k;\mathfrak{g}) = C_W(2k,\mathfrak{g})\pi^{2kn} \tag{1.4}$$

for any $k \in \mathbb{N}$, where n is the number of all positive roots of $\mathfrak{g}$ and $C_W(2k,\mathfrak{g}) \in \mathbb{Q}$ (Witten [50], Zagier [51]). In their work, the value of $C_W(2k,\mathfrak{g})$ is not explicitly given.

Let Δ be the set of all roots of $\mathfrak{g}$ in the vector space V equipped with an inner product $\langle \cdot,\cdot \rangle$, Δ_+ the set of all positive roots of $\mathfrak{g}$, $\Psi = \{\alpha_1,\ldots,\alpha_r\}$ the fundamental system of Δ, and $\alpha_j^\vee$ the coroot associated with α_j ($1 \le j \le r$). Let $\lambda_1,\ldots,\lambda_r$ be the fundamental weights satisfying $\langle \alpha_i^\vee, \lambda_j\rangle = \lambda_j(\alpha_i^\vee) = \delta_{ij}$ (Kronecker's delta). A more explicit form of $\zeta_W(s;\mathfrak{g})$ can be written down in terms of roots and weights by using Weyl's dimension formula (see (1.4) of [18]). Inspired by that form, we introduced in [18] the multi-variable version of Witten zeta-function

$$\zeta_r(\mathbf{s};\mathfrak{g}) = \sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\prod_{\alpha\in\Delta_+}\langle \alpha^\vee, m_1\lambda_1+\cdots+m_r\lambda_r\rangle^{-s_\alpha}, \tag{1.5}$$

where $\mathbf{s} = (s_\alpha)_{\alpha\in\Delta_+} \in \mathbb{C}^n$. In the case that $\mathfrak{g}$ is of type X_r, we call (1.5) the zeta-function of the root system of type X_r, and denote it by $\zeta_r(\mathbf{s};X_r)$, where $X = A,B,C,D,E,F,G$. We also use the notation $\zeta_W(s;X_r)$ and $C_W(2k,X_r)$, instead of $\zeta_W(s;\mathfrak{g})$ and $C_W(2k,\mathfrak{g})$, respectively. Note that from (1.5) and [18, (1.7)], we have

$$\zeta_W(s;X_r) = K(X_r)^s\zeta_r(s,\ldots,s;X_r), \tag{1.6}$$

where

$$K(X_r) = \prod_{\alpha\in\Delta_+}\langle \alpha^\vee, \lambda_1+\cdots+\lambda_r\rangle. \tag{1.7}$$

For example, $K(A_2) = 2$ and $K(C_2) = 6$ (see [18, (2.4) and (2.10)]).

More generally, in [18], we introduced multiple zeta-functions associated with sets of roots. In fact, we studied recursive structures in the family of those zeta-functions, which can be described in terms of Dynkin diagrams of underlying root systems. The meromorphic continuation of those zeta-functions is ensured as a special case of Essouabri's general theorem ([7,8]). It can also be proved by using the Mellin-Barnes integral formula (see [26]).

In [19], we established a general method for evaluating $\zeta_r(s,\ldots,s;X_r)$ at positive integers by considering generalizations of Bernoulli polynomials. In terms of those generalized Bernoulli polynomials, we gave a certain generalization of Witten's volume formula (1.4) with explicit determination of the constant $C_W(2k,X_r)$.

Several cases of zeta-functions of root systems had already been studied. A typical case is of A_2 type:

$$\zeta_2(s_1,s_2,s_3;A_2)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^{s_1}n^{s_2}(m+n)^{s_3}}. \tag{1.8}$$

In the 1950's, Tornheim [39] first studied the value $\zeta_2(d_1,d_2,d_3;A_2)$ for $d_1,d_2,d_3\in\mathbb{N}$, which is called the Tornheim double sum. Independently, Mordell [33] studied the value $\zeta_2(2d,2d,2d;A_2)$ ($d\in\mathbb{N}$) and proved, for example,

$$\zeta_2(2,2,2;A_2)=\frac{1}{2835}\pi^6. \tag{1.9}$$

This determines the value of $C_W(2k,A_2)$ in (1.4). Following their works, several value-relations for $\zeta_2(s_1,s_2,s_3;A_2)$ were obtained by several authors (see [6,14,37,40,51]), and also those for its alternating analogues ([41,43,48]). On the other hand, from the analytic viewpoint, the second-named author [24] studied the multi-variable function $\zeta_2(s_1,s_2,s_3;A_2)$ for $s_1,s_2,s_3\in\mathbb{C}$ which is also called the Mordell-Tornheim double zeta-function, denoted by $\zeta_{MT,2}(s_1,s_2,s_3)$.

Using $\zeta_2(s_1,s_2,s_3;A_2)$, we can give an "answer" to the Problem, that is, functional relations, for example,

$$\zeta(s+1,1)-\zeta_2(s,1,1;A_2)+\zeta(s+2)=0 \tag{1.10}$$

which holds for all $s\in\mathbb{C}$ except for singularities of the three functions on the left-hand side. In fact, letting $s=k-2$ for $k\geq 3$ in (1.10) and considering partial fraction decompositions, we can obtain the sum formula (1.2). This implies that (1.10) is an answer to the Problem. More generally the third-named author ([47]) proved functional relations for $\zeta_2(s_1,s_2,s_3;A_2)$ which include (1.10) (see Theorem 3.1), and for its alternating analogues ([45]), and its χ-analogues ([46]). A little later, Nakamura gave simple proofs of these results ([34,35]) whose method was inspired by Zagier's lecture.

As for the case of C_2 type, the second-named author defined $\zeta_2(s_1,s_2,s_3,s_4;C_2)$ and studied its analytic properties (see [26]). A little later, the third-named author gave some evaluation formulas for $\zeta_2(k_1,k_2,k_3,k_4;C_2)$ ($k_1,k_2,k_3,k_4\in\mathbb{N}$) when $k_1+k_2+k_3+k_4$ is odd ([44]).

As for the case of A_3 type, Gunnells and Sczech [10] gave explicit forms of Witten's volume formulas of this type. Recently the second and the third-named authors [30] studied $\zeta_3(\mathbf{s}; A_3)$, and gave certain functional relations for them.

Based on Zagier's work [51] and, in particular, on Nakamura's observation mentioned above, we found that the structural background of those functional relations is given by the symmetry with respect to Weyl groups. From this viewpoint, we considered this structure in [16–19]. In particular, in [19], we gave general forms of functional relations for zeta-functions of root systems. We will recall this result in Section 5.

In the first half of this paper, we summarize known results on functional relations for zeta-functions of root systems, which can be regarded as answers to the Problem. In Section 2, we recall a method of studying relations among Dirichlet series, which is called the 'u-method', introduced in [42]. In Section 3, we summarize known results on functional relations for $\zeta_2(\mathbf{s}; A_2)$. In Section 4, we introduce another method to construct functional relations for multiple Dirichlet series ([31]) which was inspired by Hardy's method of proving the functional equation for $\zeta(s)$ ([11]). In Section 5, we recall general forms of functional relations for $\zeta_r(\mathbf{s}; X_r)$ which we gave in [19]. This is the most general result stated in the present paper, but in general, from this theorem, it is not easy to deduce explicit forms of functional relations in each case. Therefore in the latter half of the paper we give a different method of constructing explicit functional relations. In Section 6, we prove a key lemma (Lemma 6.2) to give a certain procedure to construct functional relations systematically, which has the same flavour as the u-method. In Section 7, by using this lemma combined with a new idea of making use of polylogarithms, we give a functional relation for $\zeta_3(\mathbf{s}; A_3)$ which includes the explicit form of Witten's volume formula of A_3 type. (By "explicit form" we mean that the exact value of $C_W(2k, \mathfrak{g})$ is also determined.) In Sections 8 and 9, by a combination of the methods in Section 4 and in Section 7, we give functional relations for $\zeta_2(\mathbf{s}; C_2)$, for $\zeta_3(\mathbf{s}; B_3)$, and for $\zeta_3(\mathbf{s}; C_3)$ which include explicit forms of Witten's volume formulas.

2. A method to evaluate the Riemann zeta-function

In this section, we introduce a method for evaluating the (multiple) Dirichlet series at positive integers from the information of its trivial zeros, which is called the 'u-method'. In [42], the third-named author first established a method to prove Euler's formula for $\zeta(2k)$ ($k \in \mathbb{N}$). By applying this method to multiple series, several value-relations and functional relations for them have been given (see [40,44–47]). Here we briefly explain this method and recover Euler's formula for $\zeta(s)$:

$$\zeta(2m) = \frac{(-1)^{m-1}2^{2m-1}\pi^{2m}}{(2m)!}B_{2m} \quad (m \in \mathbb{N}), \tag{2.11}$$

where B_n is the nth Bernoulli number defined by $t/(e^t-1) = \sum_{n\geq 0} B_n t^n/n!$. For a small $\delta > 0$ and $u \in [1, 1+\delta]$, we let

$$F(t;u) := \frac{2e^t}{e^t+u} = \sum_{n=0}^{\infty} \mathcal{E}_n(u)\frac{t^n}{n!} \qquad (|t| < \pi), \tag{2.12}$$

where each $\mathcal{E}_n(u)$ is a rational function in u and is continuous for $u \in [1, 1+\delta]$ because $(\partial^k/\partial t^k)F(t;u)$ is continuous for $(t,u) \in \{|t| < \pi\} \times [1, 1+\delta]$. Let $\gamma \in \mathbb{R}$ with $0 < \gamma < \pi$, and $\mathcal{C}_\gamma : z = \gamma e^{it}$ for $0 \leq t \leq 2\pi$, where $i = \sqrt{-1}$. From (2.12), we have

$$\int_{\mathcal{C}_\gamma} F(z;u) z^{-n-1} dz = \frac{(2\pi i)\mathcal{E}_n(u)}{n!} \quad (n \in \mathbb{N}_0). \tag{2.13}$$

Let $M = M(\gamma) := \max |F(z,u)|$ for $(z,u) \in \mathcal{C}_\gamma \times [1, 1+\delta]$, which is independent of $u \in [1, 1+\delta]$. Then we obtain

$$\frac{|\mathcal{E}_n(u)|}{n!} \leq \frac{1}{2\pi} \int_{\mathcal{C}_\gamma} |F(z;u)|\ |z|^{-n-1} |dz| \leq \frac{M(\gamma)}{\gamma^n} \quad (n \in \mathbb{N}_0).$$

We let $\phi(s;u) = \sum_{n\geq 1}(-u)^{-n}n^{-s}$ for $s \in \mathbb{C}$. As is well known, $\phi(s;u)$ is convergent for $\Re s > 0$ when $u = 1$ and is convergent for any $s \in \mathbb{C}$ when $u > 1$. Furthermore, we see that $\phi(s;1) = (2^{1-s}-1)\zeta(s)$. When $u > 1$, the second member of (2.12) can be expanded as $-2\sum_{n\geq 1}(-u)^{-n}e^{nt}$. Hence we have $\mathcal{E}_m(u) = -2\phi(-m;u)$ for $m \in \mathbb{N}_0$.

For any $k \in \mathbb{N}$ and $\theta \in (-\pi, \pi)$, we set

$$I_k(\theta;u) := i\sum_{n=1}^{\infty} \frac{(-u)^{-n}\sin(n\theta)}{n^{2k+1}}. \tag{2.14}$$

Suppose $u \in (1, 1+\delta]$ and $\theta \in (-\pi, \pi)$, then

$$\begin{aligned} &I_k(\theta;u) \\ &= \sum_{j=0}^{\infty} \phi(2k-2j;u)\frac{(i\theta)^{2j+1}}{(2j+1)!} \\ &= \sum_{j=0}^{k-1} \phi(2k-2j;u)\frac{(i\theta)^{2j+1}}{(2j+1)!} - \frac{1}{2}\sum_{m=0}^{\infty} \mathcal{E}_{2m}(u)\frac{(i\theta)^{2m+2k+1}}{(2m+2k+1)!}. \end{aligned} \tag{2.15}$$

If $|\theta| < \gamma < \pi$, we see that the right-hand side of (2.15) is uniformly convergent with respect to u on $[1, 1+\delta]$, so is continuous on $u \in [1, 1+\delta]$ (see Remark 2.1 (ii)). On the other hand, the left-hand side of (2.15) is also continuous on $u \in [1, 1+\delta]$ from the definition of $I_k(\theta; u)$. Hence we can let $u \to 1$ on both sides of (2.15).

Now we arrive at the crucial point of the argument. We use the fact that $\zeta(-2m) = 0$, that is, $\mathcal{E}_{2m}(1) = 0$ for $m \in \mathbb{N}$ and $\mathcal{E}_0(1) = 1$ (see Remark 2.1 (i)). Then, for $\theta \in (-\pi, \pi)$, we have

$$I_k(\theta; 1) = \sum_{j=0}^{k-1} \phi(2k-2j; 1) \frac{(i\theta)^{2j+1}}{(2j+1)!} - \frac{(i\theta)^{2k+1}}{2(2k+1)!}. \tag{2.16}$$

Since $k \geq 1$, each side of (2.16) is continuous in $\theta \in [-\pi, \pi]$. Hence we can let $\theta \to \pi$ on both sides of (2.16) to obtain

$$0 = I_k(\pi; 1) = \sum_{j=0}^{k-1} \phi(2k-2j; 1) \frac{(i\pi)^{2j+1}}{(2j+1)!} - \frac{(i\pi)^{2k+1}}{2(2k+1)!}.$$

For simplicity, we define

$$\mathcal{A}_{2m} = \phi(2m; 1) \frac{(2m)!}{(i\pi)^{2m}} = (2^{1-2m} - 1)\zeta(2m) \frac{(2m)!}{(i\pi)^{2m}} \quad (m \in \mathbb{N}_0), \tag{2.17}$$

and $\mathcal{A}_0 = -1/2$. Then (2.16) implies that

$$\sum_{j=0}^{k} \binom{2k+1}{2j+1} \mathcal{A}_{2k-2j} = 0$$

for $k \in \mathbb{N}$. Since $\mathcal{A}_0 = -1/2$, we obtain

$$-\frac{t}{2} = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \binom{2k+1}{2j+1} \mathcal{A}_{2k-2j} \right) \frac{t^{2k+1}}{(2k+1)!} = \left(\sum_{m=0}^{\infty} \mathcal{A}_{2m} \frac{t^{2m}}{(2m)!} \right) \frac{e^t - e^{-t}}{2}.$$

We can easily check that

$$\frac{2t}{e^t - e^{-t}} = \frac{2te^t}{e^{2t} - 1} = \sum_{m=0}^{\infty} \left(2 - 2^{2m}\right) B_{2m} \frac{t^{2m}}{(2m)!},$$

so we have $\mathcal{A}_{2m} = (2^{2m-1} - 1)B_{2m}$ for any nonnegative integer m. In view of (2.17), we obtain Euler's formula (2.11).

Remark 2.1. (i) It should be noted that the fact

$$\begin{aligned} &-2\phi(-2m; 1) \left(= -2\left(2^{2m+1} - 1\right)\zeta(-2m)\right) \\ &= \mathcal{E}_{2m}(1) \, (= 0) \quad (m \in \mathbb{N}) \end{aligned} \tag{2.18}$$

(the trivial zeros of zeta-function!) plays a vital role in the above argument. In fact, equation (2.18) can be obtained by proving

$$\begin{aligned}\sum_{n=0}^{\infty}\{-2\phi(-n;1)\}\frac{t^n}{n!}&=\sum_{n=0}^{\infty}\left\{-2\left(2^{n+1}-1\right)\zeta(-n)\right\}\frac{t^n}{n!}\\&=1+2\sum_{n=1}^{\infty}\left(2^{n+1}-1\right)B_{n+1}\frac{t^n}{(n+1)!}\\&=2+\frac{4}{e^{2t}-1}-\frac{2}{e^t-1}=\frac{2e^t}{e^t+1},\end{aligned}\tag{2.19}$$

because $\zeta(1-k)=-B_k/k$ ($k\in\mathbb{N}$; $k\geq 2$) and $\zeta(0)=-1/2$. (ii) Also we note that $\phi(s;u)$ is continuous in u as $u\to 1+0$ for any $s\in\mathbb{C}$. In fact, similarly to the case of $\zeta(s)$, we can easily see that

$$\phi(s;u)=\frac{1}{(e^{2\pi is}-1)\,\Gamma(s)}\int_C\frac{e^t}{e^t+u}t^{s-1}dt,\tag{2.20}$$

where C is the contour, that is, the path which starts at $+\infty$, passes through the real axis, goes around the origin counterclockwise and goes back to $+\infty$. From (2.20), we immediately obtain the desired continuity.

3. Functional relations for $\zeta_2(s_1,s_2,s_3;A_2)$

By applying the method introduced in Section 2 to $\zeta_2(s_1,s_2,s_3;A_2)$, the third-named author gave value-relation formulas for $\zeta_2(s_1,s_2,s_3;A_2)$ (see [40]). Moreover, applying the above method to the double series in complex variables, he gave functional relations for $\zeta_2(s_1,s_2,s_3;A_2)$ (see [47]). The original form in [47, Theorem 4.5] is a little complicated. By using a certain transformation formula (see Lemma 6.1, which is [28, Lemma 2.1]), we obtain the following simpler form.

Theorem 3.1. *For $k,l\in\mathbb{N}_0$,*

$$\begin{aligned}&\zeta_2(k,l,s;A_2)+(-1)^k\zeta_2(k,s,l;A_2)+(-1)^l\zeta_2(l,s,k;A_2)\\&\quad=2\sum_{\rho=0}^{[k/2]}\binom{k+l-2\rho-1}{l-1}\zeta(2\rho)\zeta(s+k+l-2\rho)\\&\qquad+2\sum_{\rho=0}^{[l/2]}\binom{k+l-2\rho-1}{k-1}\zeta(2\rho)\zeta(s+k+l-2\rho)\end{aligned}\tag{3.21}$$

holds for all $s \in \mathbb{C}$ except for singularities of functions on both sides.

