A SURVEY ON THE THEORY OF UNIVERSALITY FOR ZETA AND L-FUNCTIONS

KOHJI MATSUMOTO

ABSTRACT. We survey the results and the methods in the theory of universality for various zeta and L-functions, obtained in these forty years after the first discovery of the universality for the Riemann zeta-function by Voronin.

CONTENTS

1. VORONIN'S UNIVERSALITY THEOREM

Let $\mathbb{N}$ be the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the ring of rational integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{R}$ the field of real

numbers, and $\mathbb{C}$ the field of complex numbers. In the present article, the letter p denotes a prime number.

For any open region $D \subset \mathbb{C}$, denote by $H(D)$ the space of $\mathbb{C}$-valued holomorphic functions defined on D, equipped with the topology of uniform convergence on compact sets.

For any subset $K \subset \mathbb{C}$, let $H^c(K)$ be the set of continuous functions defined on K, and are holomorphic in the interior of K, and let $H_0^c(K)$ be the set of all elements of $H^c(K)$ which are non-vanishing on K. By meas S we mean the usual Lebesgue measure of the set S, and by $\#S$ the cardinality of S.

Let $s = \sigma + it \in \mathbb{C}$ (where $\sigma = \Re s$, $t = \Im s$, and $i = \sqrt{-1}$), and $\zeta(s)$ the Riemann zeta-function. This function is defined by the infinite series $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ in the half-plane $\sigma > 1$, and can be continued meromorphically to the whole of $\mathbb{C}$. It is well known that the investigation of $\zeta(s)$ in the critical strip $0 < \sigma < 1$ is extremely important in number theory, but its behaviour there still remains quite mysterious. A typical example of expressing one of such mysterious features of $\zeta(s)$ is Voronin's *universality theorem*.

Consider the closed disc $K(r)$ with center $3/4$ and radius r, where $0 < r < 1/4$. Then Voronin [205] proved that for any $f \in H_0^c(K(r))$ and any $\varepsilon > 0$, there exists a positive number τ for which

$$(1.1) \qquad \max_{s \in K(r)} |\zeta(s + i\tau) - f(s)| < \varepsilon$$

holds. Roughly speaking, *any non-vanishing holomorphic function can be approximated uniformly by a certain shift of* $\zeta(s)$.

Actually, Voronin's proof essentially includes the fact that the set of such τ has a positive lower density. Moreover, now it is known that the disc can be replaced by more general type of sets. The modern statement of Voronin's theorem is as follows. Let $D(a, b)$ $(a < b)$ indicate a vertical strip

$$D(a, b) = \{s \in \mathbb{C} \mid a < \sigma < b\}.$$

Theorem 1. (Voronin's universality theorem) *Let K be a compact subset of $D(1/2, 1)$ with connected complement, and $f \in H_0^c(K)$. Then, for any $\varepsilon > 0$,*

$$(1.2) \quad \liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] \;\middle|\; \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0$$

holds.

Let $\varphi(s)$ be a Dirichlet series, and let K be a compact subset of $D(a,b)$ with connected complement. If

$$(1.3) \quad \liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau\in[0,T] \;\middle|\; \sup_{s\in K}|\varphi(s+i\tau)-f(s)|<\varepsilon\right\}>0$$

holds for any $f\in H_0^c(K)$ and any $\varepsilon>0$, then we say that the *universality* holds for $\varphi(s)$ in the region $D(a,b)$. Theorem 1 implies that the universality holds for the Riemann zeta-function in the region $D(1/2,1)$.

The Riemann zeta-function has the Euler product expression $\zeta(s)=\prod_p(1-p^{-s})^{-1}$, where p runs over all prime numbers. This is valid only in the region $\sigma>1$, but even in the region $D(1/2,1)$, it is possible to show that a finite truncation of the Euler product "approximates" $\zeta(s)$ in a certain mean-value sense. On the other hand, since $\{\log p\}_p$ is linearly independent over $\mathbb{Q}$, we can apply the Kronecker-Weyl approximation theorem to obtain that any target function $f(s)$ can be approximated by the above finite truncation. This is the basic structure of the proof of Theorem 1.

Remark 1. Here we recall the statement of the Kronecker-Weyl theorem. For $x\in\mathbb{R}$, the symbol $||x||$ stands for the distance from x to the nearest integer. Let $\alpha_1,\ldots,\alpha_m$ be real numbers, linearly independent over $\mathbb{Q}$. Then, for any real numbers $\theta_1,\ldots,\theta_m$ and any $\varepsilon>0$,

$$(1.4) \quad \lim_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau\in[0,T] \;|\; ||\tau\alpha_k-\theta_k||<\varepsilon\;(1\le k\le m)\right\}>0$$

holds.

So far, three proofs are known for Theorem 1. Needless to say, one of them is Voronin's original proof, which is also reproduced in [64]. This proof is based on, besides the above facts, Pecherskiĭ's rearrangement theorem [185] in Hilbert spaces. The second proof is given by Good [41], which will be discussed in Section 16. Gonek was inspired by the idea of Good to write his thesis [40], in which he gave a modified version of Good's argument. The third is a more probabilistic proof due to Bagchi [3] [4]. Bagchi [3] is an unpublished thesis, but its contents are carefully expounded in [76]. A common feature of the work of Gonek and Bagchi is that they both used the approximation theorem of Mergelyan [141] [142].

Remark 2. Mergelyan's theorem asserts that, when K is a compact subset of $\mathbb{C}$, any $f\in H^c(K)$ can be approximated by polynomials uniformly on K. This is a complex analogue of the classical approximation theorem of Weierstrass.

Here we mention some pre-history. In 1914, Bohr and Courant [13] proved that, for any σ satisfying $1/2 < \sigma \le 1$, the set

$$\{\zeta(\sigma + i\tau) \mid \tau \in \mathbb{R}\}$$

is dense in $\mathbb{C}$. In the next year, Bohr [11] proved that the same result holds for $\log \zeta(\sigma + i\tau)$. These results are called *denseness theorems.*

Before obtaining his universality theorem, Voronin [204] discovered the following multi-dimensional analogue of the theorem of Bohr and Courant.

Theorem 2. (Voronin [204]) *For any σ satisfying $1/2 < \sigma \le 1$, the set*

$$\{(\zeta(\sigma + i\tau), \zeta'(\sigma + i\tau), \dots, \zeta^{(m-1)}(\sigma + i\tau)) \mid \tau \in \mathbb{R}\}$$

(*where $\zeta^{(k)}$ denotes the k-th derivative*) *is dense in $\mathbb{C}^m$.*

Remark 3. Actually Voronin proved a stronger result. For any $s = \sigma + it$ with $1/2 < \sigma \le 1$ and for any $h > 0$, the set

$$\{(\zeta(s + inh), \zeta'(s + inh), \dots, \zeta^{(m-1)}(s + inh)) \mid n \in \mathbb{N}\}$$

is dense in $\mathbb{C}^m$.

The universality theorem of Voronin may be regarded as a natural next step, bacause it is a kind of infinite-dimensional analogue of the theorem of Bohr and Courant, or a *denseness theorem in a function space.*

Another refinement of the denseness theorem of Bohr and Courant is the *limit theorem* on $\mathbb{C}$, due to Bohr and Jessen [14]. Recently, this theorem is usually formulated by probabilistic terminology. Let $\sigma > 1/2$. For any Borel subset $A \subset \mathbb{C}$, put

$$P_{T,\sigma}(A) = \frac{1}{T}\text{meas} \left\{ \tau \in [0, T] \mid \zeta(\sigma + i\tau) \in A \right\}.$$

This $P_{T,\sigma}$ is a probability measure on $\mathbb{C}$. Then the modern formulation of the limit theorem of Bohr and Jessen is that there exists a probability measure P_σ, to which $P_{T,\sigma}$ is convergent weakly as $T \to \infty$ (see [76, Chapter 4]).

Bagchi [3] proved an analogue of the above theorem of Bohr and Jessen on a certain function space, and used it in his alternative proof of the universality theorem. Therefore, to prove some universality-type theorem by Bagchi's method, it is necessary to obtain some *functional limit theorem* similar to that of Bagchi. There are indeed a lot of papers devoted to the proofs of various functional limit theorems, for the purpose of showing various universality theorems. However in the following

sections we do not explicitly mention this closely related topic of functional limit theorems, and for the details we refer to Laurinčikas [76] or J. Steuding [201]. The connection between the theory of universality and functional limit theorems is also discussed in the author's survey articles [136] [137].

After the publication of Voronin's theorem in 1975, now almost forty years have passed. Voronin's theorem attracted a lot of mathematicians, and hence, after Voronin, quite many papers on universality theory have been published. The aim of the present article is to survey the developments in this theory in these forty years. The developments in these years can be divided into three stages.

(I) The first stage: 1975 $\sim$ 1987.
(II) The second stage: 1996 $\sim$ 2007.
(III) The third stage: 2007 $\sim$ present.
In the next section we will give a brief discussion what were the main topics in each of these stages.

2. A ROUGH SKETCH OF THE HISTORY

(I) The first stage.
This is the first decade after Voronin's paper. The original impact of Voronin's discovery was still fresh. Mathematicians who were inspired by Voronin's paper tried to discuss various generalizations, analogies, refinements and so on. Here is the list of main results obtained in this decade.
• Alternative proofs (mentioned in Section 1).
• Generalizations to the case of Dirichlet L-functions, Dedekind zeta-functions etc.
• The joint universality.
• The strong universality (for Hurwitz zeta-functions).
• The strong recurrence.
• The discrete universality.
• The χ-universality.
• The hybrid universality.
• A quantitative result.
It is really amazing that many important aspects in universality theory, developed extensively in later decades, had already been introduced

and studied in this first decade. Unfortunately, however, many of those results were written only in the theses of Voronin [207], Gonek [40] and Bagchi [3], all of them remain unpublished. This situation is probably one of the reasons why in the next several years there were so few publications on universality.

(II) The second stage.

During several years around 1990, the number of publications concerning universality is very small. Of course it is not completely empty. For example, the book of Karatsuba and Voronin [64] was published in this period. But the author prefers to choose the year 1996 as the starting point of the second stage, because in this year the important book [76] of Laurinčikas was published. This is the first textbook which is mainly devoted to the theory of universality and related topics, and especially, provides the details of unpublished work of Bagchi. Thanks to the existence of this book, many mathematicians of younger generation can now easily go into the theory of universality. In fact, in this second stage, a lot of students of Laurinčikas started to publish their papers, and they formed the strong Lithuanian school.

The main topic in this decade was probably the attempt to extend the class of zeta and L-functions for which the universality property holds. It is now known that the universality property is valid for a rather wide class of zeta-functions. J. Steuding's lecture note [201] includes the exposition of this result, and also of many other results obtained after the publication of the book of Laurinčikas [76]. Therefore the publication year of this lecture note is appropriate to the end of the second stage.

(III) The third stage.

Now comes the third, present stage. The theory of universality is now developing into several new directions. The notions of

- the mixed universality,
- the composite universality,
- the ergodic universality,

were introduced recently. Other topics in universality theory have also been discussed extensively.

In the following sections, we will discuss more closely each topic in universality theory.

3. Generalization to zeta and L-functions with Euler products

Is it possible to prove the universality property for other zeta and L-functions? This is surely one of the most fundamental question. In

Section 1 we explained that a key point in the proof of Theorem 1 is the Euler product expression. Therefore we can expect the universality property for other zeta and L-functions which have Euler products.

The universality of the following zeta and L-functions were proved in the first decade.

• Dirichlet L-functions $L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$ (χ is a Dirichlet character; see Voronin [205]) in $D(1/2, 1)$,

• certain Dirichlet series with multiplicative coefficients (see Reich [188], Laurinčikas [68] [69] [70] [71] [72]),

• Dedekind zeta-functions $\zeta_F(s) = \sum_{\mathfrak{a}} N(\mathfrak{a})^{-s}$, where F is a number field, $\mathfrak{a}$ denotes a non-zero integral ideal, and $N(\mathfrak{a})$ its norm (see Voronin [207] [208], Gonek [40], Reich [189] [190]). Here, the universality can be proved in $D(1/2, 1)$ if F is an Abelian extension of $\mathbb{Q}$ (Gonek [40]), but for general $F(\neq \mathbb{Q})$, the proof is valid only in the narrower region $D(1 - d_F^{-1}, 1)$, where $d_F = [F : \mathbb{Q}]$. The reason is that the mean value estimate for $\zeta_F(s)$, applicable to the proof of universality, is known at present only in $D(1 - d_F^{-1}, 1)$.

Later, Laurinčikas also obtained universality theorems for Matsumoto zeta-functions[1] ([78] under a strong assumption), and for the zeta-function attached to Abelian groups ([82] [83]). Laurinčikas and Šiaučiūnas [122] proved the universality for the periodic zeta-function $\zeta(s, \mathfrak{A}) = \sum_{n=1}^{\infty} a_n n^{-s}$ (where $\mathfrak{A} = \{a_n\}_{n=1}^{\infty}$ is a multiplicative periodic sequence of complex numbers) when the technical condition

$$(3.1) \qquad \sum_{m=1}^{\infty} |a_{p^m}| p^{-m/2} \leq c$$

(with a certain constant $c < 1$) holds for any prime p. Schwarz, R. Steuding and J. Steuding [192] proved another universality theorem on certain general Euler products with conditions on the asymptotic behaviour of coefficients.

However, there was an obstacle when we try to generalize further. A typical class of L-functions with Euler products is that of automorphic L-functions. Let $g(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$ be a holomorphic normalized Hecke-eigen cusp form of weight κ with respect to $SL(2, \mathbb{Z})$ and let $L(s, g) = \sum_{n=1}^{\infty} a(n) n^{-s}$ be the associated L-function. The universality of $L(s, g)$ was first discussed by Kačėnas and Laurinčikas [53], but they showed the universality only under a very strong assumption.