Proof. For $\theta, r, u \in \mathbb{R}$ with $r > 1$ and $u \in [1, 1+\delta]$, and $k, p \in \mathbb{N}_0$, we let

$$\mathfrak{F}(i\theta; r; u) = \sum_{n=1}^{\infty} \frac{(-u)^{-n} e^{int}}{n^r},$$

$$\mathcal{J}_p(i\theta; k; u) = \frac{i^{p-1}}{2} \left\{ \mathfrak{F}(i\theta; k; u) + (-1)^{p-1} \mathfrak{F}(-i\theta; k; u) \right\} - \sum_{j=0}^{k} \phi(k-j; u)\ \varepsilon_{p+1+j} \frac{(i\theta)^j}{j!},$$

where $\varepsilon_m = \{1 + (-1)^m\}/2$ for $m \in \mathbb{Z}$. Then, similarly to the proof of (2.16), we see that if $k \not\equiv p \pmod 2$ and $\theta \in (-\pi, \pi)$ then $\mathcal{J}_p(i\theta; k; u) \to 0$ as $u \to 1$. Let

$$R(s_1, s_2; s_3; u) = \sum_{m,n=1}^{\infty} \frac{(-u)^{-2m-n}}{m^{s_1} n^{s_2} (m+n)^{s_3}},$$

$$S(s_1, s_2; s_3; u) = \sum_{m,n=1}^{\infty} \frac{(-u)^{-m-n}}{m^{s_1} n^{s_2} (m+n)^{s_3}},$$

which are double analogues of $\phi(s; u)$. Then, for $u \in (1, 1+\delta]$,

$$\begin{aligned}
&\mathcal{J}_p(i\theta; k; u) \mathfrak{F}(i\theta; r; u) \\
&\quad = i^{p-1} \sum_{N=0}^{\infty} \frac{1}{2} \Big\{ S(k, r; -N; u) + (-1)^{p-1} R(k, -N; r; u) \\
&\qquad\qquad + (-1)^{p-1+N} R(r, -N; k; u) \Big\} \frac{(i\theta)^N}{N!} \\
&\quad - \sum_{N=0}^{\infty} \sum_{j=0}^{k} \binom{N}{j} \phi(k-j; u) \phi(r+j-N; u) \varepsilon_{p+1+j} \frac{(i\theta)^N}{N!} \\
&\quad + \frac{(-i)^{p-1}}{2} \sum_{m=1}^{\infty} \frac{u^{-2m}}{m^{k+r}}.
\end{aligned}$$

As noted above, the left-hand side tends to 0 as $u \to 1$ when $k \not\equiv p \pmod 2$ and $\theta \in (-\pi, \pi)$. Therefore, similarly to the case of $\zeta(s)$, we can obtain the original form of the functional relation ([47, Lemma 4.5]):

$$\zeta_2(k, l, s; A_2) + (-1)^k \zeta_2(k, s, l; A_2) + (-1)^l \zeta_2(l, s, k; A_2)$$

$$
\begin{aligned}
&= 2 \sum_{\substack{j=0 \\ j\equiv k\ (2)}}^{k} \left(2^{1-k+j}-1\right)\zeta(k-j) \\
&\quad \times \sum_{\mu=0}^{[j/2]} \frac{(i\pi)^{2\mu}}{(2\mu)!} \binom{l-1+j-2\mu}{j-2\mu} \zeta(l+j+s-2\mu) \\
&- 4 \sum_{\substack{j=0 \\ j\equiv k\ (2)}}^{k} \left(2^{1-k+j}-1\right)\zeta(k-j) \sum_{\mu=0}^{[(j-1)/2]} \frac{(i\pi)^{2\mu}}{(2\mu+1)!} \sum_{\substack{\nu=0 \\ \nu\equiv l\ (2)}}^{l} \zeta(l-\nu) \\
&\quad \times \binom{\nu-1+j-2\mu}{j-2\mu-1} \zeta(\nu+j+s-2\mu)
\end{aligned}
$$

holds for all $s \in \mathbb{C}$ except for singularities of functions on both sides, where $k, l \in \mathbb{N}$. Additionally, using a transformation formula in Lemma 6.1 below ([28, Lemma 2.1]), we obtain (3.21). □

Example 3.1. Setting $(k,l) = (2,2),\ (3,2)$ in (3.21), we have

$$\zeta_2(2,2,s;A_2) + 2\zeta_2(2,s,2;A_2) = 4\zeta(2)\zeta(s+2) - 6\zeta(s+4), \qquad (3.22)$$

$$
\begin{aligned}
&\zeta_2(3,s,2;A_2) - \zeta_2(3,2,s;A_2) - \zeta_2(2,s,3;A_2) \\
&\quad = 10\zeta(s+5) - 6\zeta(2)\zeta(s+3). \qquad (3.23)
\end{aligned}
$$

Setting $s = 2$ in (3.22) and (3.23), we have (1.9) and

$$\zeta_2(2,2,3;A_2) = 6\zeta(2)\zeta(5) - 10\zeta(7), \qquad (3.24)$$

respectively, where (3.24) was given by Tornheim [39]. Note that $\zeta_2(k,0,l;A_2) = \zeta(l,k)$. Then, setting $s = 0$ in (3.23), we have $\zeta(2,3) - \zeta(3,2) = 10\zeta(5) - 5\zeta(2)\zeta(3)$. On the other hand, it is well-known that $\zeta(3,2) + \zeta(2,3) = \zeta(2)\zeta(3) - \zeta(5)$. Combining these results, we obtain the known results

$$\zeta(2,3) = \frac{9}{2}\zeta(5) - 2\zeta(2)\zeta(3); \quad \zeta(3,2) = -\frac{11}{2}\zeta(5) + 3\zeta(2)\zeta(3),$$

which were originally obtained by double shuffle relations.

Remark 3.1. In [32], the second and the third-named authors generalized the result in Theorem 3.1 to the case of polylogarithmic analogues, that is,

$$\begin{aligned}\sum_{m,n=1}^{\infty}\frac{x^n}{m^k n^l (m+n)^s}&+(-1)^k\sum_{m,n=1}^{\infty}\frac{x^n}{m^k n^s (m+n)^l}+(-1)^l\sum_{m,n=1}^{\infty}\frac{x^{m+n}}{m^l n^s (m+n)^k}\\ &=2\sum_{\rho=0}^{[k/2]}\binom{k+l-2\rho-1}{k-2\rho}\zeta(2\rho)\sum_{m=1}^{\infty}\frac{x^m}{m^{s+k+l-2\rho}}\\ &\quad+2\sum_{\rho=0}^{[l/2]}\binom{k+l-2\rho-1}{l-2\rho}\zeta(2\rho)\sum_{m=1}^{\infty}\frac{x^m}{m^{s+k+l-2\rho}},\end{aligned}\tag{3.25}$$

for $x\in\mathbb{C}$ with $|x|\le 1$. The idea of this generalization gives an important key to construct functional relations for zeta-functions of the type of A_3, C_2, B_3 and C_3 (see Remark 7.1).

In [34], Nakamura gave an alternative simple proof of (3.21) whose method was inspired by Zagier's lecture. We explain this method. We denote by $\{B_n(x)\}$ the Bernoulli polynomials defined by $te^{xt}/(e^t-1)=\sum_{n\ge 0}B_n(x)t^n/n!$ $(|t|<2\pi)$. It is known (see [2, p.266 - p.267]) that $B_{2j}(0)=B_{2j}$ for $j\in\mathbb{N}_0$ and

$$B_j(x-[x])=-\frac{j!}{(2\pi i)^j}\lim_{K\to\infty}\sum_{\substack{k=-K\\ k\ne 0}}^{K}\frac{e^{2\pi ikx}}{k^j}\qquad (j\in\mathbb{N}),\tag{3.26}$$

where $[\,\cdot\,]$ is the integer part. For $k\in\mathbb{Z}$, $j\in\mathbb{N}$ we have

$$\int_0^1 e^{-2\pi ikx}B_j(x)\,dx=\begin{cases}0 & (k=0),\\ -(2\pi ik)^{-j}j! & (k\ne 0),\end{cases}\tag{3.27}$$

by (3.26). We further quote [2, p.276 19.(b)], for $p,q\ge 1$, which is

$$\begin{aligned}B_p(x)B_q(x)=&\sum_{k=0}^{\max(p,q)/2}\left\{p\binom{q}{2k}+q\binom{p}{2k}\right\}\frac{B_{2k}B_{p+q-2k}(x)}{p+q-2k}\\ &-(-1)^p\frac{p!q!}{(p+q)!}B_{p+q}.\end{aligned}\tag{3.28}$$

On the other hand, for $a,b\ge 2$, and $\Re(s)>1$, we have

$$\int_0^1\sum_{l=1}^{\infty}\frac{e^{2\pi ilx}}{l^a}\sum_{m=1}^{\infty}\frac{e^{2\pi imx}}{m^b}\sum_{n=1}^{\infty}\frac{e^{-2\pi inx}}{n^s}dx=\zeta_2(a,b,s;A_2),$$

$$\int_0^1 \sum_{l,m=1}^{\infty} \frac{e^{2\pi imx}}{(m+l)^a l^b} \sum_{n=1}^{\infty} \frac{e^{-2\pi inx}}{n^s} dx = \zeta_2(b,s,a;A_2),$$

$$\int_0^1 \sum_{l=1}^{\infty} \frac{e^{2\pi ilx}}{l^{a+b-j}} \sum_{m=1}^{\infty} \frac{e^{-2\pi inx}}{n^s} dx = \zeta(a+b+s-j).$$

Combining these relations and (3.26)-(3.28), we see that

$$\begin{aligned} &\zeta_2(a,b,s;A_2) + (-1)^b \zeta_2(b,s,a;A_2) + (-1)^a \zeta_2(s,a,b;A_2) \\ &= \frac{2}{a!b!} \sum_{k=0}^{\max(a,b)/2} \left\{ a\binom{b}{2k} + b\binom{a}{2k} \right\} \\ &\quad \times (a+b-2k-1)!(2k)!\zeta(2k)\zeta(a+b-s-2k), \end{aligned} \tag{3.29}$$

which coincides with (3.21). Nakamura also gave some more generalized formulas for double zeta and L-functions and triple zeta-functions of Mordell and Tornheim type ([35,36]). Furthermore triple zeta and L-functions were studied by Nakamura, Ochiai, and the second and the third-named authors ([28,29]). The aforementioned Lemma 6.1 first appeared in those studies.

4. Another method to construct functional relations for Dirichlet series

In this section, we introduce another method to study functional relations for Dirichlet series, whose basic idea was originally introduced by Hardy. Hardy gave an alternative proof of the functional equation for $\zeta(s)$ ([11], see also [38] Section 2.2). By generalizing this method, we can give functional relations for multiple zeta-functions, for example, (3.21) in Theorem 3.1.

First we consider a general Dirichlet series $Z(s) = \sum_{m=1}^{\infty} a_m m^{-s}$ where $\{a_n\} \subset \mathbb{C}$. Let $\Re s = \rho$ ($\rho \in \mathbb{R}$) be the abscissa of convergence of $Z(s)$. This means that if $\Re s > \rho$ then $Z(s)$ is convergent and if $\Re s < \rho$ then $Z(s)$ is not convergent. We further assume that $0 \le \rho < 1$.

Theorem 4.1 ([31], Theorem 3.1). *Assume that* $\sum_{m=1}^{\infty} a_m \sin(mt) = 0$ *is boundedly convergent for* $t > 0$ *and that, for* $\rho < s < 1$,

$$\lim_{\lambda \to \infty} \sum_{m=1}^{\infty} a_m \int_{\lambda}^{\infty} t^{s-1} \sin(mt) dt = 0. \tag{4.30}$$

Then $Z(s)$ *can be continued meromorphically to* $\mathbb{C}$, *and actually* $Z(s) = 0$ *for all* $s \in \mathbb{C}$. *The same conclusion holds if we assume the formulas similar to the above but "*$\sin$*" (two places) is replaced by "*$\cos$*".*

We give a simple example showing how to apply this theorem. From (2.16) and the formula obtained by differentiating both sides of (2.16), we have

$$\sum_{l=1}^{\infty} \frac{(-1)^l \cos(l\theta)}{l^{2p}} = \sum_{\nu=0}^{p} \phi(2p-2\nu) \frac{(-1)^\nu \theta^{2\nu}}{(2\nu)!}, \tag{4.31}$$

$$\sum_{l=1}^{\infty} \frac{(-1)^l \sin(l\theta)}{l^{2q+1}} = \sum_{\nu=0}^{q} \phi(2q-2\nu) \frac{(-1)^\nu \theta^{2\nu+1}}{(2\nu+1)!} \tag{4.32}$$

for $p \in \mathbb{N}$, $q \in \mathbb{N}_0$ and $\theta \in (-\pi, \pi)$. Note that the case $q = 0$ in (4.32) is a little delicate. To prove this case, we define $I_0(\theta; u)$ for $\theta \in (-\pi, \pi)$ and $u \in [1, 1+\delta]$ by (2.14). Then equation (2.15) in the case $q = 0$ holds for $u \in (1, 1+\delta]$. From [49, § 3.35] (see also [31, Lemma 4.1]) and Abel's theorem (see [49, § 3.71]), we can let $u \to 1$ in (2.15) for $I_0(\theta; u)$. Then, as well as (2.16), we obtain the case $q = 0$ in (4.32). Additionally we note that if $p, q \in \mathbb{N}$ then (4.31) and (4.32) hold for $\theta \in [-\pi, \pi]$ because both sides are continous for $\theta \in [-\pi, \pi]$.

Combining these results and putting $t = \theta + \pi$, we obtain, for $t \in \mathbb{R}\setminus 2\pi\mathbb{Z}$,

$$\begin{aligned} &\sum_{m,n=1}^{\infty} \frac{\cos((m+n)t)}{m^2 n^2} + 2 \sum_{m,n=1}^{\infty} \frac{\cos(mt)}{n^2(m+n)^2} \\ &\quad + 6 \sum_{m=1}^{\infty} \frac{\cos(mt)}{m^4} - 4\zeta(2) \sum_{m=1}^{\infty} \frac{\cos(mt)}{m^2} = 0 \end{aligned} \tag{4.33}$$

(see [31, Lemma 2.2]). We denote by $f(t)$ the left-hand side of (4.33). Note that each sum on the left-hand side of (4.33) is absolutely and uniformly convergent for $t \in \mathbb{R}$. Hence, for $s \in \mathbb{R}$ with $0 < s < 1$, we have

$$\begin{aligned} 0 &= \int_0^\infty t^{s-1} f(t) dx \\ &= \int_0^\infty t^{s-1} \Bigg\{ \sum_{m,n=1}^{\infty} \frac{\cos((m+n)t)}{m^2 n^2} + 2 \sum_{m,n=1}^{\infty} \frac{\cos(mt)}{n^2(m+n)^2} \\ &\quad + 6 \sum_{m=1}^{\infty} \frac{\cos(mt)}{m^4} - 4\zeta(2) \sum_{m=1}^{\infty} \frac{\cos(mt)}{m^2} \Bigg\} dt. \end{aligned} \tag{4.34}$$

By the same argument as in [38, Section 2.1], we have

$$\int_\lambda^\infty \frac{\cos(Nx)}{x^{1-s}} dx = \left[\frac{\sin(Nx)}{N x^{1-s}} \right]_\lambda^\infty - \frac{s-1}{N} \int_\lambda^\infty \frac{\sin(Nx)}{x^{2-s}} dx = O\left(\frac{1}{N \lambda^{1-s}} \right)$$

for $N \in \mathbb{N}$. Using this result, we see that

$$\lim_{\lambda\to\infty}\sum_{m,n=1}^{\infty}\frac{1}{m^2n^2}\int_{\lambda}^{\infty}\frac{\cos((m+n)x)}{x^{1-s}}dx=0,$$
$$\lim_{\lambda\to\infty}\sum_{m,n=1}^{\infty}\frac{1}{n^2(m+n)^2}\int_{\lambda}^{\infty}\frac{\cos(mx)}{x^{1-s}}dx=0,$$
$$\lim_{\lambda\to\infty}\sum_{m=1}^{\infty}\frac{1}{m^l}\int_{\lambda}^{\infty}\frac{\cos(mx)}{x^{1-s}}dx=0\quad(l=2,4)$$

hold for $0<s<1$. Hence we can justify term-by-term integration on the right-hand side of (4.34). Therefore it follows from Theorem 4.1 and the facts

$$\int_0^{\infty}\frac{\cos bx}{x^{1-s}}dx=\frac{\pi}{2}b^{-s}\ \frac{\sec(\pi(1-s)/2)}{\Gamma(1-s)},$$
$$\int_0^{\infty}\frac{\sin bx}{x^{1-s}}dx=\frac{\pi}{2}b^{-s}\ \frac{\mathrm{cosec}(\pi(1-s)/2)}{\Gamma(1-s)}$$

for $b>0$ and $0<s<1$ (see [49, Chapter 12]) that the functional relation

$$\zeta_2(2,2,s;A_2)+2\zeta_2(s,2,2;A_2)+6\zeta(s+4)-4\zeta(2)\zeta(s+2)=0 \tag{4.35}$$

holds for $0<s<1$ (and then for any $s\in\mathbb{C}$ by analytic continuation). This coincides with (3.22).

By using this method, we can give functional relations for more general types of multiple zeta-functions (see [31]).

5. A general form of functional relations

In the previous sections, we present various methods to obtain functional relations. However in those methods, it is not clear why these functional relations exist. From the viewpoint of Weyl group symmetry in the underlying Lie algebra structure, we can give a certain explanation of this phenomenon. In fact, in view of the Weyl group symmetry, we can show a general form of functional relations for zeta-functions of root systems. For the details, see [19,22].