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[1]This notion was first introduced by the author [133], to which the limit theorem of Bohr and Jessen was generalized. See also Kačinskaitė's survey article [56].

What was the obstacle? The asymptotic formula

$$(3.2) \qquad \sum_{p \leq x} \frac{1}{p} = \log \log x + a_1 + O(\exp(-a_2 \sqrt{\log x}))$$

(where a_1, a_2 are constants) is classically well known, and is used in the proof of Theorem 1. However the corresponding asymptotic formula for the sum $\sum_{p \leq x} |a(p)| p^{-1}$ is not known. To avoid this obstacle, Laurinčikas and the author [111] invented a new method of using (3.2), combined with the known asymptotic formula

$$(3.3) \qquad \sum_{p \leq x} |\widetilde{a}(p)|^2 = \frac{x}{\log x}(1 + o(1)),$$

where $\widetilde{a}(p) = a(p) p^{-(\kappa-1)/2}$. This method is called the *positive density method*. Modifying Bagchi's argument by virtue of this positive density method, one can show the following unconditional result.

Theorem 3. (Laurinčikas and Matsumoto [111]) *The universality holds for $L(s, g)$ in the region $D(\kappa/2, (\kappa + 1)/2)$.*

The positive density method was then applied to prove the universality for more general class of L-functions; certain Dirichlet series with multiplicative coefficients (Laurinčikas and Šleževičienė [126]), L-functions attached to new forms with respect to congruence subgroups (Laurinčikas, Matsumoto and J. Steuding [116]), L-functions attached to a cusp form with character (Laurinčikas and Macaitienė [105]), and a certain subclass of the Selberg class[2] (J. Steuding [197]). J. Steuding extended his result further in his lecture note [201]. He introduced a wide class $\widetilde{S}$ of L-functions defined axiomatically and proved the universality for elements of $\widetilde{S}$. The class $\widetilde{S}$, now sometimes called the Steuding class, is not included in the Selberg class, but is a subclass of the class of Matsumoto zeta-functions.

Since the Shimura-Taniyama conjecture has been established, we now know that the L-function $L(s, E)$ attached to a non-singular elliptic curve E over $\mathbb{Q}$ is an L-function attached to a new form. Therefore the universality for $L(s, E)$ is included in [116]. The universality of positive powers of $L(s, E)$ was studied in Garbaliauskienė and Laurinčikas [31].

Mishou [144] [145] used a variant of the positive density method to show the universality for Hecke L-functions of algebraic number fields

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[2]The notion of Selberg class was introduced by Selberg [193]. For basic definitions and results in this theory, consult a survey [63] of Kaczorowski and Perelli, or J. Steuding [201].

in the region $D(1-d_F^{-1}, 1)$. Lee [127] showed that, under the assumption of a certain density estimate of the number of zeros, it is possible to prove the universality for Hecke L-functions in the region $D(1/2, 1)$. The universality for Artin L-functions was proved by Bauer [7] by a different method, based on Voronin's original idea.

Let g_1 and g_2 be cusp forms. The universality for the Rankin-Selberg L-function $L(s, g_1 \otimes g_1)$ was shown by the author [134], and for $L(s, g_1 \otimes g_2)$ $(g_1 \neq g_2)$ was by Nagoshi [165] (both in the narrower region $D(3/4, 1)$). The latter proof is based on the above general result of J. Steuding [197] [201]. The universality of symmetric m-th power L-functions $(m \leq 4)$ and their Rankin-Selberg L-functions was studied by Li and Wu [129].

Another general result obtained by the positive density method is the following theorem, which is an extension of the result of J. Steuding [197].

Theorem 4. (Nagoshi and J. Steuding [166]) *Let $\varphi(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ be a Dirichlet series belonging to the Selberg class. Denote the degree of $\varphi(s)$ by d_φ, and put $\sigma_\varphi = \max\{1/2, 1 - d_\varphi^{-1}\}$. Assume that*

$$(3.4) \qquad \sum_{p \leq x} |a(p)|^2 = \frac{x}{\log x}(\lambda + o(1))$$

holds with a certain positive constant λ. Then the universality holds for $\varphi(s)$ in the region $D(\sigma_\varphi, 1)$.

Remark 4. When φ is the Riemann zeta-function, the formula (3.4) (with $\lambda = 1$) is nothing but the prime number theorem. Therefore we may say that the positive density method enables us to prove the universality for zeta-functions with Euler products, provided an asymptotic formula of the prime-number-theorem type is known.

Let h be a Hecke-eigen Maass form, and let $L(s, h)$ be the associated L-function. The universality for $L(s, h)$ was proved by Nagoshi [163] [164]. In the proof of Theorem 3 in [111], Deligne's estimate (Ramanujan's conjecture) $|\widetilde{a}(p)| \leq 2$ is essentially used. Assumptions of the same type are required in Steuding's general result [197] [201] and also in Theorem 4. Since Ramanujan's conjecture for Maass forms has not yet been proved, Nagoshi in his first paper [163] assumed the validity of Ramanujan's conjecture for h to show the universality. Then in the second paper [164] he succeeded to remove this assumption, by invoking the asymptotic formula for the fourth power mean of the coefficients due to M. Ram Murty.

Theorem 5. (Nagoshi [164]) *The universality holds for $L(s, h)$ in the region $D(1/2, 1)$.*

Another important class of zeta-functions which have Euler products is the class of Selberg zeta-functions. In this case, instead of the prime-number-theorem type of results, the prime geodesic theorem plays an important role. Let

$$\mathcal{D} = \{d \in \mathbb{N} \mid d \equiv 0 \text{ or } 1 \ (\text{mod } 4), d \text{ is not a square}\}.$$

For each $d \in \mathcal{D}$, let $h^+(d)$ be the number of inequivalent primitive quadratic forms of discriminant d, and $\varepsilon(d) = (u(d) + v(d)\sqrt{d})/2$, where $(u(d), v(d))$ is the fundamental solution of the Pell equation $u^2 - v^2 d = 4$. Then, the prime geodesic theorem for $SL(2, \mathbb{Z})$ implies

$$(3.5) \qquad \sum_{\substack{d \in \mathcal{D} \\ \varepsilon(d)^2 \leq x}} h^+(d) = \int_0^x \frac{dt}{\log t} + O(x^\alpha)$$

with a certain $\alpha < 1$.

Theorem 6. (Drungilas, Garunkštis and Kačėnas [23]) *Let $Z(s)$ be the Selberg zeta-function attached to $SL(2, \mathbb{Z})$. If (3.5) holds, then the universality holds for $Z(s)$ in the region $D(1/2 + \alpha/2, 1)$.*

As for the value of α, it is known that one can take $\alpha = 71/102 + \varepsilon$ for any $\varepsilon > 0$ (Cai [19]). It is conjectured that one could take $\alpha = 1/2 + \varepsilon$. If the conjecture is true, then Theorem 6 implies that $Z(s)$ has the universality property in $D(3/4, 1)$. The paper [23] includes a discussion which suggests that $D(3/4, 1)$ is the widest possible region where the universality for $Z(s)$ holds.

4. The joint universality for zeta and L-functions with Euler products

The results presented in the previous sections give approximation properties of some single zeta or L-function. Here we discuss simultaneous approximations by several zeta or L-functions.

Let $K_1, \ldots, K_r$ be compact subsets of $D(a, b)$ with connected complements, and $f_j \in H_0^c(K_j)$ $(1 \leq j \leq r)$. If Dirichlet series $\varphi_1(s), \ldots, \varphi_r(s)$ satisfy

$$(4.1) \qquad \liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] \ \middle| \ \sup_{s \in K_j} |\varphi_j(s + i\tau) - f_j(s)| < \varepsilon \right.$$

$$\left. (1 \leq j \leq r) \right\} > 0$$

for any $\varepsilon > 0$, we call the *joint universality* holds for $\varphi_1(s), \ldots, \varphi_r(s)$ in the region $D(a, b)$. The joint universality for Dirichlet L-functions was already obtained in the first decade by Voronin [206] [207] [208], Gonek [40], and Bagchi [3] [4], independently of each other:

Theorem 7. (Voronin, Gonek, Bagchi) *Let $\chi_1, \ldots, \chi_r$ be pairwise non-equivalent Dirichlet characters, and $L(s, \chi_1), \ldots, L(s, \chi_r)$ the corresponding Dirichlet L-functions. Then the joint universality holds for $L(s, \chi_1), \ldots, L(s, \chi_r)$ in the region $D(1/2, 1)$.*

To prove such a theorem of simultaneous approximations, it is obviously necessary that the behaviour of L-functions appearing in the theorem should be "independent" of each other. In the situation of Theorem 7, this is embodied by the orthogonality relation of Dirichlet characters, which is essentially used in the proof.

The joint universality for Dedekind zeta-functions was studied by Voronin [207] [208]. Bauer's work [7] mentioned in the preceding section actually proves a joint universality theorem on Artin L-functions, in the region $D(1 - (2d_F)^{-1}, 1)$. Lee [128] extended the region to $D(1/2, 1)$ under the assumption of a certain density estimate.

Let g_j $(1 \leq j \leq r)$ be multiplicative arithmetic functions. Laurinčikas [74] considered the joint universality of the associated Dirichlet series $\sum_{n=1}^{\infty} g_j(n) n^{-s}$ $(1 \leq j \leq r)$. In this case, the "independence" condition is given by the following matrix condition. Let $P_1, \ldots, P_k$ $(k \geq r)$ be certain sets of prime numbers, with the condition that $\sum_{p \leq x, p \in P_i} p^{-1}$ satisfies a good asymptotic formula, and assume that $g_j(n)$ is a constant g_{jl} on the set P_l $(1 \leq j \leq r, 1 \leq l \leq k)$. Laurinčikas [74] proved a joint universality theorem under the condition that the rank of the matrix $(g_{jl})_{1 \leq j \leq r, 1 \leq l \leq k}$ is equal to r.

Laurinčikas frequently used various matrix conditions to obtain joint universality theorems. A joint universality theorem on Matsumoto zeta-functions under a certain matrix condition was proved in [79]. A joint universality for automorphic L-functions under a certain matrix condition was discussed in [112].

A matrix condition naturally appears in the joint universality theory of periodic zeta-functions (see [107] [103]). Let $\mathfrak{A}_j = \{a_{jn}\}_{n=1}^{\infty}$ be a multiplicative periodic sequence (whose least period we denote by k_j) of complex numbers, and $\zeta(s, \mathfrak{A}_j)$ the associated periodic zeta-function $(1 \leq j \leq r)$. Let k be the least common multiple of $k_1, \ldots, k_r$. Define the matrix $A = (a_{jl})_{j,l}$, where $1 \leq j \leq r$ and $1 \leq l \leq k$, $(l, k) = 1$. Then Laurinčikas and Macaitienė [103] proved the joint universality for $\zeta(s, \mathfrak{A}_1), \ldots, \zeta(s, \mathfrak{A}_r)$ in the region $D(1/2, 1)$, if we assume $\mathrm{rank}(A) = r$ and a technical condition similar to (3.1).

Using the positive density method, it is possible to prove a joint universality theorem for twisted automorphic L-functions. Let $g(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi i n z}$ be a holomorphic normalized Hecke-eigen cusp form, $\chi_1, \ldots, \chi_r$ be pairwise non-equivalent Dirichlet characters, and $L(s, g, \chi_j) = \sum_{n=1}^{\infty} a(n)\chi_j(n)n^{-s}$ the associated χ_j-twisted L-function.

Theorem 8. (Laurinčikas and Matsumoto [113]) *The joint universality holds for* $L(s, g, \chi_1), \ldots, L(s, g, \chi_r)$ *in the region* $D(\kappa/2, (\kappa + 1)/2)$.

To prove this result, we need a prime number theorem for $a(p)$ in arithmetic progressions, that is

$$(4.2) \qquad \sum_{\substack{p \leq x \\ p \equiv h \,(\mathrm{mod}\, q)}} |\widetilde{a}(p)|^2 = \frac{1}{\varphi(q)} \frac{x}{\log x}(1 + o(1)),$$

where $(h, q) = 1$ and $\varphi(q)$ is Euler's totient function.

J. Steuding [201, Theorem 12.8] generalized Theorem 8 to the Steuding class $\widetilde{S}$. A joint version of [126] was given by Šleževičienė [194]. A joint universality theorem on L-functions of elliptic curves, under a certain matrix condition, was given by Garbaliauskienė, Kačinskaitė and Laurinčikas [29].

Let $\varphi_j(s) = \sum_{n=1}^{\infty} a_j(n)n^{-s}$ $(j = 1, 2)$ be elements of the Selberg class. The following orthogonality conjecture of Selberg [193] is well known: if $\varphi_1(s), \varphi_2(s)$ are primitive, then

$$(4.3) \qquad \sum_{p \leq x} \frac{a_1(p)\overline{a_2(p)}}{p} = \begin{cases} \log \log x + O(1) & \text{if} \quad \varphi_1 = \varphi_2, \\ O(1) & \text{otherwise.} \end{cases}$$

Inspired by this conjecture, J. Steuding [201, Section 12.5] proposed:

Conjecture 1. (J. Steuding) *Any finite collection of distinct primitive functions in the Selberg class is jointly universal* [3].