First we prepare some notation. Let V be an r-dimensional real vector space equipped with an inner product $\langle\cdot,\cdot\rangle$. Let Δ be a finite reduced root system in V of X_r type and $\Psi=\{\alpha_1,\ldots,\alpha_r\}$ its fundamental system. Let Δ_+ and Δ_- be the set of all positive roots and negative roots respectively. Then we have a decomposition of the root system $\Delta=\Delta_+\coprod\Delta_-$. Let $Q^\vee$

be the coroot lattice, P the weight lattice, P_+ the set of integral dominant weights and P_{++} the set of integral strongly dominant weights respectively defined by

$$Q^\vee = \bigoplus_{i=1}^r \mathbb{Z}\,\alpha_i^\vee, \quad P = \bigoplus_{i=1}^r \mathbb{Z}\,\lambda_i, \quad P_+ = \bigoplus_{i=1}^r \mathbb{N}_0\,\lambda_i, \quad P_{++} = \bigoplus_{i=1}^r \mathbb{N}\,\lambda_i, \tag{5.36}$$

where the fundamental weights $\{\lambda_j\}_{j=1}^r$ form a basis dual to $\Psi^\vee$ satisfying $\langle\alpha_i^\vee, \lambda_j\rangle = \delta_{ij}$. The reflection $\sigma_\alpha : V \to V$ with respect to a root $\alpha \in \Delta$ is defined by $\sigma_\alpha(v) = v - \langle\alpha^\vee, v\rangle\alpha$. For a subset $A \subset \Delta$, let $W(A)$ be the group generated by reflections σ_α for $\alpha \in A$. Let $W = W(\Delta)$ be the Weyl group. Then $\sigma_j = \sigma_{\alpha_j}$ $(1 \le j \le r)$ generates W. We denote the fundamental domain called the fundamental Weyl chamber by $C = \{v \in V \mid \langle\Psi^\vee, v\rangle \ge 0\}$, where $\langle\Psi^\vee, v\rangle$ means any of $\langle\alpha^\vee, v\rangle$ for $\alpha^\vee \in \Psi^\vee$. Then W acts on the set of Weyl chambers $WC = \{wC \mid w \in W\}$ simply transitively. Moreover if $wx = y$ for $x, y \in C$, then $x = y$ holds. The stabilizer W_x of a point $x \in V$ is generated by the reflections which stabilize x. We see that $P_+ = P \cap C$. For $w \in W$, we set $\Delta_w = \Delta_+ \cap w^{-1}\Delta_-$.

Let $I \subset \{1, \ldots, r\}$ and $\Psi_I = \{\alpha_i \mid i \in I\} \subset \Psi$. Let V_I be the linear subspace spanned by Ψ_I. Then $\Delta_I = \Delta \cap V_I$ is a root system in V_I whose fundamental system is Ψ_I. For the root system Δ_I, we denote the corresponding coroot lattice (resp. weight lattice etc.) by $Q_I^\vee = \bigoplus_{i\in I} \mathbb{Z}\,\alpha_i^\vee$ (resp. $P_I = \bigoplus_{i\in I} \mathbb{Z}\,\lambda_i$ etc.). Let $\Delta_+^\vee$ be the set of all positive coroots, and $W^I = \{w \in W \mid \Delta_{I+}^\vee \subset w\Delta_+^\vee\}$.

Let $\mathbf{y} \in V$ and $\mathbf{s} = (s_\alpha)_{\alpha\in\overline{\Delta}} \in \mathbb{C}^{|\Delta_+|}$, where $\overline{\Delta}$ is the quotient of Δ obtained by identifying α and $-\alpha$. Define an action of W to $\mathbf{s}$ by $(w\mathbf{s})_\alpha = s_{w^{-1}\alpha}$. Now we introduce the "twisted" multiple zeta-function of the form

$$\zeta_r(\mathbf{s}, \mathbf{y}; \Delta) = \sum_{\lambda\in P_{++}} e^{2\pi\sqrt{-1}\langle\mathbf{y},\lambda\rangle} \prod_{\alpha\in\Delta_+} \frac{1}{\langle\alpha^\vee, \lambda\rangle^{s_\alpha}}. \tag{5.37}$$

A motivation of introducing such a generalized form with exponential factors is to study multiple L-functions of root systems (see [17,21]). When $\mathbf{y} = 0$ in (5.37), the function $\zeta_r(\mathbf{s}; \Delta) = \zeta_r(\mathbf{s}, 0; \Delta)$ coincides with the zeta-function of the root system Δ, defined by (1.5).

For $s \in \mathbb{C}$, $\Re s > 1$ and $x, c \in \mathbb{R}$, let

$$\mathcal{L}_s(x, c) = -\frac{\Gamma(s+1)}{(2\pi\sqrt{-1})^s} \sum_{\substack{n\in\mathbb{Z}\\ n+c\neq 0}} \frac{e^{2\pi\sqrt{-1}(n+c)x}}{(n+c)^s}. \tag{5.38}$$

Then we obtain the following general form of functional relations for zeta-functions of root systems.

Theorem 5.1 ([19], Theorem 4.3). *When $I \neq \emptyset$, for $\mathbf{s} \in \mathcal{S}$ and $\mathbf{y} \in V$, we have*

$$\begin{aligned} S(\mathbf{s},\mathbf{y};I;\Delta) & \qquad (5.39) \\ := \sum_{w\in W^I}\Big(\prod_{\alpha\in\Delta_{w^{-1}}}(-1)^{-s_\alpha}\Big)\zeta_r(w^{-1}\mathbf{s},w^{-1}\mathbf{y};\Delta) \\ = (-1)^{|\Delta_+\setminus\Delta_{I+}|}\Bigg(\prod_{\alpha\in\Delta_+\setminus\Delta_{I+}}\frac{(2\pi\sqrt{-1})^{s_\alpha}}{\Gamma(s_\alpha+1)}\Bigg)\sum_{\lambda\in P_{I++}}e^{2\pi\sqrt{-1}\langle\mathbf{y},\lambda\rangle}\prod_{\alpha\in\Delta_{I+}}\frac{1}{\langle\alpha^\vee,\lambda\rangle^{s_\alpha}} \\ \times\int_0^1\cdots\int_0^1\exp\Big(-2\pi\sqrt{-1}\sum_{\alpha\in\Delta_+\setminus(\Delta_{I+}\cup\Psi)}x_\alpha\langle\alpha^\vee,\lambda\rangle\Big) \\ \times\Big(\prod_{\alpha\in\Delta_+\setminus(\Delta_{I+}\cup\Psi)}\mathcal{L}_{s_\alpha}(x_\alpha,0)\Big) \\ \times\Big(\prod_{i\in I^c}\mathcal{L}_{s_{\alpha_i}}\Big(\langle\mathbf{y},\lambda_i\rangle-\sum_{\alpha\in\Delta_+\setminus(\Delta_{I+}\cup\Psi)}x_\alpha\langle\alpha^\vee,\lambda_i\rangle,0\Big)\Big)\prod_{\alpha\in\Delta_+\setminus(\Delta_{I+}\cup\Psi)}dx_\alpha. \end{aligned}$$

Remark 5.1. We also studied the case $I = \emptyset$ and gave an integral expression of $S(\mathbf{s},\mathbf{y};\emptyset;\Delta)$ similar to (5.39) (see [19, Theorem 4.4]).

Example 5.1. Here we give an alternative proof of (3.23). Set $\Delta_+ = \Delta_+(A_2) = \{\alpha_1,\alpha_2,\alpha_1+\alpha_2\}$, and $\mathbf{y} = 0$, $\mathbf{s} = (2,s,3)$ for $s \in \mathbb{C}$ with $\Re s > 1$, $I = \{2\}$, that is, $\Delta_{I+} = \{\alpha_2\}$. Then we see that the left-hand side of (5.39) is

$$\begin{aligned} S(\mathbf{s},\mathbf{y};I;\Delta) &= \sum_{m,n=1}^\infty\frac{1}{m^2n^s(m+n)^3}-\sum_{\substack{m,n=1\\m\neq n}}^\infty\frac{1}{m^2n^s(-m+n)^3} \\ &= \zeta_2(2,s,3;A_2)-\zeta_2(3,2,s;A_2)+\zeta_2(3,s,2;A_2). \end{aligned}$$

On the other hand, the right-hand side of (5.39) is

$$\begin{aligned} &\left(\frac{(2\pi\sqrt{-1})^2}{2!}\right)\left(\frac{(2\pi\sqrt{-1})^3}{3!}\right)\sum_{m=1}^\infty\frac{1}{m^s}\int_0^1 e^{-2\pi\sqrt{-1}mx}\mathcal{L}_2(x,0)\mathcal{L}_3(-x,0)dx \\ &= \left(\frac{(2\pi\sqrt{-1})^2}{2!}\right)\left(\frac{(2\pi\sqrt{-1})^3}{3!}\right)\sum_{m=1}^\infty\frac{1}{m^s}\int_0^1 e^{-2\pi\sqrt{-1}mx}B_2(x)B_3(1-x)dx, \end{aligned}$$

by $\mathcal{L}_k(x,0) = B_k(x-[x])$ for $x \in \mathbb{R}$ (see (3.26)). Hence, by using (3.27) and (3.28), we obtain (3.23).

6. Some lemmas for explicit construction of functional relations

From the general form of functional relations in Theorem 5.1, it is possible to deduce explicit formulas of functional relations for zeta-functions of root systems, e.g., as in Example 5.1. However, if a rank of the root system is high, then it seems quite hard to give explicit forms directly from Theorem 5.1. Therefore now we introduce a different procedure to construct explicit functional relations. For this aim, we give some general preparatory lemmas. We first quote the following lemma from our previous paper. Let $\phi(s) := \sum_{n\geq 1}(-1)^n n^{-s} = \left(2^{1-s}-1\right)\zeta(s)$, and $\varepsilon_\nu := (1+(-1)^\nu)/2$ $(\nu \in \mathbb{Z})$.

Lemma 6.1 ([28] Lemma 2.1). *Let $f, g : \mathbb{N}_0 \to \mathbb{C}$ be arbitrary functions. Then, for $a \in \mathbb{N}$, we have*

$$\sum_{k=0}^{a} \phi(a-k)\varepsilon_{a-k} \sum_{\mu=0}^{[k/2]} f(k-2\mu)\frac{(-1)^\mu \pi^{2\mu}}{(2\mu)!} = \sum_{\xi=0}^{[a/2]} \zeta(2\xi) f(a-2\xi), \tag{6.40}$$

and

$$\sum_{k=1}^{a} \phi(a-k)\varepsilon_{a-k} \sum_{\mu=0}^{[(k-1)/2]} g(k-2\mu)\frac{(-1)^\mu \pi^{2\mu}}{(2\mu+1)!} = -\frac{1}{2}g(a). \tag{6.41}$$

Corollary 6.1. *With the same notation as in Lemma 6.1, put*

$$h(d) := \sum_{\mu=0}^{[d/2]} g(d-2\mu)\frac{(-1)^\mu \pi^{2\mu}}{(2\mu+1)!} \qquad (d \in \mathbb{N}_0).$$

Then we have

$$g(d) = -2\sum_{\mu=0}^{d} \phi(d-\mu)\varepsilon_{d-\mu} h(\mu) \qquad (d \in \mathbb{N}_0).$$

Proof. In (6.41), we replace $g(x)$ by $g(x-1)$. Then (6.41) implies that

$$\sum_{k=1}^{a} \phi(a-k)\varepsilon_{a-k} h(k-1) = -\frac{1}{2}g(a-1)$$

for $a \in \mathbb{N}$. Replacing a by $d+1$ and $k-1$ by μ, respectively, we obtain the desired assertion. □

Using Lemma 6.1, we prove the following lemma which is a key to construct functional relations. Let $h \in \mathbb{N}$, and

$$\begin{aligned}
&\mathfrak{C} := \{C(l) \in \mathbb{C} \,|\, l \in \mathbb{Z},\ l \neq 0\}, \\
&\mathfrak{D} := \{D(N;m;\eta) \in \mathbb{R} \,|\, N, m, \eta \in \mathbb{Z},\ N \neq 0,\ m \geq 0,\ 1 \leq \eta \leq h\}, \\
&\mathfrak{A} := \{a_\eta \in \mathbb{N} \,|\, 1 \leq \eta \leq h\}
\end{aligned}$$

be sets of numbers indexed by integers. We let

$$\binom{x}{k} := \begin{cases} \frac{x(x-1)\cdots(x-k+1)}{k!} & (k \in \mathbb{N}), \\ 1 & (k = 0). \end{cases}$$

Lemma 6.2. *With the above notation, we assume that the infinite series appearing in*

$$\begin{aligned}
&\sum_{\substack{N\in\mathbb{Z}\\ N\neq 0}} (-1)^N C(N) e^{iN\theta} - 2\sum_{\eta=1}^{h}\sum_{k=0}^{a_\eta} \phi(a_\eta - k)\varepsilon_{a_\eta - k} \\
&\qquad \times \sum_{\xi=0}^{k} \left\{ \sum_{\substack{N\in\mathbb{Z}\\ N\neq 0}} (-1)^N D(N; k-\xi; \eta) e^{iN\theta} \right\} \frac{(i\theta)^\xi}{\xi!} \qquad (6.42)
\end{aligned}$$

are absolutely convergent for $\theta \in [-\pi, \pi]$, and that (6.42) is a constant function for $\theta \in [-\pi, \pi]$. Then, for $d \in \mathbb{N}_0$,

$$\begin{aligned}
&\sum_{\substack{N\in\mathbb{Z}\\ N\neq 0}} \frac{(-1)^N C(N) e^{iN\theta}}{N^d} = 2\sum_{\eta=1}^{h}\sum_{k=0}^{a_\eta} \phi(a_\eta - k)\varepsilon_{a_\eta - k} \\
&\times \sum_{\xi=0}^{k} \left\{ \sum_{\omega=0}^{k-\xi} \binom{\omega+d-1}{\omega} (-1)^\omega \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \frac{(-1)^m D(m; k-\xi-\omega; \eta) e^{im\theta}}{m^{d+\omega}} \right\} \frac{(i\theta)^\xi}{\xi!} \\
&- 2\sum_{k=0}^{d} \phi(d-k)\varepsilon_{d-k} \sum_{\xi=0}^{k} \left\{ \sum_{\eta=1}^{h}\sum_{\omega=0}^{a_\eta - 1} \binom{\omega+k-\xi}{\omega} (-1)^\omega \right. \\
&\qquad \left. \times \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \frac{D(m; a_\eta - 1 - \omega; \eta)}{m^{k-\xi+\omega+1}} \right\} \frac{(i\theta)^\xi}{\xi!} \qquad (6.43)
\end{aligned}$$

holds for $\theta \in [-\pi, \pi]$, where the infinite series appearing on both sides of (6.43) are absolutely convergent for $\theta \in [-\pi, \pi]$.

Proof. For $d \in \mathbb{N}_0$, put

$$
\begin{aligned}
&G_d(\theta;\mathfrak{C};\mathfrak{D};\mathfrak{A}) \qquad (6.44)\\
&:= \frac{1}{i^d}\Bigg[\sum_{\substack{l\in\mathbb{Z}\\ l\neq 0}} \frac{(-1)^l C(l) e^{il\theta}}{l^d} - 2\sum_{\eta=1}^{h}\sum_{k=0}^{a_\eta}\phi(a_\eta - k)\varepsilon_{a_\eta-k}\\
&\quad\times\sum_{\xi=0}^{k}\Bigg\{\sum_{\nu=0}^{\xi}\binom{d-1+\xi-\nu}{\xi-\nu}(-1)^\xi\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\frac{(-1)^m D(m;k-\xi;\eta)e^{im\theta}}{m^{d+\xi-\nu}}\frac{(-i\theta)^\nu}{\nu!}\Bigg\}\Bigg]\\
&= \frac{1}{i^d}\Bigg[\sum_{\substack{l\in\mathbb{Z}\\ l\neq 0}} \frac{(-1)^l C(l) e^{il\theta}}{l^d} - 2\sum_{\eta=1}^{h}\sum_{k=0}^{a_\eta}\phi(a_\eta - k)\varepsilon_{a_\eta-k}\\
&\quad\times\sum_{\nu=0}^{k}\Bigg\{\sum_{\omega=0}^{k-\nu}\binom{d-1+\omega}{\omega}(-1)^\omega\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\frac{(-1)^m D(m;k-\nu-\omega;\eta)e^{im\theta}}{m^{d+\omega}}\Bigg\}\frac{(i\theta)^\nu}{\nu!}\Bigg].
\end{aligned}
$$

Note that the second equality of (6.44) follows by putting $\omega = \xi - \nu$. Then the assumption of (6.42) implies that

$$G_0(\theta;\mathfrak{C};\mathfrak{D};\mathfrak{A}) = R_0 \quad (\theta \in [-\pi,\pi]), \qquad (6.45)$$

where there exists a constant $R_0 = R_0(\mathfrak{C};\mathfrak{D};\mathfrak{A}) \in \mathbb{C}$, because $\binom{-1+\xi-\nu}{\xi-\nu} = 0$ if $\xi > \nu$. Furthermore we can check that $G_d(\theta;\mathfrak{C};\mathfrak{D};\mathfrak{A})$ is absolutely convergent with respect to $\theta \in [-\pi,\pi]$ and that

$$\frac{d}{d\theta}G_d(\theta;\mathfrak{C};\mathfrak{D};\mathfrak{A}) = G_{d-1}(\theta;\mathfrak{C};\mathfrak{D};\mathfrak{A}) \quad (d \in \mathbb{N}). \qquad (6.46)$$

In fact, if we differentiate the second member of (6.44) with respect to θ, then we have

$$
\begin{aligned}
&\frac{d}{d\theta}G_d(\theta;\mathfrak{C};\mathfrak{D};\mathfrak{A})\\
&= \frac{1}{i^{d-1}}\Bigg[\sum_{\substack{l\in\mathbb{Z}\\ l\neq 0}} \frac{(-1)^l C(l) e^{il\theta}}{l^{d-1}} - 2\sum_{\eta=1}^{h}\sum_{k=0}^{a_\eta}\phi(a_\eta - k)\varepsilon_{a_\eta-k}\\
&\quad\times\sum_{\xi=0}^{k}\Bigg\{\sum_{\nu=0}^{\xi}\binom{d-1+\xi-\nu}{\xi-\nu}(-1)^\xi\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\frac{(-1)^m D(m;k-\xi;\eta)e^{im\theta}}{m^{d+\xi-\nu-1}}\frac{(-i\theta)^\nu}{\nu!}\\
&\quad-\sum_{\nu=1}^{\xi}\binom{d-1+\xi-\nu}{\xi-\nu}(-1)^\xi\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\frac{(-1)^m D(m;k-\xi;\eta)e^{im\theta}}{m^{d+\xi-\nu}}\frac{(-i\theta)^{\nu-1}}{(\nu-1)!}\Bigg\}\Bigg].
\end{aligned}
$$

Replacing $\nu-1$ by μ in the last member of the above summation, and using the well-known relation

$$-\binom{m-1}{l-1}+\binom{m}{l}=\binom{m-1}{l} \quad (l,m\in\mathbb{N}),$$

we obtain (6.46).

By integrating both sides of (6.45) and multiplying by i on both sides, we have $iG_1(\theta;\mathfrak{C};\mathfrak{D};\mathfrak{A})=R_0(i\theta)+R_1$ with some constant $R_1=R_1(\mathfrak{C};\mathfrak{D};\mathfrak{A})$. Repeating this operation, and by (6.46), we obtain

$$i^d G_d(\theta;\mathfrak{C};\mathfrak{D};\mathfrak{A})=\sum_{k=0}^{d} R_{d-k}\frac{(i\theta)^k}{k!}, \tag{6.47}$$

where there exist constants $R_k=R_k(\mathfrak{C};\mathfrak{D};\mathfrak{A})$ $(0\leq k\leq d)$. We can explicitly determine $\{R_k\}$ as follows. Putting $\theta=\pm\pi$ in (6.47) with $d+1$ $(d\in\mathbb{N}_0)$, we have

$$\begin{aligned}&\frac{i^{d+1}}{2(i\pi)}\left\{G_{d+1}(\pi;\mathfrak{C};\mathfrak{D};\mathfrak{A})-G_{d+1}(-\pi;\mathfrak{C};\mathfrak{D};\mathfrak{A})\right\}\\&=\sum_{\mu=0}^{[d/2]} R_{d-2\mu}\frac{(i\pi)^{2\mu}}{(2\mu+1)!}.\end{aligned} \tag{6.48}$$

It follows from (6.44) that the left-hand side of (6.48) is equal to

$$\begin{aligned}&-2\sum_{\eta=1}^{h}\sum_{k=0}^{a_\eta}\phi(a_\eta-k)\varepsilon_{a_\eta-k} \\ &\times\sum_{\tau=0}^{[(k-1)/2]}\left\{\sum_{\omega=0}^{k-2\tau-1}\binom{d+\omega}{\omega}(-1)^\omega\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\frac{D(m;k-2\tau-1-\omega;\eta)}{m^{d+\omega+1}}\right\}\frac{(i\pi)^{2\tau}}{(2\tau+1)!}.\end{aligned} \tag{6.49}$$

Applying (6.41) with

$$g(x)=\sum_{\omega=0}^{x-1}\binom{d+\omega}{\omega}(-1)^\omega\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\frac{D(m;x-1-\omega;\eta)}{m^{d+\omega+1}},$$

we can rewrite (6.48) as

$$\begin{aligned}&\sum_{\eta=1}^{h}\sum_{\omega=0}^{a_\eta-1}\binom{d+\omega}{\omega}(-1)^\omega\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\frac{D(m;a_\eta-1-\omega;\eta)}{m^{d+\omega+1}}\\&=\sum_{\nu=0}^{[d/2]} R_{d-2\nu}\frac{(i\pi)^{2\nu}}{(2\nu+1)!}.\end{aligned} \tag{6.50}$$

Hence, by Corollary 6.1, we have

$$\begin{aligned} R_d &= R_d(\mathfrak{C};\mathfrak{D};\mathfrak{A}) \qquad (6.51)\\ &= -2\sum_{\nu=0}^{d}\phi(d-\nu)\varepsilon_{d-\nu}\sum_{\eta=1}^{h}\sum_{\omega=0}^{a_\eta-1}\binom{\nu+\omega}{\omega}(-1)^\omega\sum_{\substack{m\in\mathbb{Z}\\m\neq0}}\frac{D(m;a_\eta-1-\omega;\eta)}{m^{\nu+\omega+1}}\end{aligned}$$

for $d\in\mathbb{N}_0$. Therefore, combining (6.44),(6.47) and (6.51), we have

$$\begin{aligned} &\sum_{\substack{l\in\mathbb{Z}\\l\neq0}}\frac{(-1)^l C(l)e^{il\theta}}{l^d} - 2\sum_{\eta=1}^{h}\sum_{k=0}^{a_\eta}\phi(a_\eta-k)\varepsilon_{a_\eta-k} \qquad (6.52)\\ &\quad\times\sum_{\nu=0}^{k}\left\{\sum_{\omega=0}^{k-\nu}\binom{d-1+\omega}{\omega}(-1)^\omega\sum_{\substack{m\in\mathbb{Z}\\m\neq0}}\frac{(-1)^m D(m;k-\nu-\omega;\eta)e^{im\theta}}{m^{d+\omega}}\right\}\frac{(i\theta)^\nu}{\nu!}\\ &= -2\sum_{\mu=0}^{d}\sum_{\nu=0}^{d-\mu}\phi(d-\mu-\nu)\varepsilon_{d-\mu-\nu}\\ &\quad\times\sum_{\eta=1}^{h}\sum_{\omega=0}^{a_\eta-1}\binom{\nu+\omega}{\omega}(-1)^\omega\sum_{\substack{m\in\mathbb{Z}\\m\neq0}}\frac{D(m;a_\eta-1-\omega;\eta)}{m^{\nu+\omega+1}}\frac{(i\theta)^\mu}{\mu!}.\end{aligned}$$

Changing the running indices (μ,ν) into (k,ξ) with $k=\mu+\nu$ and $\xi=\mu\leq k$, we find that the right-hand side of (6.52) is equal to the second term on the right-hand side of (6.43). □

7. Functional relations for $\zeta_3(\mathbf{s};A_3)$

In the rest of this paper, we will give explicit forms of functional relations for zeta-functions of root systems by using lemmas proved in Section 6. In this section, we consider the case of A_r type.

Fix $p\in\mathbb{N}$ and $s\in\mathbb{R}$ with $s>1$ and $x\in\mathbb{C}$ with $|x|=1$. From (4.31), we have

$$\left(\sum_{\substack{l\in\mathbb{Z}\\l\neq0}}\frac{(-1)^l e^{il\theta}}{l^{2p}} - 2\sum_{j=0}^{p}\phi(2p-2j)\frac{(i\theta)^{2j}}{(2j)!}\right)\sum_{m=1}^{\infty}\frac{(-1)^m x^m e^{im\theta}}{m^s}=0 \qquad (7.53)$$

for $\theta \in [-\pi, \pi]$. Hence we have

$$\sum_{\substack{l\in\mathbb{Z},\ l\neq 0\\ m\geq 1\\ l+m\neq 0}} \frac{(-1)^{l+m}x^m e^{i(l+m)\theta}}{l^{2p}m^s} - 2\sum_{j=0}^{p} \phi(2p-2j)\left\{\sum_{m=1}^{\infty}\frac{(-1)^m x^m e^{im\theta}}{m^s}\right\}\frac{(-1)^j\theta^{2j}}{(2j)!} = -\sum_{m=1}^{\infty}\frac{x^m}{m^{s+2p}} \tag{7.54}$$

for $\theta \in [-\pi, \pi]$. Now we use Lemma 6.2 with $h = 1$, $a_1 = 2p$,

$$C(N) = \sum_{\substack{l\neq 0,\ m\geq 1\\ l+m=N}} \frac{x^m}{l^{2p}m^s} \quad (N \in \mathbb{Z},\ N \neq 0),$$

and $D(N;\mu;1) = x^N N^{-s}$ (if $\mu = 0$ and $N \geq 1$), or $= 0$ (otherwise). Under these choices, we see that the left-hand side of (7.54) is of the form (6.42) because $\varepsilon_{2p-k} = 1$ $(0 \leq k \leq 2p)$ implies $k = 2j$ $(0 \leq j \leq p)$. Furthermore the right-hand side of (7.54) is a constant, because we fix s, x and p. Therefore we can apply Lemma 6.2 with $d = 2q$ for $q \in \mathbb{N}$. Then (6.43) gives that

$$\begin{aligned} 0 = &\sum_{\substack{l\neq 0,\ m\geq 1\\ l+m\neq 0}} \frac{(-1)^{l+m}x^m e^{i(l+m)\theta}}{l^{2p}m^s(l+m)^{2q}} \\ &- 2\sum_{j=0}^{p}\phi(2p-2j)\sum_{\xi=0}^{2j}\binom{2q-1+2j-\xi}{2q-1}(-1)^{2j-\xi} \\ &\quad\times\sum_{m=1}^{\infty}\frac{(-1)^m x^m e^{im\theta}}{m^{s+2q+2j-\xi}}\frac{(i\theta)^\xi}{\xi!} \\ &+ 2\sum_{j=0}^{q}\phi(2q-2j)\sum_{\xi=0}^{2j}\binom{2p-1+2j-\xi}{2p-1}(-1)^{2p-1} \\ &\quad\times\sum_{m=1}^{\infty}\frac{x^m}{m^{s+2p+2j-\xi}}\frac{(i\theta)^\xi}{\xi!} = 0 \end{aligned} \tag{7.55}$$

for $\theta \in [-\pi, \pi]$, where we replace k by $2j$ in (6.43) because $(a_1, d) = (2p, 2q)$ as mentioned above. This relation will play an important role in the next section. Here we apply Lemma 6.1 to the real part of (7.55) in the case $\theta = \pi$ and $x = 1$. Then we have the following.

Proposition 7.1. *For* $p, q \in \mathbb{N}$,

$$\sum_{\substack{l\in\mathbb{Z},\ l\neq 0\\ m\geq 1\\ l+m\neq 0}} \frac{1}{l^{2p}m^s(l+m)^{2q}} = 2\sum_{\nu=0}^{p}\binom{2p+2q-2\nu-1}{2q-1}\zeta(2\nu)\zeta(s+2p+2q-2\nu) + 2\sum_{\nu=0}^{q}\binom{2p+2q-2\nu-1}{2p-1}\zeta(2\nu)\zeta(s+2p+2q-2\nu) \tag{7.56}$$

holds for $s \in \mathbb{C}$ *except for singularities of functions on both sides.*

Note that (7.56) essentially coincides with (3.21) in the case $(k,l) = (2p, 2q)$, because the left-hand side of (7.56) can be easily transformed to that of (3.21) in the case $(k,l) = (2p, 2q)$. This implies that, from relation (7.53) which is given by multiplying two quantities of A_1 type, we can obtain relation (7.56) for zeta-functions of A_2 and A_1 type. From the view point of Dynkin diagrams, we may say that (7.53) corresponds to two vertices, and the above procedure of applying Lemma 6.2 to obtain (7.56) corresponds to the fact that the Dynkin diagram of A_2 can be produced by joining those two vertices. Based on this observation, instead of (7.53), we next combine a quantity of A_2 type and a quantity of A_1 type to get a relation for zeta-functions of A_3 and of A_2 type. From (1.5), we see that the zeta-function of root system of A_3 type is defined by

$$\zeta_3(s_1, s_2, s_3, s_4, s_5, s_6; A_3) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{l^{s_1}m^{s_2}n^{s_3}(l+m)^{s_4}(m+n)^{s_5}(l+m+n)^{s_6}}. \tag{7.57}$$

Fix $p, q, b \in \mathbb{N}$ and $s \in \mathbb{R}$ with $s > 1$ and $x \in \mathbb{C}$ with $|x| = 1$. From (4.31), we have

$$\left(\sum_{\substack{l\in\mathbb{Z}\\ l\neq 0}}\frac{(-1)^l e^{il\theta}}{l^{2p}} - 2\sum_{j=0}^{p}\ \phi(2p-2j)\ \frac{(i\theta)^{2j}}{(2j)!}\right) \times \sum_{\substack{m\in\mathbb{Z},\ m\neq 0\\ n\geq 1\\ m+n\neq 0}}\frac{(-1)^{m+n}x^n e^{i(m+n)\theta}}{m^{2q}n^s(m+n)^{2b}} = 0 \tag{7.58}$$

for $\theta \in [-\pi, \pi]$. This formula corresponds to a diagram of A_2 and another vertex. Next we use Lemma 6.2, which gives the procedure of joining these two figures to obtain the diagram of A_3. First, by separating the terms corresponding to $l+m+n=0$, we have

$$\begin{aligned}
&\sum_{\substack{l,m\neq 0,\ n\geq 1\\ m+n\neq 0,\ l+m+n\neq 0}} \frac{(-1)^{l+m+n}x^n e^{i(l+m+n)\theta}}{l^{2p}m^{2q}n^s(m+n)^{2b}}\\
&\quad -2\sum_{j=0}^{p}\phi(2p-2j)\left\{\sum_{\substack{m\neq 0\\ n\geq 1\\ m+n\neq 0}}\frac{(-1)^{m+n}x^n e^{i(m+n)\theta}}{m^{2q}n^s(m+n)^{2b}}\right\}\frac{(-1)^j\theta^{2j}}{(2j)!}\\
&\quad =-\sum_{\substack{m\neq 0\\ n\geq 1\\ m+n\neq 0}}\frac{x^n}{m^{2q}n^s(m+n)^{2p+2b}} \qquad (7.59)
\end{aligned}$$

for $\theta\in[-\pi,\pi]$. We can apply Lemma 6.2 with $h=1$, $a_1=2p$,

$$C(N)=\sum_{\substack{l,m\neq 0\\ n\geq 1\\ m+n\neq 0\\ l+m+n=N}}\frac{x^n}{l^{2p}m^{2q}n^s(m+n)^{2b}},$$

$$D(N;\mu;1)=\begin{cases}\displaystyle\sum_{\substack{m\neq 0\\ n\geq 1\\ m+n=N}}\frac{x^n}{m^{2q}n^s(m+n)^{2b}} & (\nu=0,\ N\neq 0),\\ 0 & (\nu\geq 1,\ N\neq 0),\end{cases}$$

and $d=2c$ ($c\in\mathbb{N}$). Formula (7.59) implies that the assumptions of Lemma 6.2 are satisfied, so consequently we have

$$\begin{aligned}
&\sum_{\substack{l,m\neq 0,\ n\geq 1\\ m+n\neq 0,\ l+m+n\neq 0}} \frac{(-1)^{l+m+n}x^n e^{i(l+m+n)\theta}}{l^{2p}m^{2q}n^s(m+n)^{2b}(l+m+n)^{2c}}\\
&=2\sum_{j=0}^{p}\phi(2p-2j)\sum_{\xi=0}^{2j}\binom{2j-\xi+2c-1}{2c-1}(-1)^{2j-\xi}\\
&\quad\times\sum_{\substack{m\neq 0\\ n\geq 1\\ m+n\neq 0}}\frac{(-1)^{m+n}x^n e^{i(m+n)\theta}}{m^{2q}n^s(m+n)^{2b+2c+2j-\xi}}\frac{(i\theta)^{\xi}}{\xi!}\\
&-2\sum_{j=0}^{c}\phi(2c-2j)\sum_{\xi=0}^{2j}\binom{2j-\xi+2p-1}{2p-1}(-1)^{2p-1}\\
&\quad\times\sum_{\substack{m\neq 0\\ n\geq 1\\ m+n\neq 0}}\frac{x^n}{m^{2q}n^s(m+n)^{2p+2b+2j-\xi}}\frac{(i\theta)^{\xi}}{\xi!}.
\end{aligned}$$

Putting $x = -e^{-i\theta}$ ($\theta \in \mathbb{R}$) on both sides and separating the terms corresponding to $l + m = 0$, we have

$$\begin{aligned}
&\sum_{\substack{l,m\neq 0,\ n\geq 1\\ l+m\neq 0,\ m+n\neq 0\\ l+m+n\neq 0}} \frac{(-1)^{l+m} e^{i(l+m)\theta}}{l^{2p} m^{2q} n^s (m+n)^{2b} (l+m+n)^{2c}} \\
&- 2\sum_{j=0}^{p} \phi(2p-2j) \sum_{\xi=0}^{2j} \binom{2j-\xi+2c-1}{2c-1} (-1)^{2j-\xi} \\
&\quad \times \sum_{\substack{m\neq 0\\ n\geq 1\\ m+n\neq 0}} \frac{(-1)^m e^{im\theta}}{m^{2q} n^s (m+n)^{2b+2c+2j-\xi}} \frac{(i\theta)^\xi}{\xi!} \\
&+ 2\sum_{j=0}^{c} \phi(2c-2j) \sum_{\xi=0}^{2j} \binom{2j-\xi+2p-1}{2p-1} (-1)^{2p-1} \\
&\quad \times \sum_{\substack{m\neq 0\\ n\geq 1\\ m+n\neq 0}} \frac{(-1)^n e^{-in\theta}}{m^{2q} n^s (m+n)^{2p+2b+2j-\xi}} \frac{(i\theta)^\xi}{\xi!} \\
&= -\sum_{\substack{m\neq 0\\ n\geq 1\\ m+n\neq 0}} \frac{1}{m^{2p+2q} n^{s+2c} (m+n)^{2b}}.
\end{aligned}$$