Towards this conjecture, recent progress has been mainly due to Mishou. In [153], Mishou proved the following. Consider two strips $D_1 = D(1/2, 3/4)$ and $D_2 = D(3/4, 1)$. Let K_j be a compact subset of D_j and $f_j \in H_0^c(K_j)$ $(j = 1, 2)$. Then

$$(4.4) \qquad \liminf_{T \to \infty} \frac{1}{T}\mathrm{meas}\Big\{\tau \in [0, T] \ \Big| \ \sup_{s \in K} |\zeta(s + i\tau) - f_1(s)| < \varepsilon,$$

$$|L(s + i\tau, g) - f_2(s)| < \varepsilon \Big\} > 0$$

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[3] J. Steuding also mentioned the expectation that any two functions φ_1, φ_2 in the Selberg class would be jointly universal if and only if $\sum_{p \leq x} \frac{a_1(p)\overline{a_2(p)}}{p} = O(1)$. However H. Nagoshi, and then H. Mishou, pointed out that there are counter examples to this statement.

holds, where $L(s, g)$ is the automorphic L-functions attached to a certain cusp form g. In Mishou [154], this result was generalized to the case of several L-functions belonging to the Selberg class.

The result (4.4) is weaker than the joint universality, because K_1 and K_2 are in the strips disjoint to each other. In [156], Mishou succeeded in removing this restriction to obtain the following theorem. Let g, g_1, g_2 be holomorphic normalized Hecke-eigen cusp forms.

Theorem 9. (Mishou [156]) (i) $\zeta(s)$ *and* $L(s, g)$ *are jointly universal in* $D(1/2, 1)$.

(ii) *If* g_1 *and* g_2 *are distinct, then* $L(s, g_1)$ *and* $L(s, g_2)$ *are jointly universal in* $D(1/2, 1)$.

(iii) $\zeta(s)$ *and* $L(s, \mathrm{sym}^2 g)$ *are jointly universal in* $D(2/3, 1)$.

(iv) *If* g_1 *and* g_2 *are distinct, then* $\zeta(s)$ *and* $L(s, g_1 \otimes g_2)$ *are jointly universal in* $D(3/4, 1)$.

(v) $L(s, g_1)$ *and* $L(s, g_1 \otimes g_2)$ *are jointly universal in* $D(3/4, 1)$.

Remark 5. The universality theorem for $L(s, g \otimes g)$ by the author [134] (mentioned in Section 3) was proved in $D(3/4, 1)$, but the above theorem of Mishou especially implies that the universality for $L(s, g \otimes g)$ is valid in the wider region $D(2/3, 1)$.

A remarkable feature of Mishou's method is that it does not depend on any periodicity of coefficients. His proof is based on orthogonality relations of Fourier coefficients. Theorem 9 is a strong support to Conjecture 1.

5. THE STRONG UNIVERSALITY

So far we have talked about universality only for zeta and L-functions with Euler products. However already in the first decade, the universality for zeta-functions without Euler products was also studied. Let $0 < \alpha \leq 1$. The Hurwitz zeta-function with the parameter α is defined by $\zeta(s, \alpha) = \sum_{n=0}^{\infty} (n + \alpha)^{-s}$, and does not have the Euler product (except for the special cases $\alpha = 1, 1/2$). The known universality theorem for $\zeta(s, \alpha)$ is as follows.

Theorem 10. (Bagchi [3], Gonek [40]) *Let* K *be a compact subset of* $D(1/2, 1)$ *with connected complement, and* $f \in H^c(K)$. *Then for any* $\varepsilon > 0$,

$$(5.1) \quad \liminf_{T \to \infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] \; \middle| \; \sup_{s \in K} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon \right\} > 0$$

holds, provided α *is transcendental or rational* $(\neq 1, 1/2)$.

To prove this theorem, when α is transcendental, we use the fact that the elements of the set

$$\{\log(n + \alpha) \mid n \in \mathbb{N}_0\}$$

are linearly independent over $\mathbb{Q}$. On the other hand, when $\alpha = a/b$ is rational, then in view of the formula

$$(5.2) \qquad \zeta(s, a/b) = \frac{b^s}{\varphi(b)} \sum_{\chi \,(\mathrm{mod}\, b)} \overline{\chi}(a) L(s, \chi),$$

we can reduce the problem to the joint universality of Dirichlet L-functions, so we can apply Theorem 7. The case of algebraic irrational α is still open.

A remarkable point is that, in the statement of Theorem 10, we do not assume that the target function $f(s)$ is non-vanishing on K. This is a big difference from Theorem 1, and when a universality-type theorem holds without the non-vanishing assumption, we call it a *strong universality theorem.*

Strong universality has an important application to the theory of zero-distribution. Let $a < \sigma_1 < \sigma_2 < b$, and let

$$N(t; \sigma_1, \sigma_2; \varphi) = \#\{\rho \in \mathbb{C} \mid \sigma_1 \leq \Re\rho \leq \sigma_2, 0 \leq \Im\rho \leq T, \varphi(\rho) = 0\}$$

for a function φ. (In the above definition, zeros are counted with multiplicity.) Then we have the following consequence.

Theorem 11. *If $\varphi(s)$ is strongly universal in the region $D(a, b)$, then there exists a positive constant C for which*

$$(5.3) \qquad N(T; \sigma_1, \sigma_2; \varphi) \geq CT$$

holds for any σ_1, σ_2 satisfying $a < \sigma_1 < \sigma_2 < b$.

Proof. Let δ be a small positive number, $0 < \varepsilon < \delta$, $\sigma_1 < \sigma_0 < \sigma_2$, and $K = \{s \in \mathbb{C} \mid |s - \sigma_0| < \delta\}$. We choose δ so small that $K \subset D(a, b)$. We apply the strong universality to this K, $f(s) = s - \sigma_0$ and ε to obtain that the set of real numbers τ such that

$$\sup_{|s-\sigma_0| \leq \delta} |\varphi(s + i\tau) - f(s)| < \varepsilon$$

is of positive lower density. Then, for such τ,

$$\sup_{|s-\sigma_0| \leq \delta} |\varphi(s + i\tau) - f(s)| < \delta = \inf_{|s-\sigma_0| = \delta} |f(s)|.$$

Therefore by Rouché's theorem we see that $f(s) + (\varphi(s + i\tau) - f(s)) = \varphi(s + i\tau)$ has the same number of zeros as that of $f(s)$ in the region $|s - \sigma_0| < \delta$, but the latter is obviously 1. That is, for each τ in the above set, $\varphi(s)$ has one zero in $|s - (\sigma_0 + i\tau)| < \delta$. $\qquad \square$

Corollary 1. *If α is transcendental or rational ($\neq 1, 1/2$), then*

$$C_1 T \leq N(T; \sigma_1, \sigma_2; \zeta(s, \alpha)) \leq C_2 T$$

holds for any $1/2 < \sigma_1 < \sigma_2 < 1$.

As for the upper bound part of this corollary, see [102, Chapter 8, Theorem 4.10].

Further topics on the application of universality to the distribution of zeros will be discussed in Section 9 and Section 15.