Again we apply Lemma 6.2 with $h = 2$, $a_1 = 2p$, $a_2 = 2c$ and $d = 2a$ for $a \in \mathbb{N}$. Then

$$\begin{aligned}
&\sum_{\substack{l,m\neq 0,\ n\geq 1\\ l+m\neq 0,\ m+n\neq 0\\ l+m+n\neq 0}} \frac{(-1)^{l+m} e^{i(l+m)\theta}}{l^{2p} m^{2q} n^s (l+m)^{2a} (m+n)^{2b} (l+m+n)^{2c}} \\
&= 2\sum_{j=0}^{p} \phi(2p-2j) \sum_{\xi=0}^{2j} \sum_{\omega=0}^{2j-\xi} \binom{\omega+2a-1}{\omega} (-1)^\omega \binom{2j-\xi-\omega+2c-1}{2c-1} \\
&\quad \times (-1)^{2j-\xi-\omega} \sum_{\substack{m\neq 0\\ n\geq 1\\ m+n\neq 0}} \frac{(-1)^m e^{im\theta}}{m^{2q+2a+\omega} n^s (m+n)^{2b+2c+2j-\xi-\omega}} \frac{(i\theta)^\xi}{\xi!} \\
&\quad - 2\sum_{j=0}^{c} \phi(2c-2j) \sum_{\xi=0}^{2j} \sum_{\omega=0}^{2j-\xi} \binom{\omega+2a-1}{\omega} (-1)^\omega \binom{2j-\xi-\omega+2p-1}{2p-1} \\
&\quad \times (-1)^{2p-1} (-1)^{2a+\omega} \sum_{\substack{m\neq 0\\ n\geq 1\\ m+n\neq 0}} \frac{(-1)^n e^{-in\theta}}{m^{2q} n^{s+2a+\omega} (m+n)^{2p+2b+2j-\xi-\omega}} \frac{(i\theta)^\xi}{\xi!}
\end{aligned}$$

$$
\begin{aligned}
&-2\sum_{j=0}^{a}\phi(2a-2j)\sum_{\xi=0}^{2j}\sum_{\omega=0}^{2p-1}\binom{\omega+2j-\xi}{\omega}(-1)^{\omega}\binom{2p+2c-2-\omega}{2c-1}\\
&\times(-1)^{2p-1-\omega}\sum_{\substack{m\neq 0\\ n\geq 1\\ m+n\neq 0}}\frac{1}{m^{2q+2j-\xi+\omega+1}n^{s}(m+n)^{2p+2b+2c-1-\omega}}\frac{(i\theta)^{\xi}}{\xi!}\\
&+2\sum_{j=0}^{a}\phi(2a-2j)\sum_{\xi=0}^{2j}\sum_{\omega=0}^{2c-1}\binom{\omega+2j-\xi}{\omega}(-1)^{\omega}\binom{2p+2c-2-\omega}{2p-1}\\
&\times(-1)^{2p-1}(-1)^{2j-\xi+\omega+1}\sum_{\substack{m\neq 0\\ n\geq 1\\ m+n\neq 0}}\frac{1}{m^{2q}n^{s+2j-\xi+\omega+1}(m+n)^{2p+2b+2c-1-\omega}}\frac{(i\theta)^{\xi}}{\xi!}
\end{aligned}
$$

holds for $\theta\in[-\pi,\pi]$. Now we put $\theta=\pi$ in this equation and take its real part. For simplicity, we denote the obtained equation by $J_1=J_2+J_3+J_4+J_5$. First we consider J_1. This can be divided into the following:

$$
\sum_{\substack{l\geq 1\\ m\geq 1\\ n\geq 1}}+\sum_{\substack{l\leq -1\\ m\geq 1\\ n\geq 1\\ l+m\neq 0\\ l+m+n\neq 0}}+\sum_{\substack{l\geq 1\\ m\leq -1\\ n\geq 1\\ l+m\neq 0\\ m+n\neq 0\\ l+m+n\neq 0}}+\sum_{\substack{l\leq -1\\ m\leq -1\\ n\geq 1\\ m+n\neq 0\\ l+m+n\neq 0}},
$$

which we denote by $J_{11}+J_{12}+J_{13}+J_{14}$. We can immediately see that $J_{11}=\zeta_3(2p,2q,s,2a,2b,2c;A_3)$. For J_{12}, replacing l by $-l$, we have

$$
J_{12}=\sum_{\substack{l\geq 1,\ m\geq 1\\ n\geq 1,\ l\neq m\\ l\neq m+n}}\frac{1}{(-l)^{2p}m^{2q}n^{s}(-l+m)^{2a}(m+n)^{2b}(-l+m+n)^{2c}}.
$$

Here, putting $j=-l+m$ if $l<m$ and $k=l-m$ if $l>m$, respectively, we have

$$
\begin{aligned}
J_{12}=&\sum_{\substack{l\geq 1,\ j\geq 1\\ n\geq 1}}\frac{1}{l^{2p}(l+j)^{2q}n^{s}j^{2a}(l+j+n)^{2b}(j+n)^{2c}}\\
&+\sum_{\substack{k\geq 1,\ m\geq 1\\ n\geq 1,\ k\neq n}}\frac{1}{(k+m)^{2p}m^{2q}n^{s}k^{2a}(m+n)^{2b}(-k+n)^{2c}},
\end{aligned}
$$

where the first term on the right-hand side is $\zeta_3(2p,2a,s,2q,2c,2b;A_3)$. Furthermore, putting $j'=-k+n$ if $k<n$ and $k'=k-n$ if $k>n$, respectively, in the second term on the right-hand side, we can obtain

$$
\begin{aligned}
J_{12}=&\zeta_3(2p,2a,s,2q,2c,2b;A_3)+\zeta_3(2q,2a,2c,2p,s,2b;A_3)\\
&+\zeta_3(2q,s,2c,2b,2a,2p;A_3).
\end{aligned}
$$

Similarly we can express J_{13} and J_{14} as sums of values of the zeta-function of A_3 type. Therefore J_1 can be transformed to the left-hand side of the following theorem. On the other hand, if we apply (6.40) to $J_2+J_3+J_4+J_5$, then it can be transformed to the right-hand side of the following theorem with

$$\mathcal{T}(2d,s,2e)=\sum_{\substack{m\neq 0,\ n\geq 1\\ m+n\neq 0}}\frac{1}{m^{2d}n^s(m+n)^{2e}}$$

for $d,e\in\mathbb{N}$. From Proposition 7.1, we see that $\mathcal{T}(2d,s,2e)$ can be written as (7.60) below.

Theorem 7.1. *For $p,q,a,b,c\in\mathbb{N}$,*

$$\begin{aligned}
&\zeta_3(2p,2q,s,2a,2b,2c;A_3)+\zeta_3(2p,2a,s,2q,2c,2b;A_3)\\
&+\zeta_3(2q,2a,2c,2p,s,2b;A_3)+\zeta_3(2q,s,2c,2b,2a,2p;A_3)\\
&+\zeta_3(2a,2p,2b,2q,2c,s;A_3)+\zeta_3(2a,2c,2b,s,2p,2q;A_3)\\
&+\zeta_3(s,2c,2p,2a,2b,2q;A_3)+\zeta_3(2b,2q,2a,s,2p,2c;A_3)\\
&+\zeta_3(2b,s,2a,2q,2c,2p;A_3)+\zeta_3(2p,2b,s,2c,2q,2a;A_3)\\
&+\zeta_3(2c,2p,2q,2b,2a,s;A_3)+\zeta_3(2c,2b,2q,2p,s,2a;A_3)\\
&=2\sum_{\xi=0}^{p}\zeta(2\xi)\sum_{\omega=0}^{2p-2\xi}\binom{\omega+2a-1}{\omega}\binom{2p+2c-2\xi-\omega-1}{2c-1}\\
&\quad\times\mathcal{T}(2q+2a+\omega,s,2p+2b+2c-2\xi-\omega)\\
&\quad+2\sum_{\xi=0}^{c}\zeta(2\xi)\sum_{\omega=0}^{2c-2\xi}\binom{\omega+2a-1}{\omega}\binom{2p+2c-2\xi-\omega-1}{2p-1}\\
&\quad\times\mathcal{T}(2q,s+2a+\omega,2p+2b+2c-2\xi-\omega)\\
&\quad+2\sum_{\xi=0}^{a}\zeta(2\xi)\sum_{\omega=0}^{2p-1}\binom{\omega+2a-2\xi}{\omega}\binom{2p+2c-2-\omega}{2c-1}\\
&\quad\times\mathcal{T}(2q+2a-2\xi+\omega+1,s,2p+2b+2c-1-\omega)\\
&\quad+2\sum_{\xi=0}^{a}\zeta(2\xi)\sum_{\omega=0}^{2c-1}\binom{\omega+2a-2\xi}{\omega}\binom{2p+2c-\omega-2}{2p-1}\\
&\quad\times\mathcal{T}(2q,s+2a-2\xi+\omega+1,2p+2b+2c-\omega-1)
\end{aligned}$$

holds for $s \in \mathbb{C}$ except for singularities of functions on both sides, where

$$\begin{aligned}\mathcal{T}(2d,s,2e) = 2\sum_{\nu=0}^{d}\binom{2d+2e-2\nu-1}{2e-1}\zeta(2\nu)\zeta(s+2d+2e-2\nu)\\ +2\sum_{\nu=0}^{e}\binom{2d+2e-2\nu-1}{2d-1}\zeta(2\nu)\zeta(s+2d+2e-2\nu).\end{aligned} \tag{7.60}$$

Example 7.1. In the case when $(p,q,a,b,c) = (k,k,k,k,k)$ and $s = 2k$ for $k \in \mathbb{N}$ in Theorem 7.1, we recover the explicit expression for Witten's volume formula of A_3 type, which has been proved by Gunnells and Sczech ([10, Proposition 8.5]). For example, in the case when $(p,q,a,b,c) = (1,1,1,1,1)$, we obtain

$$\begin{aligned}&4\zeta_3(2,2,s,2,2,2;A_3) + 2\zeta_3(2,s,2,2,2,2;A_3)\\ &\quad + 4\zeta_3(2,2,2,s,2,2;A_3) + 2\zeta_3(2,2,2,2,2,s;A_3)\\ &= 678\zeta(s+10) - 512\zeta(2)\zeta(s+8)\\ &\quad + 148\zeta(4)\zeta(s+6) + 4\zeta(6)\zeta(s+4),\end{aligned} \tag{7.61}$$

because $\zeta_3(s_1,s_2,s_3,s_4,s_5,s_6;A_3) = \zeta_3(s_3,s_2,s_1,s_5,s_4,s_6;A_3)$. In particular when $s = 2$, we obtain the explicit value of $C_W(2,A_3)$, that is,

$$\zeta_3(2,2,2,2,2,2;A_3) = \frac{23}{2554051500}\pi^{12}. \tag{7.62}$$

In our previous work [16, Theorem 3.4], we already obtained the functional relation between $\zeta_3(\mathbf{s};A_3)$ and $\zeta_2(\mathbf{s};A_2)$, and checked that the functional relation implicitly implies (7.62), by using the properties of $\zeta_2(\mathbf{s};A_2)$. On the other hand, we can see that the above formula in Theorem 7.1 itself includes the explicit form of Witten's volume formula of A_3 type.

Example 7.2. By the same method as above, we can obtain the following formulas ([30]):

$$\begin{aligned}\zeta_3(1,1,1,2,1,2;A_3) &= -\frac{29}{175}\zeta(2)^4 + \zeta(3)\zeta(5) - \frac{1}{2}\zeta(6,2),\\ \zeta_3(1,1,2,1,2,1;A_3) &= \frac{2683}{1050}\zeta(2)^4 + \frac{1}{2}\zeta(2)\zeta(3)^2 - 16\zeta(3)\zeta(5) + \frac{29}{4}\zeta(6,2),\\ \zeta_3(1,1,1,2,1,3;A_3) &= \frac{2}{5}\zeta(2)^2\zeta(5) + 10\zeta(2)\zeta(7) - \frac{53}{3}\zeta(9).\end{aligned}$$

Remark 7.1. Here we summarize the method developed in this section. The starting point is the simple identity (4.31) (and (4.32)), which is based on the fact $\zeta(-2n) = 0$ ($n \in \mathbb{N}$). One basic idea is to multiply (4.31) by an infinite series (see (7.53))) to obtain a new identity (see (7.54)). Then we apply the argument of repeated integration, embodied in Lemma 6.2, to deduce the functional relations. This procedure is the essence of the "u-method" mentioned in Sections 2 and 3, though the parameter $u > 1$ does not appear in this section.

However, the original u-method (developed, for instance, in [30]) is unsatisfactory because it only produces functional relations in which some of the variables should be equal to 0. In order to remove this restriction, we introduce the idea of considering the infinite series of polylogarithm type (that is, with an additional parameter x in the numerators). This idea, inspired by the method in [32] (see Remark 3.3), was first successfully used in [16] under the name of the "polylogarithm technique". This additional flexibility enables us to deduce more general type of functional relations such as Theorem 7.1. We will also use this technique in the following sections.

We may proceed further. Next we combine a quantity of A_3 type and a quantity of A_1 type to obtain

$$\left(\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}}\frac{(-1)^k e^{ik\theta}}{k^{2p}}-2\sum_{j=0}^{p}\phi(2p-2j)\,\frac{(i\theta)^{2j}}{(2j)!}\right)\times\sum_{\substack{l,m\in\mathbb{Z},\,n\geq 1,\\ l,m\neq 0,\,l+m\neq 0\\ m+n\neq 0,\,l+m+n\neq 0}}\frac{(-1)^{l+m+n}x^m y^n e^{i(l+m+n)\theta}}{l^{2q}m^{2r}n^s(l+m)^{2a}(m+n)^{2b}(l+m+n)^{2c}}=0$$

for $p,q,r,a,b,c\in\mathbb{N}$ and $x,y\in\mathbb{C}$ with $|x|=1$ and $|y|=1$. Again, by using Lemma 6.2 repeatedly, we will be able to obtain the functional relation for zeta-functions of A_4 and A_3 type. Then, by using the result in Theorem 7.1, we will be able to obtain functional relations for zeta-functions of A_4 and A_1 type, which include explicit forms of Witten's volume formulas of A_4 type, for example,

$$\zeta_4(2,2,2,2,2,2,2,2,2,2;A_4)=\frac{1}{650970015609375}\pi^{20}. \tag{7.63}$$

By continuing this procedure inductively, it seems to be possible to obtain functional relations which include explicit forms of Witten's volume formulas of A_r type for any $r\in\mathbb{N}$.

8. Functional relations for $\zeta_2(\mathbf{s}; C_2)$

In this section, we study

$$\zeta_2(s_1, s_2, s_3, s_4; C_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3} (m+2n)^{s_4}} \tag{8.64}$$

(see [18, (6.1)], also [19, Example 7.3]). As noted in [18, Section 2], we know that

$$\zeta_2(s_1, s_2, s_3, s_4; B_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3} (2m+n)^{s_4}} \tag{8.65}$$

(see [18, (2.11)]), which coincides with $\zeta_2(s_2, s_1, s_3, s_4; C_2)$. This fact is the natural consequence of the isomorphism $B_2 \simeq C_2$.

Here we consider $\zeta_2(\mathbf{s}; C_2)$ and construct explicit functional relations which include explicit forms of Witten's volume formulas of C_2 type.

As we mentioned in the previous section, the procedure of producing a functional relation for $\zeta_2(\mathbf{s}; A_2)$ corresponds to the fact that the Dynkin diagram of A_2 can be produced by adding one edge which joins two vertices. From this viewpoint, we should step on the procedure corresponding to adding another edge to the Dynkin diagram of A_2 to obtain the diagram of C_2, by using Lemma 6.2.

Replacing x by $-xe^{i\theta}$ on the left-hand side of (7.55), we have

$$\begin{aligned}
&\sum_{\substack{l\in\mathbb{Z},\, l\neq 0\\ m\geq 1,\, l+m\neq 0}} \frac{(-1)^l x^m e^{i(l+2m)\theta}}{l^{2p} m^s (l+m)^{2q}} \\
&\quad - 2\sum_{j=0}^{p} \phi(2p-2j) \sum_{\xi=0}^{2j} \binom{2q-1+2j-\xi}{2q-1} (-1)^{2j-\xi} \sum_{m=1}^{\infty} \frac{x^m e^{2im\theta}}{m^{s+2q+2j-\xi}} \frac{(i\theta)^{\xi}}{\xi!} \\
&\quad - 2\sum_{j=0}^{q} \phi(2q-2j) \sum_{\xi=0}^{2j} \binom{2p-1+2j-\xi}{2p-1} \sum_{m=1}^{\infty} \frac{(-1)^m x^m e^{im\theta}}{m^{s+2p+2j-\xi}} \frac{(i\theta)^{\xi}}{\xi!} = 0
\end{aligned}$$

for $\theta \in [-\pi, \pi]$. From the first sum, we separate the terms corresponding to the condition $l + 2m = 0$ and move them to the right-hand side. Then, as well as in the case of (7.54), applying Lemma 6.2 with $(h, a_1, a_2, d) =$

$(2, 2p, 2q, 2r)$ for $r \in \mathbb{N}$, we obtain

$$\sum_{\substack{l\in\mathbb{Z},\, l\neq 0\\ m\geq 1,\ l+m\neq 0\\ l+2m\neq 0}} \frac{(-1)^l x^m e^{i(l+2m)\theta}}{l^{2p} m^s (l+m)^{2q} (l+2m)^{2r}}$$

$$- 2\sum_{j=0}^{p} \phi(2p-2j) \sum_{\xi=0}^{2j}\sum_{\omega=0}^{2j-\xi} \binom{\omega+2r-1}{\omega}(-1)^{\omega}$$

$$\times \binom{2q-1+2j-\xi-\omega}{2q-1}(-1)^{2j-\xi-\omega}\frac{1}{2^{2r+\omega}}\sum_{m=1}^{\infty}\frac{x^m e^{2im\theta}}{m^{s+2q+2j-\xi+2r}}\frac{(i\theta)^{\xi}}{\xi!}$$

$$- 2\sum_{j=0}^{q} \phi(2q-2j) \sum_{\xi=0}^{2j}\sum_{\omega=0}^{2j-\xi} \binom{\omega+2r-1}{\omega}(-1)^{\omega}$$

$$\times \binom{2p-1+2j-\xi-\omega}{2p-1}\sum_{m=1}^{\infty}\frac{(-1)^m x^m e^{im\theta}}{m^{s+2p+2j-\xi+2r}}\frac{(i\theta)^{\xi}}{\xi!}$$

$$+ 2\sum_{j=0}^{r} \phi(2r-2j) \sum_{\xi=0}^{2j}\sum_{\omega=0}^{2p-1} \binom{\omega+2j-\xi}{\omega}(-1)^{\omega}$$

$$\times \binom{2p+2q-2-\omega}{2q-1}(-1)^{2p-1-\omega}\frac{1}{2^{2j-\xi+\omega+1}}\sum_{m=1}^{\infty}\frac{x^m}{m^{s+2q+2j-\xi+2p}}\frac{(i\theta)^{\xi}}{\xi!}$$

$$+ 2\sum_{j=0}^{r} \phi(2r-2j) \sum_{\xi=0}^{2j}\sum_{\omega=0}^{2q-1} \binom{\omega+2j-\xi}{\omega}(-1)^{\omega}$$

$$\times \binom{2p+2q-2-\omega}{2p-1}\sum_{m=1}^{\infty}\frac{x^m}{m^{s+2p+2j-\xi+2q}}\frac{(i\theta)^{\xi}}{\xi!} = 0 \qquad (8.66)$$

for $\theta \in [-\pi, \pi]$. Then, putting $(x, \theta) = (1, \pi)$ in (8.66) and applying Lemma 6.1 to the real part of this equation, we obtain the following relation which holds for $s > 1$, and furthermore for $s \in \mathbb{C}$ except for singularities by the meromorphic continuation of $\zeta_2(\mathbf{s}; C_2)$.