Now strong unversality theorems are known for many other zeta-functions. The Estermann zeta-function is defined by

$$(5.4) \qquad E\left(s, \frac{k}{l}, \alpha\right) = \sum_{n=1}^{\infty} \sigma_\alpha(n) \exp\left(2\pi i \frac{k}{l} n\right) n^{-s},$$

where k and l are coprime integers and $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$. The strong universality for $E(s, k/l, \alpha)$ was studied in Garunkštis, Laurinčikas, Šleževičienė and J. Steuding [36]. The method is to write $E(s, k/l, \alpha)$ as a linear combination of $E(s, \chi, \alpha) = \sum_{n=1}^{\infty} \sigma_\alpha(n) \chi(n) n^{-s}$, and apply a joint universality theorem for $E(s, \chi, \alpha)$ which follows from Šleževičienė [194].

The Lerch zeta-function is defined by

$$\zeta(s; \alpha, \lambda) = \sum_{n=0}^{\infty} e^{2\pi i \lambda n} (n + \alpha)^{-s},$$

where $0 < \alpha \leq 1$ and λ is real. When λ is an integer, then $\zeta(s; \alpha, \lambda)$ reduces to the Hurwitz zeta-function, so we may assume $0 < \lambda < 1$. The strong universality for $\zeta(s; \alpha, \lambda)$ was proved by Laurinčikas [77] when α is transcendental. The case when α is rational was discussed by Laurinčikas [80]. See also the textbook [102] of Laurinčikas and Garunkštis.

Let $\mathfrak{B} = \{b_n\}$ is a periodic sequence, not necessarily multiplicative. The universality for periodic zeta-functions $\zeta(s, \mathfrak{B}) = \sum_{n=1}^{\infty} b_n n^{-s}$ was first studied by J. Steuding [198] (see [201, Chapter 11]). Kaczorowski [60] proved that there exists a constant $c_0 = c_0(\mathfrak{B})$ such that the universality holds for $\zeta(s, \mathfrak{B})$, provided that

$$\max_{s \in K} \Im(s) - \min_{s \in K} \Im(s) \leq c_0.$$

This result is a consequence of the hybrid joint universality theorem of Kaczorowski and Kulas [61] (see Section 15). Javtokas and Laurinčikas [49] [50] studied the strong universality for periodic Hurwitz

zeta-function

$$\zeta(s, \alpha, \mathfrak{B}) = \sum_{n=0}^{\infty} b_n (n + \alpha)^{-s}.$$

They proved that the strong universality holds for $\zeta(s, \alpha, \mathfrak{B})$, when α is transcendental.

A more general situation was considered by Laurinčikas, Schwarz and J. Steuding [121]. Let $\{\lambda_n\}_{n=1}^{\infty}$ be an increasing sequence of real numbers, linearly independent over $\mathbb{Q}$, and $\lambda_n \to \infty$ as $n \to \infty$. Define the general Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n s)$, which is assumed to be convergent absolutely in the region $\sigma > \sigma_a$. Put $r(x) = \sum_{\lambda_n \leq x} 1$ and $c_n = a_n \exp(-\lambda_n \sigma_a)$. We suppose

(i) $f(s)$ cannot be represented as an Euler product,

(ii) $f(s)$ can be continued meromorphically to $\sigma > \sigma_1$, and holomorphic in $D(\sigma_1, \sigma_a)$,

(iii) For $\sigma > \sigma_1$ it holds that $f(s) = O(|t|^{\alpha})$ with some $\alpha > 0$,

(iv) For $\sigma > \sigma_1$ it holds that

$$\int_{-T}^{T} |f(\sigma + it)|^2 dt = O(T),$$

(v) $r(x) = Cx^{\kappa} + O(1)$ with a $\kappa > 1$,

(vi) $|c_n|$ is bounded and $\sum_{\lambda_n \leq x} |c_n|^2 = \theta r(x)(1 + o(1))$ with a $\theta > 0$.

Then we have

Theorem 12. (Laurinčikas, Schwarz and J. Steuding [121]) *If $f(s)$ satisfies all the above conditions, then the strong universality holds for $f(s)$ in the region $D(\sigma_1, \sigma_a)$.*

In Section 3 and Section 5, we have seen a lot of examples of zeta and L-functions, with or without Euler products, for which the universality property holds. How general is this property expected to hold? The following conjecture predicts that any "reasonable" Dirichlet series would satisfy the universality property.

Conjecture 2. (Yu. V. Linnik and I. A. Ibragimov) *All functions given by Dirichlet series and meromorphically continuable to the left of the half-plane of absolute convergence are universal in some suitable region.*

Remark 6. Actually this conjecture has trivial counter-examples. For example, let $a_n = 1$ if n is a power of 2 and $a_n = 0$ otherwise. The series $\sum_{n=1}^{\infty} a_n n^{-s}$ can be continued to $(2^s - 1)^{-1}$, which is obviously not universal. Therefore some additional condition should be added to make the rigorous statement of the above conjecture.

6. The joint strong universality

The joint universality property is also possible to be valid among zeta-functions without Euler products. The first attempt to this direction is a series of papers of Laurinčikas and the author [110] [114] [115] on the joint universality for Lerch zeta-functions. Here, a matrix condition again appears. Let $\lambda_1, \ldots, \lambda_r$ be rational numbers. Write $\lambda_j = a_j/q_j$, $(a_j, q_j) = 1$, and let k be the least common multiple of $q_1, \ldots, q_r$. Define the matrix $L = (\exp(2l\pi i\lambda_j))_{1 \leq l \leq k, 1 \leq j \leq r}$. Then, by virtue of a variant of the positive density method, we have

Theorem 13. (Laurinčikas and Matsumoto [110] [114]) *Suppose that $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and that* $\mathrm{rank}(L) = r$. *Then the joint strong universality holds for $\zeta(s, \alpha_1, \lambda_1), \ldots, \zeta(s, \alpha_r, \lambda_r)$ in the region $D(1/2, 1)^4$.*

The Lerch zeta-function $\zeta(s, \alpha, \lambda)$ with rational λ is a special case of periodic Hurwitz zeta-functions. The joint strong universality of periodic Hurwitz zeta-functions was first studied by Laurinčikas [88] [89]. Let $\mathfrak{B}_j = \{a_{nj}\}_{n=0}^{\infty}$ be periodic sequences with period k_j, k be the least common multiple of $k_1, \ldots, k_r$, and define $B = (a_{jl})_{1 \leq j \leq r, 1 \leq l \leq k}$.

Theorem 14. (Laurinčikas [89]) *If α is transcendental and $\mathrm{rank}(B) = r$, then the joint strong universality holds for $\zeta(s, \alpha, \mathfrak{B}_1), \ldots, \zeta(s, \alpha, \mathfrak{B}_r)$ in the region $D(1/2, 1)$.*

Next in [51] [124], the joint universality for $\zeta(s, \alpha_j, \mathfrak{B}_j)$ $(1 \leq j \leq r)$ was discussed. Some matrix conditions were still assumed in [51], but finally in [124], a joint universality theorem free from any matrix condition was obtained.