Theorem 8.1. *For $p, q, r \in \mathbb{N}$,*

$$\zeta_2(2p, s, 2q, 2r; C_2) + \zeta_2(2p, 2q, s, 2r; C_2)$$
$$+ \zeta_2(2r, 2q, s, 2p; C_2) + \zeta_2(2r, s, 2q, 2p; C_2)$$

$$
\begin{aligned}
&= 2\sum_{\nu=0}^{p}\zeta(2\nu)\zeta(2p+2q+2r-2\nu+s) \\
&\quad\times\sum_{\mu=0}^{2p-2\nu}\frac{1}{2^{2r+\mu}}\binom{2p+2q-2\nu-\mu-1}{2q-1}\binom{2r-1+\mu}{2r-1} \\
&\quad+2\sum_{\nu=0}^{q}\zeta(2\nu)\zeta(2p+2q+2r-2\nu+s) \\
&\quad\times\sum_{\mu=0}^{2q-2\nu}(-1)^{\mu}\binom{2p+2q-2\nu-\mu-1}{2p-1}\binom{2r-1+\mu}{2r-1} \\
&\quad+2\sum_{\nu=0}^{r}\zeta(2\nu)\zeta(2p+2q+2r-2\nu+s) \\
&\quad\times\sum_{\mu=0}^{2p-1}\frac{1}{2^{2r-2\nu+\mu+1}}\binom{2p+2q-\mu-2}{2q-1}\binom{2r-2\nu+\mu}{2r-2\nu} \\
&\quad+2\sum_{\nu=0}^{r}\zeta(2\nu)\zeta(2p+2q+2r-2\nu+s) \\
&\quad\times\sum_{\mu=0}^{2q-1}(-1)^{\mu+1}\binom{2p+2q-\mu-2}{2p-1}\binom{2r-2\nu+\mu}{2r-2\nu}
\end{aligned}
$$

holds for all $s\in\mathbb{C}$ except for singularities of functions on both sides. Note that singularities of $\zeta_2(\mathbf{s},C_2)$ have been determined in [18, Theorem 6.2].

Example 8.1. Putting $(p,q,r)=(1,1,1)$ in Theorem 8.1, we can obtain

$$\zeta_2(2,s,2,2;C_2)+\zeta_2(2,2,s,2;C_2)=-\frac{39}{16}\zeta(s+6)+\frac{3}{2}\zeta(2)\zeta(s+4). \quad (8.67)$$

In particular when $s=2$, we obtain

$$\zeta_2(2,2,2,2;C_2)=\frac{\pi^8}{302400}, \quad (8.68)$$

which have already been obtained in [19, (7.24)]. It should be noted that Equations (8.67) and (8.70) mentioned below coincide with Equations (2.6) and (2.7) in [16], respectively. Note that, in [16], we used the notation $\zeta_2(s_1,s_2,s_3,s_4;B_2)$ defined by

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(2m+n)^{s_1}n^{s_2}m^{s_3}(m+n)^{s_4}},$$

different from (8.65) (see [18, Section 2]).

Remark 8.1. Using the same method as in the proof of Theorem 8.1, we can prove that

$$\begin{aligned}
&\zeta_2(p,s,q,r;C_2)+(-1)^p\zeta_2(p,q,s,r;C_2)+(-1)^{p+q}\zeta_2(r,q,s,p;C_2)\\
&\quad+(-1)^{p+q+r}\zeta_2(r,s,q,p;C_2)\\
&\quad=2(-1)^p\Bigg\{\sum_{\nu=0}^{[p/2]}\zeta(2\nu)\zeta(p+q+r-2\nu+s)\\
&\qquad\times\sum_{\mu=0}^{p-2\nu}\frac{1}{2^{r+\mu}}\binom{p+q-2\nu-\mu-1}{q-1}\binom{r-1+\mu}{r-1}\\
&\quad+\sum_{\nu=0}^{[q/2]}\zeta(2\nu)\zeta(p+q+r-2\nu+s)\\
&\qquad\times\sum_{\mu=0}^{q-2\nu}(-1)^{\mu}\binom{p+q-2\nu-\mu-1}{p-1}\binom{r-1+\mu}{r-1}\\
&\quad+\sum_{\nu=0}^{[r/2]}\zeta(2\nu)\zeta(p+q+r-2\nu+s)\\
&\qquad\times\sum_{\mu=0}^{p-1}\frac{1}{2^{r-2\nu+\mu+1}}\binom{p+q-\mu-2}{q-1}\binom{r-2\nu+\mu}{r-2\nu}\\
&\quad+\sum_{\nu=0}^{[r/2]}\zeta(2\nu)\zeta(p+q+r-2\nu+s)\\
&\qquad\times\sum_{\mu=0}^{q-1}(-1)^{\mu+1}\binom{p+q-\mu-2}{p-1}\binom{r-2\nu+\mu}{r-2\nu}\Bigg\}
\end{aligned}\tag{8.69}$$

holds for all $s\in\mathbb{C}$ except for singularities of functions on both sides, where $p,q,r\in\mathbb{N}$. For example, we have

$$\begin{aligned}
&\zeta_2(2,s,2,1;C_2)+\zeta_2(2,2,s,1;C_2)+\zeta_2(1,2,s,2;C_2)-\zeta_2(1,s,2,2;C_2)\\
&\qquad=3\zeta(2)\zeta(s+3)-\frac{39}{8}\zeta(s+5).
\end{aligned}\tag{8.70}$$

In particular, putting $s=2$ in (8.70), we have

$$\zeta_2(2,2,2,1;C_2)=\frac{3}{2}\zeta(2)\zeta(5)-\frac{39}{16}\zeta(7),$$

which coincides with our previous result in [44, Example in §3]. Note that the left-hand side of (8.69) is equal to $S(\mathbf{s},\mathbf{y};I;\Delta)$ for $\Delta=\Delta(C_2)$, $\mathbf{s}=(p,s,q,r)$, $\mathbf{y}=0$ and $I=\{2\}$. Therefore we can see that Theorem 8.1 corresponds to the case C_2 of Theorem 5.1.

9. Functional relations for $\zeta_3(\mathrm{s};B_3)$ and for $\zeta_3(\mathrm{s};C_3)$

In this section, we consider $\zeta_3(\mathbf{s};B_3)$ and $\zeta_3(\mathbf{s};C_3)$ defined by

$$\begin{aligned}\zeta_3(s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8,s_9;B_3)&\\ =\sum_{m_1=1}^\infty\sum_{m_2=1}^\infty\sum_{m_3=1}^\infty & m_1^{-s_1}m_2^{-s_2}m_3^{-s_3}(m_1+m_2)^{-s_4}(m_2+m_3)^{-s_5}\\ &\times(2m_2+m_3)^{-s_6}(m_1+m_2+m_3)^{-s_7}(m_1+2m_2+m_3)^{-s_8}\\ &\times(2m_1+2m_2+m_3)^{-s_9},\end{aligned}\tag{9.71}$$

and

$$\begin{aligned}\zeta_3(s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8,s_9;C_3)&\\ =\sum_{m_1=1}^\infty\sum_{m_2=1}^\infty\sum_{m_3=1}^\infty & m_1^{-s_1}m_2^{-s_2}m_3^{-s_3}(m_1+m_2)^{-s_4}(m_2+m_3)^{-s_5}\\ &\times(m_2+2m_3)^{-s_6}(m_1+m_2+m_3)^{-s_7}(m_1+m_2+2m_3)^{-s_8}\\ &\times(m_1+2m_2+2m_3)^{-s_9},\end{aligned}\tag{9.72}$$

which have been continued meromorphically to the whole space whose possible singularities have been determined in [18, Theorems 6.1 and 6.3]. Note that $\zeta_3(\mathbf{s};D_3)$ essentially coincides with $\zeta_3(\mathbf{s};A_3)$ which has been considered in [16,30].

We aim to prove functional relations for these functions, namely generalize the result in Theorems 7.1 and 8.1 to the cases B_3 and C_3. However, it seems too complicated to treat these cases in full generality. Hence we study some special cases as follows.

First we prove the following functional relation for $\zeta_3(\mathbf{s};C_3)$. The basic structure of the proof, based on Lemma 6.2, is similar to that in the proof of Theorem 7.1 for $\zeta_3(\mathbf{s};A_3)$. A novel point here is that we will also use the result described in Section 4.

Theorem 9.1. *The functional relation*

$$\begin{aligned}
&8\zeta_3(2,2,s,2,2,2,2,2,2;C_3)+8\zeta_3(2,2,2,2,s,2,2,2,2;C_3)\\
&\quad+8\zeta_3(2,2,2,2,2,2,s,2,2;C_3)\\
&=\frac{184775}{512}\zeta(s+16)-\frac{16875}{64}\zeta(2)\zeta(s+14)+\frac{513}{8}\zeta(4)\zeta(s+12)\\
&\quad+\frac{25}{8}\zeta(6)\zeta(s+10)+\frac{1}{4}\zeta(8)\zeta(s+8)
\end{aligned}\tag{9.73}$$

holds for $s\in\mathbb{C}$ except for singularities of functions on both sides. In particular when $s=2$,

$$\zeta_3(2,2,2,2,2,2,2,2,2;C_3)=\frac{19}{8403115488768000}\pi^{18},\tag{9.74}$$

hence $C_W(2,C_3)=19/16209713520$ in Witten's volume formula (1.4).

Proof. Instead of (7.53) or (7.58), we start the same argument as in the proof of Proposition 7.1 or Theorem 7.1 from the relation

$$\{G(\theta;2,2,2;x)+G(-\theta;2,2,2;x^{-1})\}\sum_{n=1}^{\infty}\frac{(-1)^n y^n e^{in\theta}}{n^s}=0\tag{9.75}$$

for $s>1$, where we denote by $G(\theta;2,2,2;x)$ the left-hand side of (7.55) in the case $(2p,2q,s)=(2,2,2)$. Then, by replacing $-m$ $(m\geq 1)$ by m $(m\leq -1)$ on the left-hand side of (9.75), we can rewrite (9.75) to

$$\begin{aligned}
&\sum_{\substack{l\neq 0,\, m\neq 0\\ n\geq 1,\, l+m\neq 0}}\frac{(-1)^{l+m+n}x^m y^n e^{i(l+m+n)\theta}}{l^2m^2n^s(l+m)^2}\\
&-2\sum_{j=0}^{1}\phi(2-2j)\sum_{\xi=0}^{2j}\binom{1+2j-\xi}{1}(-1)^{2j-\xi}\\
&\qquad\times\sum_{\substack{m\neq 0\\ n\geq 1}}\frac{(-1)^{m+n}x^m y^n e^{i(m+n)\theta}}{m^{4+2j-\xi}n^s}\frac{(i\theta)^\xi}{\xi!}\\
&+2\sum_{j=0}^{1}\phi(2-2j)\sum_{\xi=0}^{2j}\binom{1+2j-\xi}{1}(-1)\sum_{\substack{m\neq 0\\ n\geq 1}}\frac{(-1)^n x^m y^n e^{in\theta}}{m^{4+2j-\xi}n^s}\frac{(i\theta)^\xi}{\xi!}=0.
\end{aligned}$$

As well as in the proof of Theorem 7.1, separate the constant terms corresponding to $l+m+n=0$ in the first term and to $m+n=0$ in the second term on the left-hand side, move them to the right-hand side, and apply

Lemma 6.2 with $d = 2$. Then we obtain

$$\begin{aligned}
&\sum_{\substack{l\neq 0,\ m\neq 0\\ n\geq 1,\ l+m\neq 0\\ l+m+n\neq 0}} \frac{(-1)^{l+m+n}x^m y^n e^{i(l+m+n)\theta}}{l^2m^2n^s(l+m)^2(l+m+n)^2}\\
&-2\sum_{j=0}^{1}\phi(2-2j)\sum_{\xi=0}^{2j}\sum_{\omega=0}^{2j-\xi}\binom{\omega+1}{\omega}(-1)^{\omega}\binom{1+2j-\xi-\omega}{1}(-1)^{2j-\xi-\omega}\\
&\times\sum_{\substack{m\neq 0\\ n\geq 1}}\frac{(-1)^{m+n}x^m y^n e^{i(m+n)\theta}}{m^{4+2j-\xi-\omega}n^s(m+n)^{2+\omega}}\frac{(i\theta)^{\xi}}{\xi!}\\
&+\cdots=0,
\end{aligned}$$

where we omit three terms on the left-hand side, which are of the form similar to the second term on the left-hand side. Note that each of their denominators is of the form of A_2 type. Next we replace y by $-ye^{i\theta}$, move the constant terms to the right-hand side and apply Lemma 6.2 with $d = 2$. Then we have

$$\begin{aligned}
&\sum_{\substack{l\neq 0,\ m\neq 0\\ n\geq 1,\ l+m\neq 0\\ l+m+n\neq 0\\ l+m+2n\neq 0}} \frac{(-1)^{l+m}x^m y^n e^{i(l+m+2n)\theta}}{l^2m^2n^s(l+m)^2(l+m+n)^2(l+m+2n)^2}\\
&-2\sum_{j=0}^{1}\phi(2-2j)\sum_{\xi=0}^{2j}\sum_{\sigma=0}^{2j-\sigma}\binom{\sigma+1}{\sigma}(-1)^{\sigma}\sum_{\omega=0}^{2j-\xi-\sigma}\binom{\omega+1}{\omega}(-1)^{2j-\xi-\sigma}\\
&\times\binom{1+2j-\xi-\sigma-\omega}{1}\sum_{\substack{m\neq 0\\ n\geq 1}}\frac{(-1)^{m}x^m y^n e^{i(m+2n)\theta}}{m^{4+2j-\xi-\omega}n^s(m+n)^{2+\omega}(m+2n)^{2+\sigma}}\frac{(i\theta)^{\xi}}{\xi!}\\
&+\cdots=0,
\end{aligned}$$

where we omit seven terms of the forms similar to the second term. Each denominator of these terms is of the form of C_2 type. Replacing x by $-xe^{i\theta}$, applying Lemma 6.2 with $d = 2$, and putting $\theta = \pi$, we obtain

$$\begin{aligned}
&\sum_{\substack{l\neq 0,\ m\neq 0\\ n\geq 1,\ l+m\neq 0\\ l+m+n\neq 0\\ l+m+2n\neq 0\\ l+2m+2n\neq 0}} \frac{x^m y^n}{l^2m^2n^s(l+m)^2(l+m+n)^2(l+m+2n)^2(l+2m+2n)^2}\\
&+\cdots=0, \qquad (9.76)
\end{aligned}$$

where the omitted terms are of the form of C_2 type. Next, we replace (x, y) by $\left(xe^{i\theta}, ye^{i\theta}\right)$ and $\left(xe^{-i\theta}, ye^{-i\theta}\right)$ respectively, and subtract these terms.

Then we have

$$\sum_{\substack{l\neq 0,\ m\neq 0\\ n\geq 1,\ l+m\neq 0\\ m+n\neq 0\\ l+m+n\neq 0\\ l+m+2n\neq 0\\ l+2m+2n\neq 0}} \frac{x^m y^n \sin((m+n)\theta)}{l^2 m^2 n^s (l+m)^2 (l+m+n)^2 (l+m+2n)^2 (l+2m+2n)^2} + \cdots = 0, \tag{9.77}$$

where the omitted terms are double series of similar forms.

Finally, in order to complete the proof of this theorem, we need to apply Theorem 4.1 to each term on the left-hand side of (9.77) with $s=2$. Then we consequently obtain

$$\sum_{\substack{l\neq 0,\ m\neq 0\\ n\geq 1,\ l+m\neq 0\\ m+n\neq 0\\ l+m+n\neq 0\\ l+m+2n\neq 0\\ l+2m+2n\neq 0}} \frac{x^m y^n}{l^2 m^2 n^s (l+m)^2 (m+n)^2 (l+m+n)^2 (l+m+2n)^2 (l+2m+2n)^2} + \cdots = 0. \tag{9.78}$$

We further replace (x,y) by $\left(e^{i\theta}, e^{2i\theta}\right)$ and $\left(e^{-i\theta}, e^{-2i\theta}\right)$ respectively, subtract these terms, and apply Theorem 4.1 with $s=2$. Then we obtain

$$\begin{aligned}\sum_{\substack{l\neq 0,\ m\neq 0\\ n\geq 1,\ l+m\neq 0\\ m+n\neq 0,\ m+2n\neq 0\\ l+m+n\neq 0\\ l+m+2n\neq 0\\ l+2m+2n\neq 0}} & \frac{1}{l^2 m^2 n^s (l+m)^2 (m+n)^2 (m+2n)^2 (l+m+n)^2} \\ & \times \frac{1}{(l+m+2n)^2 (l+2m+2n)^2} \\ & + \cdots = 0,\end{aligned} \tag{9.79}$$

where the omitted terms are finite sums of zeta values of C_2 type. Though we omit their explicit forms, we can also apply Theorem 8.1 to these terms, and can express them as the right-hand side of (9.73). On the other hand, similarly to the case of Theorem 7.1, we can transform the first term on the left-hand side of (9.79) to the left-hand side of (9.73).

Moreover, from (2.16) in [18], we can easily check that $K(C_3)=720$ (see definition (1.7)). Hence, combining (1.6) and (9.74), we obtain the value of $C_W(2, C_3)$. □

Similarly we can obtain the following formula in the case B_3.

Theorem 9.2. *The functional relation*

$$\begin{aligned}
&4\zeta_3(2,s,2,2,2,2,2,2,2;B_3)+4\zeta_3(s,2,2,2,2,2,2,2,2;B_3) \\
&\quad+4\zeta_3(2,2,2,s,2,2,2,2,2;B_3)+4\zeta_3(2,2,2,2,s,2,2,2,2;B_3) \\
&\quad+4\zeta_3(2,2,2,2,2,2,s,2,2;B_3)+4\zeta_3(2,2,2,2,2,2,2,s,2;B_3) \\
&=\left(9\cdot 2^{-s-6}+\frac{5626955}{256}\right)\zeta(s+16)+\left(5\cdot 2^{-s-5}-\frac{59131}{4}\right)\zeta(2)\zeta(s+14) \\
&\quad+\left(5\cdot 2^{-s-5}+\frac{17155}{8}\right)\zeta(4)\zeta(s+12)+\frac{241}{16}\zeta(6)\zeta(s+10)+\frac{1}{8}\zeta(8)\zeta(s+8)
\end{aligned} \tag{9.80}$$

holds for $s\in\mathbb{C}$ except for singularities of functions on both sides. In particular when $s=2$, we have

$$\zeta_3(2,2,2,2,2,2,2,2,2;B_3)=\frac{19}{8403115488768000}\pi^{18}, \tag{9.81}$$

hence $C_W(2,B_3)=19/16209713520$ in Witten's volume formula (1.4).