Theorem 15. (Laurinčikas and Skerstonaitė [124]) *Assume that the elements of the set*

$$(6.1) \qquad \{\log(n + \alpha_j) \mid 1 \leq j \leq r, \ n \in \mathbb{N}_0\}$$

are linearly independent over $\mathbb{Q}$. Then the joint strong universality holds for $\zeta(s, \alpha_1, \mathfrak{B}_1), \ldots, \zeta(s, \alpha_r, \mathfrak{B}_r)$ in the region $D(1/2, 1)$.

In the case $\mathfrak{B}_j = \{1\}_{n=0}^{\infty}$ for $1 \leq j \leq r$ (that is, the case of Hurwitz zeta-functions), the above result was already given in Laurinčikas [91].

In [90] [125], a more general joint strong universality for $\zeta(s, \alpha_j, \mathfrak{B}_{jl})$ $(1 \leq j \leq r, 1 \leq l \leq l_j$ with $l_j \in \mathbb{N})$ was discussed under certain matrix conditions.

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[4]In [110], the theorem is stated under the weaker assumption that $\alpha_1, \ldots, \alpha_r$ are transcendental, but (as is pointed out in [114]) this assumption should be replaced by the algebraic independence over $\mathbb{Q}$.

Laurinčikas [84] [85] and [38] (with Genys) studied the joint strong universality for general Dirichlet series, under the same assumptions as in Theorem 12 and a certain matrix condition.

Now return to the problem of the joint universality for Lerch zeta-functions. Theorem 15 implies, especially, that the assumptions of Theorem 13 can now be replaced by just the linear independence of (6.1).

Is the assumption (6.1) indeed weaker than the assumptions of Theorem 13? The answer is yes, and the following result of Mishou [152] gives an example: Let α_1, α_2 be two transcendental numbers, $0 < \alpha_1, \alpha_2 < 1$, $\alpha_1 \neq \alpha_2$, and $\alpha_2 \in \mathbb{Q}(\alpha_1)$. Then Mishou [152] proved that the joint strong universality holds for $\zeta(s, \alpha_1)$ and $\zeta(s, \alpha_2)$. Dubickas [24] extended Mishou's result to the case of r transcendental numbers, which is also an extension of [91].

Let m_1, m_2, m_3 be relatively prime positive integers (≥ 2), and $\lambda_0 = n_3/m_3$ (with another integer n_3). Nakamura [167] proved the joint strong universality for

$$(6.2) \qquad \left\{ \zeta\left(s, \alpha, \lambda_0 + \frac{n_1}{m_1} + \frac{n_2}{m_2}\right) \;\middle|\; 0 \leq n_1 < m_1, 0 \leq n_2 < m_2 \right\}$$

when α is transcendental. He pointed out that various other types of joint universality can be deduced from the above.

In [115] and in Laurinčikas [92], the joint universality of $\zeta(s, \alpha_j, \lambda_{j\mu_j})$ ($1 \leq j \leq r$, $1 \leq \mu_j \leq m_j$, where m_j is some positive integer) is discussed. Write $\lambda_{j\mu_j} = a_{j\mu_j}/q_{j\mu_j}$, $(a_{j\mu_j}, q_{j\mu_j}) = 1$, and let k_j be the least common multiple of $q_{j\mu_j}$ ($1 \leq \mu_j \leq m_j$). Define $L_j = (\exp(2l\pi i \lambda_{j\mu_j})_{1 \leq l \leq k_j, 1 \leq \mu_j \leq m_j})$. Then in [92] it is shown that if the elements of the set (6.1) are linearly independent over $\mathbb{Q}$, and $\mathrm{rank}(L_j) = k_j$ ($1 \leq j \leq r$), then the joint strong universality holds for $\zeta(s, \alpha_j, \lambda_{j\mu_j})$.

How about the joint universality for Lerch zeta-functions when the parameter λ is not rational? Nakamura [167] noted that the joint strong universality for (6.2) also holds if we replace λ_0 by any non-rational real number. Also, Nakamura [168] extended the idea in [114] to obtain the following more general result.

Theorem 16. (Nakamura [168]) *If $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, then for any real numbers $\lambda_1, \ldots, \lambda_r$ the joint strong universality holds for $\zeta(s, \alpha_1, \lambda_1), \ldots, \zeta(s, \alpha_r, \lambda_r)$ in the region $D(1/2, 1)$.*

On the other hand, Mishou [155] proved the following measure-theoretic result.

Theorem 17. (Mishou [155]) *There exists a subset $\Lambda \subset [0,1)^r$ whose r-dimensional Lebesgue measure is 1, and for any transcendental real*

number α and $(\lambda_1, \ldots, \lambda_r) \in \Lambda$, the joint strong universality holds for $\zeta(s, \alpha, \lambda_1), \ldots, \zeta(s, \alpha, \lambda_r)$ in the region $D(1/2, 1)$.

Moreover in the same paper Mishou gives the following two explicit descriptions of $\lambda_1, \ldots, \lambda_r$ $(0 \leq \lambda_j < 1)$ for which the above joint universality holds;

(i) $\lambda_1, \ldots, \lambda_r$ are algebraic irrational and $1, \lambda_1, \ldots, \lambda_r$ are linearly independent over $\mathbb{Q}$,

(ii) $\lambda_1 = \exp(u_1), \ldots, \lambda_r = \exp(u_r)$ where $u_1, \ldots, u_r$ are distinct rational numbers.

Mishou's proof is based on two classical discrepancy estimates due to W. M. Schmidt and H. Niederreiter. These results lead Mishou to propose the following conjecture.

Conjecture 3. (Mishou [155]) *The joint strong universality holds for $\zeta(s, \alpha, \lambda_1), \ldots, \zeta(s, \alpha, \lambda_r)$ in the region $D(1/2, 1)$, for any transcendental real number α $(0 < \alpha < 1)$ and any distinct real numbers $\lambda_1, \ldots, \lambda_r$ $(0 \leq \lambda_j < 1)$.*

For single zeta or L-functions, there is Conjecture 2, which asserts that universality would hold for any "reasonable" Dirichlet series. As for the joint universality, the situation is much more complicated. If there is some relation among several Dirichlet series, then the behaviour of those Dirichlet series cannot be independent of each other, so the joint universality among them cannot be expected. Nakamura [168] pointed out that some collections of Lerch zeta-functions cannot be jointly universal, because of the inversion formula among Lerch zeta-functions. In the same paper Nakamura introduced the generalized Lerch zeta-function of the form

$$(6.3) \qquad \zeta(s, \alpha, \beta, \gamma, \lambda) = \sum_{n=0}^{\infty} \frac{e^{2\pi i \lambda n}}{(n+\alpha)^{s-\gamma}(n+\beta)^{\gamma}},$$

and showed that, under suitable choices of parameters the joint strong universality sometimes holds, and sometimes does not hold.