Proof. The argument is similar to that in the proof of Theorem 9.1. In fact, instead of (9.75), we start the same argument from the relation

$$\{H(\theta;2,2,2,2;x)+H(-\theta;2,2,2,2;x)\}\sum_{n=1}^{\infty}\frac{(-1)^n y^n e^{in\theta}}{n^s}=0 \tag{9.82}$$

for $s>1$, where we denote by $H(\theta;2,2,2,2;x)$ the left-hand side of (8.66) in the case $(2p,2q,2r,s)=(2,2,2,2)$. Repeating the same procedure as in the proof of Theorem 9.1, we can describe the left-hand side of (9.80) as a finite sum of the forms of the left-hand side of (8.69). Hence, by using (8.69), we can obtain (9.80). The value of $C_W(2,B_3)$ can be calculated from (1.6), (9.81), and the fact $K(B_3)=720$. □

Remark 9.1. Comparing the above two theorems, we see that $C_W(2,B_3)=C_W(2,C_3)$. However it does not always hold that $C_W(2k,B_3)=C_W(2k,C_3)$, that is, $\zeta_W(2k;B_3)=\zeta_W(2k;C_3)$ for $k\geq 2$. In fact, we can compute that

$$\zeta_W(4;B_3)=1.00066856607695295\cdots,$$

$$\zeta_W(4;C_3)=1.00082905650461486\cdots,$$

hence $C_W(4,B_3)\neq C_W(4,C_3)$.

Note that the left-hand side of (9.73) corresponds to $S(\mathbf{s},\mathbf{y};I;\Delta)$ for $\Delta=\Delta(C_3)$, $\mathbf{s}=(2,s,2,2,2,2,2,2,2)$, $\mathbf{y}=0$ and $I=\{3\}$ in the terminology of Section 5. Next we prove the following result which is corresponding to the case $\Delta = \Delta(C_3)$, $\mathbf{s} = (2, s, t, 2, u, v, 2, 2, 2)$, $\mathbf{y} = 0$ and $I = \{2, 3\}$.

Theorem 9.3. *The functional relation*

$$\begin{aligned}
&\zeta_3(2, s, t, 2, u, v, 2, 2, 2; C_3) + 2\zeta_3(2, 2, t, s, 2, 2, u, v, 2; C_3) \\
&\quad + 2\zeta_3(s, 2, 2, 2, t, 2, u, 2, v; C_3) \\
&= \sum_{\xi=0,1} \zeta(2\xi) \sum_{\tau=0}^{2-2\xi} (\tau + 1) \sum_{\nu=0}^{2-2\xi-\tau} (\nu + 1) \\
&\quad \times \Bigg\{ \sum_{\omega=0}^{2-2\xi-\tau-\nu} (\omega + 1)(3 - 2\xi - \tau - \nu - \omega) \\
&\qquad \times \frac{1}{2^{\tau+2}} \zeta_2(s + 2 + \omega, t, u + 6 - 2\xi - \nu - \omega, v + 2 + \nu; C_2) \\
&\quad + (-1)^{\tau+\nu} \sum_{\omega=0}^{2-2\xi-\tau-\nu} (\omega + 1)(3 - 2\xi - \tau - \nu - \omega) \\
&\qquad \times \zeta_2(s, t + 4 + \omega + \nu, u + 6 - 2\xi - \omega - \nu, v; C_2) \\
&\quad + (-1)^{\tau+\nu} \sum_{\omega=0,1} \binom{\omega + 2 - 2\xi - \tau - \nu}{\omega} (2 - \omega) \\
&\qquad \times \frac{1}{2^{\nu+2}} \zeta_2(s + 3 - 2\xi - \tau - \nu + \omega, t + 2 + \nu, u + 3 - \omega, v + 2 + \tau; C_2) \\
&\quad + \sum_{\omega=0,1} \binom{\omega + 2 - 2\xi - \tau - \nu}{\omega} (2 - \omega) \\
&\qquad \times \frac{1}{2^{\nu+2}} \zeta_2(s, t + 5 - 2\xi - \tau + \omega, u + 3 - \omega, v + 2 + \tau; C_2) \Bigg\} \\
&\quad - \sum_{\xi=0,1} \zeta(2\xi) \sum_{\tau=0}^{2-2\xi} (\tau + 1) \sum_{\nu=0,1} \binom{\nu + 2 - 2\xi - \tau}{\nu} \\
&\quad \times \Bigg\{ (-1)^{\tau+1} \sum_{\omega=0}^{1-\nu} (\omega + 1)(2 - \nu - \omega) \\
&\qquad \times \zeta_2(q, r + 3 - \nu - \omega, v + 3 - 2\xi - \tau + \nu, p + 4 + \tau + \omega; C_2) \\
&\quad + (-1)^{\tau+\nu} \sum_{\omega=0}^{1-\nu} (\omega + 1)(2 - \nu - \omega) \\
&\qquad \times \zeta_2(s + 2 + \tau, t + 5 - 2\xi - \tau + \nu + \omega, u + 3 - \nu - \omega, v; C_2)
\end{aligned}$$

$$
\begin{aligned}
&\quad + (-1)^{\tau+\nu} \sum_{\omega=0,1} \binom{\omega+1-\nu}{\omega} (2-\omega) \\
&\qquad \times \frac{1}{2^{3-2\xi-\tau+\nu}} \zeta_2(s+4+\tau-\nu+\omega, t+3-2\xi-\tau+\nu, u+3-\omega, v; C_2) \\
&\quad + (-1)^{\tau+1} \sum_{\omega=0,1} \binom{\omega+1-\nu}{\omega} (2-\omega) \\
&\qquad \times \frac{1}{2^{3-2\xi-\tau+\nu}} \zeta_2(s+2+\tau, t+5-2\xi-\tau+\omega, u+3-\omega, v; C_2) \Bigg\} \\
&- \sum_{\xi=0,1} \zeta(2\xi) \sum_{\tau=0,1} \binom{\tau+2-2\xi}{\tau} \sum_{\nu=0}^{1-\tau} (\nu+1) \\
&\quad \times \Bigg\{ - \sum_{\omega=0}^{1-\tau-\nu} (\omega+1)(2-\tau-\nu-\omega) \\
&\qquad \times \frac{1}{2^{3-2\xi+\tau}} \zeta_2(s+2+\omega, t, u+6-2\xi-\nu-\omega, v+2+\nu, ; C_2) \\
&\quad + (-1)^{\tau+\nu} \sum_{\omega=0}^{1-\tau-\nu} (\omega+1)(2-\tau-\nu-\omega) \\
&\qquad \times \zeta_2(s, t+4+\nu+\omega, u+6-2\xi-\nu-\omega, v; C_2) \\
&\quad + (-1)^{\tau+\nu} \sum_{\omega=0,1} \binom{\omega+1-\tau-\nu}{\omega} (2-\omega) \\
&\qquad \times \frac{1}{2^{\nu+2}} \zeta_2(s+2-\tau-\nu+\omega, t+2+\nu, u+3-\omega, v+3-2\xi+\tau; C_2) \\
&\quad - \sum_{\omega=0,1} \binom{\omega+1-\tau-\nu}{\omega} (2-\omega) \\
&\qquad \times \frac{1}{2^{\nu+2}} \zeta_2(s, t+4-\tau+\omega, u+3-\omega, v+3-2\xi+\tau; C_2) \Bigg\} \\
&+ \sum_{\xi=0,1} \zeta(2\xi) \sum_{\tau=0,1} \binom{\tau+2-2\xi}{\tau} \sum_{\nu=0,1} \binom{\nu+1-\tau}{\nu} \\
&\quad \times \Bigg\{ (-1)^{\tau+1} \sum_{\omega=0}^{1-\nu} (\omega+1)(2-\nu-\omega) \\
&\qquad \times \zeta_2(s+5-2\xi+\tau+\omega, t, u+3-\nu-\omega, v+2-\tau+\nu; C_2) \\
&\quad + (-1)^{\tau+\nu} \sum_{\omega=0}^{1-\nu} (\omega+1)(2-\nu-\omega) \\
&\qquad \times \zeta_2(s+3-2\xi+\tau, t+4+\omega-\tau+\nu, u+3-\nu-\omega, v; C_2)
\end{aligned}
$$

$$+ (-1)^{\tau+\nu} \sum_{\omega=0,1} \binom{\omega+1-\nu}{\omega} (2-\omega)$$
$$\times \frac{1}{2^{\nu+2-\tau}} \zeta_2(s+5-2\xi+\tau-\nu+\omega, t+2-\tau+\nu, u+3-\omega, v; C_2)$$
$$+ (-1)^{\tau+1} \sum_{\omega=0,1} \binom{\omega+1-\nu}{\omega} (2-\omega)$$
$$\times \frac{1}{2^{\nu+2-\tau}} \zeta_2(s+3-2\xi+\tau, t+4-\tau+\omega, u+3-\omega, v; C_2) \Bigg\}$$

holds for $s, t, u, v \in \mathbb{C}$ *except for singularities of functions on both sides.*

Proof. As well as (7.53) and (7.58), we begin by combining quantities of type A_1 and of type C_2, that is,

$$\left(\sum_{l=1}^{\infty} \frac{(-1)^l (e^{il\theta} + e^{-il\theta})}{l^2} - 2 \sum_{j=0,1} \phi(2-2j) \frac{(i\theta)^{2j}}{(2j)!} \right) \times \sum_{m,n=1}^{\infty} \frac{(-1)^{m+n} x^m y^n e^{i(m+n)\theta}}{m^s n^t (m+n)^u (m+2n)^v} = 0 \tag{9.83}$$

for $\theta \in [-\pi, \pi]$, where we fix $s, t, u, v \in \{z \in \mathbb{R} \,|\, z > 1\}$ and $x, y \in \{z \in \mathbb{R} \,|\, |z| = 1\}$. Then

$$\sum_{l,m,n=1}^{\infty} \frac{(-1)^{l+m+n} x^m y^n e^{i(l+m+n)\theta}}{l^2 m^s n^t (m+n)^u (m+2n)^v} + \sum_{\substack{l,m,n=1 \\ l \neq m+n}}^{\infty} \frac{(-1)^{l+m+n} x^m y^n e^{i(-l+m+n)\theta}}{l^2 m^s n^t (m+n)^u (m+2n)^v}$$
$$- 2 \sum_{j=0,1} \phi(2-2j) \frac{(-1)^j \theta^{2j}}{(2j)!} \sum_{m,n=1}^{\infty} \frac{(-1)^{m+n} x^m y^n e^{i(m+n)\theta}}{m^s n^t (m+n)^u (m+2n)^v}$$
$$= - \sum_{m,n=1}^{\infty} \frac{x^m y^n}{m^s n^t (m+n)^{u+2} (m+2n)^v} \tag{9.84}$$

for $\theta \in [-\pi, \pi]$. Applying Lemma 6.2 with $d = 2$, we have

$$\sum_{l,m,n=1}^{\infty} \frac{(-1)^{l+m+n} x^m y^n e^{i(l+m+n)\theta}}{l^2 m^s n^t (m+n)^u (m+2n)^v (l+m+n)^2}$$
$$+ \sum_{\substack{l,m,n=1 \\ l \neq m+n}}^{\infty} \frac{(-1)^{l+m+n} x^m y^n e^{i(-l+m+n)\theta}}{l^2 m^s n^t (m+n)^u (m+2n)^v (-l+m+n)^2}$$

$$
\begin{aligned}
&-2\sum_{j=0,1}\phi(2-2j)\sum_{\xi=0}^{2j}\binom{2j-\xi+1}{2j-\xi}(-1)^{2j-\xi}\frac{(i\theta)^{\xi}}{\xi!}\\
&\quad\times\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}x^m y^n e^{i(m+n)\theta}}{m^s n^t (m+n)^{u+2+2j-\xi}(m+2n)^v}\\
&-2\sum_{j=0,1}\phi(2-2j)\sum_{\xi=0}^{2j}\binom{1+2j-\xi}{1}\frac{(i\theta)^{\xi}}{\xi!}\\
&\quad\times\sum_{m,n=1}^{\infty}\frac{x^m y^n}{m^s n^t (m+n)^{u+2+2j-\xi}(m+2n)^v}=0 \quad (\theta\in[-\pi,\pi]). \qquad (9.85)
\end{aligned}
$$

For simplicity, we denote the sum of the third and the fourth terms on the left-hand side of (9.85) by

$$
\begin{aligned}
&-2\sum_{j=0,1}\phi(2-2j)\sum_{\xi=0}^{2j}\frac{(i\theta)^{\xi}}{\xi!}\\
&\times\left[\sum_{m,n=1}^{\infty}\left\{(-1)^{m+n}\mathcal{D}_1(m,n;2j-\xi)e^{i(m+n)\theta}+\mathcal{D}_2(m,n;2j-\xi)\right\}x^m y^n\right],
\end{aligned}
$$

where $\mathcal{D}_j(m,n;\nu)\in\mathbb{R}$ $(j=1,2)$. Since (9.85) holds for $y\in\mathbb{C}$ with $|y|=1$, we replace y by $-ye^{-i\theta}$ with $y\in\mathbb{C}$ $(|y|=1)$. Then we have

$$
\begin{aligned}
&\sum_{l,m,n=1}^{\infty}\frac{(-1)^{l+m}x^m y^n e^{i(l+m)\theta}}{l^2 m^s n^t (m+n)^u (m+2n)^v (l+m+n)^2}\\
&\quad+\sum_{\substack{l,m,n=1\\ l\neq m+n,\ l\neq m}}^{\infty}\frac{(-1)^{l+m}x^m y^n e^{i(-l+m)\theta}}{l^2 m^s n^t (m+n)^u (m+2n)^v (-l+m+n)^2}\\
&\quad-2\sum_{j=0,1}\phi(2-2j)\sum_{\xi=0}^{2j}\sum_{m,n=1}^{\infty}\Big\{(-1)^m\mathcal{D}_1(m,n;2j-\xi)e^{im\theta}\\
&\qquad\qquad+(-1)^n\mathcal{D}_2(m,n;2j-\xi)e^{-in\theta}\Big\}x^m y^n\frac{(i\theta)^{\xi}}{\xi!}\\
&=-\sum_{m,n=1}^{\infty}\frac{x^m y^n}{m^{s+2}n^{t+2}(m+n)^u(m+2n)^v}\quad(\theta\in[-\pi,\pi]). \qquad (9.86)
\end{aligned}
$$

Therefore, applying Lemma 6.2 with $d=2$, we have

$$\begin{aligned}
&\sum_{l,m,n=1}^{\infty} \frac{(-1)^{l+m} x^m y^n e^{i(l+m)\theta}}{l^2 m^s n^t (l+m)^2 (m+n)^u (m+2n)^v (l+m+n)^2} \qquad (9.87)\\
&= -\sum_{\substack{l,m,n=1 \\ l\neq m+n,\ l\neq m}}^{\infty} \frac{(-1)^{l+m} x^m y^n e^{i(-l+m)\theta}}{l^2 m^s n^t (-l+m)^2 (m+n)^u (m+2n)^v (-l+m+n)^2}\\
&\quad + 2\sum_{j=0,1} \phi(2-2j) \sum_{\xi=0}^{2j} \sum_{m,n=1}^{\infty} \Big\{ (-1)^m \mathcal{D}'_1(m,n;2j-\xi) e^{im\theta}\\
&\qquad + (-1)^n \mathcal{D}'_2(m,n;2j-\xi) e^{-in\theta} + \mathcal{D}'_3(m,n;2j-\xi) \Big\} x^m y^n \frac{(i\theta)^\xi}{\xi!}
\end{aligned}$$

for $\theta \in [-\pi,\pi]$ with some $\mathcal{D}'_j(m,n;\nu) \in \mathbb{R}$ $(j=1,2,3)$.

Now we repeat this procedure. Namely, replace y by $ye^{2i\theta}$ and apply Lemma 6.2 with $d=2$. Furthermore, replace x by $-xe^{i\theta}$ and apply Lemma 6.2 with $d=2$. Then we can obtain the equation

$$\begin{aligned}
&\sum_{l,m,n=1}^{\infty} \frac{(-1)^{l} x^m y^n e^{i(l+2m+2n)\theta}}{l^2 m^s n^t (l+m)^2 (m+n)^u (m+2n)^v (l+m+n)^2}\\
&\qquad \times \frac{1}{(l+m+2n)^2 (l+2m+2n)^2}\\
&= -\sum_{\substack{l,m,n=1 \\ l\neq m,\ l\neq m+n \\ l\neq m+2n,\ l\neq 2m+2n}}^{\infty} \frac{1}{l^2 m^s n^t (-l+m)^2 (m+n)^u (m+2n)^v (-l+m+n)^2}\\
&\qquad \times \frac{(-1)^{l} x^m y^n e^{i(-l+2m+2n)\theta}}{(-l+m+2n)^2 (-l+2m+2n)^2}\\
&\quad + 2\sum_{j=0,1} \phi(2-2j) \sum_{\xi=0}^{2j} \sum_{m,n=1}^{\infty} \Big\{ \widetilde{\mathcal{D}}_1(m,n;2j-\xi) e^{i(2m+2n)\theta}\\
&\quad + (-1)^{m+n} \widetilde{\mathcal{D}}_2(m,n;2j-\xi) e^{i(m+n)\theta} + (-1)^m \widetilde{\mathcal{D}}_3(m,n;2j-\xi) e^{i(m+2n)\theta}\\
&\quad + (-1)^m \widetilde{\mathcal{D}}_4(m,n;2j-\xi) e^{im\theta} + \widetilde{\mathcal{D}}_5(m,n;2j-\xi) \Big\} x^m y^n \frac{(i\theta)^\xi}{\xi!} \qquad (9.88)
\end{aligned}$$

for $\theta \in [-\pi,\pi]$ with some $\widetilde{\mathcal{D}}_j(m,n;\nu) \in \mathbb{R}$ $(j=1,2,\ldots,5)$. Put $\theta=\pi$ and

$(x, y) = (1, 1)$, and consider the real part. Then we have

$$\begin{aligned}
&\sum_{l,m,n=1}^{\infty} \frac{1}{l^2 m^s n^t (l+m)^2 (m+n)^u (m+2n)^v (l+m+n)^2} \\
&\qquad \times \frac{1}{(l+m+2n)^2 (l+2m+2n)^2} \\
&+ \sum_{\substack{l,m,n=1 \\ l\neq m,\ l\neq m+n \\ l\neq m+2n,\ l\neq 2m+2n}}^{\infty} \frac{1}{l^2 m^s n^t (-l+m)^2 (m+n)^u (m+2n)^v (-l+m+n)^2} \\
&\qquad \times \frac{1}{(-l+m+2n)^2 (-l+2m+2n)^2} \\
&- 2 \sum_{j=0,1} \phi(2-2j) \sum_{\tau=0}^{j} \sum_{m,n=1}^{\infty} \Big\{ \widetilde{\mathcal{D}}_1(m,n;2j-2\tau) + \widetilde{\mathcal{D}}_2(m,n;2j-2\tau) \\
&\qquad + \widetilde{\mathcal{D}}_3(m,n;2j-2\tau) + \widetilde{\mathcal{D}}_4(m,n;2j-2\tau) + \widetilde{\mathcal{D}}_5(m,n;2j-2\tau) \Big\} \\
&\quad \times \frac{(-1)^\tau \pi^{2\tau}}{(2\tau)!} = 0. \qquad (9.89)
\end{aligned}$$

By (9.72), we see that the first term on the left-hand side of (9.89) coincides with

$$\zeta_3(2, s, t, 2, u, v, 2, 2, 2; C_3).$$

For the second term on the left-hand side of (9.89), change the running indices of summation corresponding to the conditions $l \neq m$, $l \neq m+n$, $l \neq m+2n$, $l \neq 2m+2n$. Then we can see that the second term on the left-hand side of (9.89) coincides with

$$\begin{aligned}
&\zeta_3(2, s, t, 2, u, v, 2, 2, 2; C_3) + 2\zeta_3(2, 2, t, s, 2, 2, u, v, 2; C_3) \\
&\quad + 2\zeta_3(s, 2, 2, 2, t, 2, u, 2, v; C_3).
\end{aligned}$$

Furthermore, using Lemma 6.1, we can rewrite the third term on the left-hand side of (9.89) as

$$\begin{aligned}
&- 2 \sum_{\xi=0,1} \zeta(2\xi) \sum_{m,n=1}^{\infty} \Big\{ \widetilde{\mathcal{D}}_1(m,n;2-2\xi) + \widetilde{\mathcal{D}}_2(m,n;2-2\xi) \\
&\qquad + \widetilde{\mathcal{D}}_3(m,n;2-2\xi) + \widetilde{\mathcal{D}}_4(m,n;2-2\xi) + \widetilde{\mathcal{D}}_5(m,n;2-2\xi) \Big\}.
\end{aligned}$$

We can concretely calculate the value $\widetilde{\mathcal{D}}_j(m,n;\nu)$ in terms of $\zeta_2(\mathbf{s}; C_2)$ and $\zeta(s)$. Combining these results, we obtain the assertion. □

Example 9.1. Putting $(s,t,u,v)=(2,2,2,2)$, we obtain

$$
\begin{aligned}
&3\,\zeta_3(2,2,2,2,2,2,2,2,2;C_3)\\
&= \zeta_2(5,5,4,2;C_2)\zeta(2)-\frac{3}{2}\zeta_2(5,5,6,2;C_2)+\frac{1}{2}\zeta_2(6,4,4,2;C_2)\zeta(2)\\
&-\frac{3}{4}\zeta_2(6,4,6,2;C_2)+\zeta_2(6,5,3,2;C_2)\zeta(2)-\frac{3}{2}\zeta_2(6,5,5,2;C_2)\\
&-\frac{3}{2}\zeta_2(6,8,2,2;C_2)+\frac{3}{4}\zeta_2(7,4,3,2;C_2)\zeta(2)-\frac{9}{8}\zeta_2(7,4,5,2;C_2)\\
&-\frac{11}{16}\zeta_2(7,5,4,2;C_2)-\frac{23}{32}\zeta_2(8,4,4,2;C_2)-\zeta_2(8,6,2,2;C_2)\\
&+\zeta_2(4,5,4,3;C_2)\zeta(2)-\frac{3}{2}\zeta_2(4,5,6,3;C_2)+\frac{1}{2}\zeta_2(5,5,3,3;C_2)\zeta(2)\\
&-\frac{3}{4}\zeta_2(5,5,5,3;C_2)+\zeta_2(6,5,2,3;C_2)\zeta(2)-\frac{3}{16}\zeta_2(6,5,4,3;C_2)\\
&-\frac{3}{4}\zeta_2(7,4,2,3;C_2)\zeta(2)+\zeta_2(2,5,5,4;C_2)\zeta(2)+\frac{3}{2}\zeta_2(2,6,4,4;C_2)\zeta(2)\\
&-\frac{3}{8}\zeta_2(2,6,6,4;C_2)-\frac{7}{8}\zeta_2(2,7,5,4;C_2)-\frac{15}{16}\zeta_2(2,8,4,4;C_2)\\
&+\frac{1}{2}\zeta_2(4,4,4,4;C_2)\zeta(2)-\frac{3}{4}\zeta_2(4,4,6,4;C_2)-\frac{1}{2}\zeta_2(4,5,3,4;C_2)\zeta(2)\\
&+\frac{3}{4}\zeta_2(4,5,5,4;C_2)+\frac{1}{4}\zeta_2(5,4,3,4;C_2)\zeta(2)-\frac{3}{8}\zeta_2(5,4,5,4;C_2)\\
&-\frac{3}{2}\zeta_2(5,5,2,4;C_2)\zeta(2)+\frac{1}{4}\zeta_2(5,5,4,4;C_2)+\frac{1}{2}\zeta_2(6,4,2,4;C_2)\zeta(2)\\
&-\frac{3}{32}\zeta_2(6,4,4,4;C_2)+\frac{11}{16}\zeta_2(7,5,2,4;C_2)-\frac{23}{32}\zeta_2(8,4,2,4;C_2)\\
&-2\zeta_2(2,4,5,5;C_2)\zeta(2)-\zeta_2(2,5,4,5;C_2)\zeta(2)-\frac{1}{2}\zeta_2(2,6,5,5;C_2)\\
&-\frac{7}{8}\zeta_2(2,7,4,5;C_2)-\frac{1}{2}\zeta_2(4,4,3,5;C_2)\zeta(2)+\frac{3}{4}\zeta_2(4,4,5,5;C_2)\\
&+\frac{3}{2}\zeta_2(4,5,2,5;C_2)\zeta(2)-\frac{1}{4}\zeta_2(4,5,4,5;C_2)-\frac{1}{4}\zeta_2(5,4,2,5;C_2)\zeta(2)\\
&+\frac{1}{4}\zeta_2(5,4,4,5;C_2)-\frac{27}{16}\zeta_2(6,5,2,5;C_2)+\frac{9}{8}\zeta_2(7,4,2,5;C_2)\\
&-\frac{3}{2}\zeta_2(2,4,6,6;C_2)+4\zeta_2(2,5,3,6;C_2)\zeta(2)-\zeta_2(2,5,5,6;C_2)\\
&-\frac{3}{8}\zeta_2(2,6,4,6;C_2)-2\zeta_2(3,5,2,6;C_2)\zeta(2)+\zeta_2(4,4,2,6;C_2)\zeta(2)\\
&-\frac{3}{8}\zeta_2(4,4,4,6;C_2)+\frac{19}{8}\zeta_2(5,5,2,6;C_2)-\frac{27}{32}\zeta_2(6,4,2,6;C_2)\\
&+4\zeta_2(2,4,3,7;C_2)\zeta(2)+2\zeta_2(2,4,5,7;C_2)+3\zeta_2(2,5,4,7;C_2)
\end{aligned}
$$

$$
\begin{aligned}
&- 2\zeta_2(3,4,2,7;C_2)\zeta(2) - \frac{9}{4}\zeta_2(4,5,2,7;C_2) + \frac{1}{2}\zeta_2(5,4,2,7;C_2)\\
&- 6\zeta_2(2,5,3,8;C_2) + 3\zeta_2(3,5,2,8;C_2) - \frac{3}{2}\zeta_2(4,4,2,8;C_2)\\
&- 6\zeta_2(2,4,3,9;C_2) + 3\zeta_2(3,4,2,9;C_2).
\end{aligned}
$$

The authors also checked this equation numerically by using definitions (8.64) and (9.72).

Remark 9.2. From these considerations, we can see that our method may be applied to much wider class of multiple zeta-functions. As another example, we will consider the zeta-function $\zeta_2(\mathbf{s};G_2)$ associated with the exceptional Lie algebra of type G_2 and will give certain functional relations including explicit forms of Witten's volume formulas of type G_2 in a forthcoming paper [20].

Acknowledgements. The authors greatly thank Dr. Takuya Okamoto for his pointing out a mistake in this paper.

References

1. S. Akiyama, S. Egami, and Y. Tanigawa, Analytic continuation of multiple zeta-functions and their values at non-positive integers, *Acta Arith.* **98** (2001), 107-116.
2. T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, 1976.
3. T. Arakawa and M. Kaneko, Multiple zeta values, poly-Bernoulli numbers, and related zeta functions, *Nagoya Math. J.* **153** (1999), 189-209.
4. D. Bowman and D. M. Bradley, Multiple polylogarithms: a brief survey, in *'Conference on q-Series with Applications to Combinatorics, Number Theory, and Physics' (Urbana, IL, 2000), Contemp. Math. 291*, B. C. Berndt and K. Ono (eds.), Amer. Math. Soc., Providence, RI, 2001, 71-92.
5. D. M. Bradley, Partition identities for the multiple zeta function, in *'Zeta Functions, Topology and Quantum Physics', Developments in Mathematics 14*, T. Aoki et al. (eds.), Springer, New York, 2005, 19-29.
6. O. Espinosa and V. H. Moll, The evaluation of Tornheim double sums, Part I, *J. Number Theory* **116** (2006), 200-229.
7. D. Essouabri, *Singularités des séries de Dirichlet associées à des polynômes de plusieurs variables et applications à la théorie analytique des nombres*, Thése, Univ. Henri Poincaré - Nancy I, 1995.
8. D. Essouabri, Singularités des séries de Dirichlet associées à des polynômes de plusieurs variables et applications en théorie analytique des nombres, *Ann. Inst. Fourier* **47** (1997), 429-483.
9. L. Euler, Meditationes circa singulare serierum genus, *Novi Comm. Acad. Sci. Petropol.* **20** (1775), 140-186. Reprinted in *Opera Omnia, ser. I*, vol. 15, B. G. Teubner, Berlin, 1927, 217-267.

10. P. E. Gunnells and R. Sczech, Evaluation of Dedekind sums, Eisenstein cocycles, and special values of L-functions, *Duke Math. J.* **118** (2003), 229-260.
11. G. H. Hardy, Notes on some points in the integral calculus LV, On the integration of Fourier series, *Messenger of Math.* **51** (1922), 186-192. Reprinted in Collected papers of G. H. Hardy (including joint papers with J. E. Littlewood and others), Vol. III, Clarendon Press, Oxford, 1969, 506-512.
12. M. E. Hoffman, Multiple harmonic series, *Pacific J. Math.* **152** (1992), 275-290.
13. M. E. Hoffman, Algebraic aspects of multiple zeta values, in 'Zeta Functions, *Topology and Quantum Physics', Developments in Mathematics 14,* T. Aoki et al. (eds.), Springer, New York, 2005, 51-74.
14. J. G. Huard, K. S. Williams and N.-Y. Zhang, On Tornheim's double series, *Acta Arith.* **75** (1996), 105-117.
15. M. Kaneko, Multiple zeta values, *Sugaku Expositions* **18** (2005), 221-232.
16. Y. Komori, K. Matsumoto and H. Tsumura, Zeta-functions of root systems, in *'Proceedings of the Conference on L-functions' (Fukuoka, 2006)*, L. Weng and M. Kaneko (eds), World Scientific, 2007, 115-140.
17. Y. Komori, K. Matsumoto and H. Tsumura, Zeta and L-functions and Bernoulli polynomials of root systems, *Proc. Japan Acad., Ser. A*, **84** (2008), 57-62.
18. Y. Komori, K. Matsumoto and H. Tsumura, On Witten multiple zeta-functions associated with semisimple Lie algebras II, to appear in *J. Math. Soc. Japan.*
19. Y. Komori, K. Matsumoto and H. Tsumura, On Witten multiple zeta-functions associated with semisimple Lie algebras III, preprint, arXiv:math/0907.0955.
20. Y. Komori, K. Matsumoto and H. Tsumura, On Witten multiple zeta-functions associated with semisimple Lie algebras IV, preprint, arXiv:math/0907.0972.
21. Y. Komori, K. Matsumoto and H. Tsumura, On multiple Bernoulli polynomials and multiple L-functions of root systems, to appear in *Proc. London Math. Soc.*
22. Y. Komori, K. Matsumoto and H. Tsumura, An introduction to the theory of zeta-functions of root systems, preprint.
23. K. Matsumoto, Asymptotic expansions of double zeta-functions of Barnes, of Shintani, and Eisenstein series, *Nagoya Math J.*, **172** (2003), 59-102.
24. K. Matsumoto, On the analytic continuation of various multiple zeta-functions, in *'Number Theory for the Millennium II, Proc. Millennial Conference on Number Theory'*, M. A. Bennett et al. (eds.), A K Peters, 2002, 417-440.
25. K. Matsumoto, The analytic continuation and the asymptotic behaviour of certain multiple zeta-functions I, *J. Number Theory*, **101** (2003), 223-243.
26. K. Matsumoto, On Mordell-Tornheim and other multiple zeta-functions, in *'Proceedings of the Session in analytic number theory and Diophantine equations' (Bonn, January-June 2002)*, D. R. Heath-Brown and B. Z. Moroz

(eds.), Bonner Mathematische Schriften Nr. 360, Bonn 2003, n.25, 17pp.
27. K. Matsumoto, Analytic properties of multiple zeta-functions in several variables, in *'Number Theory: Tradition and Modernization'*, W. Zhang and Y. Tanigawa (eds.), Springer, 2006, 153-173.
28. K. Matsumoto, T. Nakamura, H. Ochiai and H. Tsumura, On value-relations, functional relations and singularities of Mordell-Tornheim and related triple zeta-functions, *Acta Arith.* **132** (2008), 99-125.
29. K. Matsumoto, T. Nakamura and H. Tsumura, Functional relations and special values of Mordell-Tornheim triple zeta and L-functions, *Proc. Amer. Math. Soc.* **136** (2008), 2135-2145.
30. K. Matsumoto and H. Tsumura, On Witten multiple zeta-functions associated with semisimple Lie algebras I, *Ann. Inst. Fourier* **56** (2006), 1457-1504.
31. K. Matsumoto and H. Tsumura, A new method of producing functional relations among multiple zeta-functions, *Quart. J. Math.* (Oxford) **59** (2008), 55-83.
32. K. Matsumoto and H. Tsumura, Functional relations among certain double polylogarithms and their character analogues, *Šiauliai Math. Semin.* **3 (11)** (2008), 189-205.
33. L. J. Mordell, On the evaluation of some multiple series, *J. London Math. Soc.* **33** (1958), 368-371.
34. T. Nakamura, A functional relation for the Tornheim double zeta function, *Acta Arith.* **125** (2006), 257-263.
35. T. Nakamura, Double Lerch series and their functional relations, *Aequationes Math.* **75** (2008), 251-259.
36. T. Nakamura, Double Lerch value relations and functional relations for Witten zeta functions, *Tokyo J. Math.* **31** (2008), 551-574.
37. M. V. Subbarao and R. Sitaramachandrarao, On some infinite series of L. J. Mordell and their analogues, *Pacific J. Math.* **119** (1985), 245-255.
38. E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, 2nd ed. (revised by D. R. Heath-Brown), Oxford University Press, 1986.
39. L. Tornheim, Harmonic double series, *Amer. J. Math.* **72** (1950), 303-314.
40. H. Tsumura, On some combinatorial relations for Tornheim's double series, *Acta Arith.* **105** (2002), 239-252.
41. H. Tsumura, On alternating analogues of Tornheim's double series, *Proc. Amer. Math. Soc.* **131** (2003), 3633-3641.
42. H. Tsumura, An elementary proof of Euler's formula for $\zeta(2m)$, *Amer. Math. Monthly* **111** (2004), 430-431.
43. H. Tsumura, Evaluation formulas for Tornheim's type of alternating double series, *Math. Comp.* **73** (2004), 251-258.
44. H. Tsumura, On Witten's type of zeta values attached to $SO(5)$, *Arch. Math. (Basel)* **84** (2004), 147-152.
45. H. Tsumura, Certain functional relations for the double harmonic series related to the double Euler numbers, *J. Austral. Math. Soc., Ser. A.* **79** (2005), 317-333.
46. H. Tsumura, On some functional relations between Mordell-Tornheim dou-

ble L-functions and Dirichlet L-functions, *J. Number Theory* **120** (2006), 161-178.

47. H. Tsumura, On functional relations between the Mordell-Tornheim double zeta functions and the Riemann zeta function, *Math. Proc. Cambridge Philos. Soc.* **142** (2007), 395-405.
48. H. Tsumura, On alternating analogues of Tornheim's double series II, *Ramanujan J.* **18** (2009), 81-90.
49. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th ed., Cambridge University Press (1958).
50. E. Witten, On quantum gauge theories in two dimensions, *Comm. Math. Phys.* **141** (1991), 153-209.
51. D. Zagier, Values of zeta functions and their applications, in *'First European Congress of Mathematics'* Vol. II, A. Joseph et al. (eds.), Progr. Math. **120**, Birkhäuser, 1994, 497-512.
52. J. Zhao, Analytic continuation of multiple zeta functions, *Proc. Amer. Math. Soc.* **128** (2000), 1275-1283.

INDEX

central interval, 16
circle method, 2

Davenport-Heilbronn Fourier transform method, 2
diagonal form, 1
Diophantine inequality, 1, 4

exceptional set, 30

Goldbach probelm, 2

interference principle, 19

linear forms in primes, 6

Waring's problem for cubes, 2