In [169], Nakamura considered more general series

$$(6.4) \qquad \zeta(s, \alpha, \mathcal{C}) = \sum_{n=0}^{\infty} \frac{c(n)}{(n+\alpha)^s},$$

where $\mathcal{C} = \{c(n)\}_{n=0}^{\infty}$ is a bounded sequence of complex numbers, and using it, constructed some counter-examples to the joint universality. The proof in [169] is based on a non-denseness property and a limit theorem on a certain function space.

The above results of Nakamura suggest that it is not easy to find a suitable joint version of Conjecture 2.

Remark 7. It is to be noted that there is the following simple principle of producing the joint universality. Let $\varphi \in H(D(\sigma_1, \sigma_2))$, and assume that φ is universal. Let K be a compact subset of $D(\sigma_1, \sigma_2)$ with connected complement, $\lambda_1, \ldots, \lambda_r$ be complex numbers, and $K_j = \{s + \lambda_j \mid s \in K\}$ $(1 \leq j \leq r)$. Assume these K_j's are disjoint. Then $\varphi_j(s) = \varphi(s + \lambda_j)$ $(1 \leq j \leq r)$ are jointly universal. If φ is strongly universal, then φ_j's are jointly strongly universal. This is the *shifts universality principle* of Kaczorowski, Laurinčikas and J. Steuding [62].

7. The universality for multiple zeta-functions

An important generalization of the notion of zeta-functions is multiple zeta-functions, defined by certain multiple sums. The history of the theory of multiple zeta-functions goes back to the days of Euler, but extensive studies started only in 1990s.

The problem of searching for universality theorems on multiple zeta-functions was first proposed by the author [135]. In this paper the author wrote that one accessible problem would be the universality of Barnes multiple zeta-functions

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} (w_1 n_1 + \cdots + w_r n_r + \alpha)^{-s}$$

(where $w_1, \ldots, w_r, \alpha$ are parameters). Nakamura [168] pointed out that his $\zeta(s, \alpha, \beta, \gamma, \lambda)$ (see (6.3)) includes the twisted Barnes double zeta-function of the form

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{e^{2\pi i \lambda (n_1 + n_2)}}{(n_1 + n_2 + \alpha)^s}$$

as a special case. Therefore [168] includes a study of universality for Barnes double zeta-functions. More generally, the series $\zeta(s, \alpha, \mathcal{C})$ (see (6.4)) studied in Nakamura [169] includes twisted Barnes r-ple zeta-functions (for any r).

The Euler-Zagier r-ple sum is defined by

$$(7.1) \qquad \sum_{n_1 > n_2 > \cdots n_r \geq 1} n_1^{-s_1} n_2^{-s_2} \cdots n_r^{-s_r},$$

where $s_1, \ldots, s_r$ are complex variables. Nakamura [170] considered the universality of the following generalization of (7.1) of Hurwitz-type:

$$(7.2) \qquad \zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r) =$$
$$\sum_{n_1 > n_2 > \cdots n_r \geq 0} (n_1 + \alpha_1)^{-s_1}(n_2 + \alpha_2)^{-s_2} \cdots (n_r + \alpha_r)^{-s_r},$$

where $0 < \alpha_j \leq 1$ ($1 \leq j \leq r$). Nakamura's results suggest that universality for the multiple zeta-function (7.2) is connected with the zero-free region. One of his main results is as follows.

Theorem 18. (Nakamura [170]) *Let $\Re s_2 > 3/2$, $\Re s_j \geq 1$ ($3 \leq j \leq r$). Assume α_1 is transcendental, and $\zeta_{r-1}(s_2, \ldots, s_r; \alpha_2, \ldots, \alpha_r) \neq 0$. Then the strong universality holds for $\zeta_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r)$ as a function in s_1 in the region $D(1/2, 1)$.*

In [173], Nakamura considered a generalization of Tornheim's double sum of Hurwitz-type and proved the strong universality for it.

8. THE MIXED UNIVERSALITY

In Section 4 we discussed the joint universality among zeta or L-functions with Euler products. Then in Section 6 we considered the joint universality for those without Euler products. Is it possible to combine these two directions to obtain certain joint universality results between two (or more) zeta-functions, one of which has Euler products and the other does not? The first affirmative answers are due to Sander and J. Steuding [191], and to Mishou [149]. The work of Sander and J. Steuding will be discussed later in Section 15. Here we state Mishou's theorem.

Theorem 19. (Mishou [149]) *Let K_1, K_2 be compact subsets of $D(1/2, 1)$ with connected complements, and $f_1 \in H_0^c(K_1)$, $f_2 \in H^c(K_2)$. Then, for any $\varepsilon > 0$, we have*

$$(8.1) \qquad \liminf_{T \to \infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] \ \bigg| \ \sup_{s \in K_1} |\zeta(s + i\tau) - f_1(s)| < \varepsilon, \right.$$
$$\left. \sup_{s \in K_2} |\zeta(s + i\tau, \alpha) - f_2(s)| < \varepsilon \right\} > 0,$$

provided α is transcendental.

This type of universality is now called the *mixed universality*. The essential point of the proof of this theorem is the fact that the elements of the set
$$\{\log(n + \alpha) \mid n \in \mathbb{N}_0\} \cup \{\log p \mid p : \text{prime}\}$$

are linearly independent over $\mathbb{Q}$.

Mishou's theorem was generalized to the periodic case by Kačinskaitė and Laurinčikas [58][5]. Let $\mathfrak{A}$ be a multiplicative periodic sequence satisfying (3.1) and $\mathfrak{B}$ a (not necessarily multiplicative) periodic sequence. They proved that if α is transcendental, then the mixed universality holds for $\zeta(s, \mathfrak{A})$ and $\zeta(s, \alpha, \mathfrak{B})$ in the region $D(1/2, 1)$.

Laurinčikas [93] proved a further generalization. Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_{r_1}$ be multiplicative periodic sequences (with inequalities similar to (3.1)) and $\mathfrak{B}_1, \ldots, \mathfrak{B}_{r_2}$ be periodic sequences. Then, under certain matrix conditions, the mixed universality for $\zeta(s, \mathfrak{A}_{j_1})$ $(1 \leq j_1 \leq r_1)$ and $\zeta(s, \alpha_{j_2}, \mathfrak{B}_{j_2})$ $(1 \leq j_2 \leq r_2)$ holds, provided $\alpha_1, \ldots, \alpha_{r_2}$ are algebraically independent over $\mathbb{Q}$.

Now mixed universality theorems are known for many pairs of zeta or L-functions.

• The Riemann zeta-function and several periodic Hurwitz zeta-functions (Genys, Macaitienė, Račkauskienė and Šiaučiūnas [39]),

• The Riemann zeta-function and several Lerch zeta-functions (Laurinčikas and Macaitienė [106]),

• Several Dirichlet L-functions and several Hurwitz zeta-functions (Janulis and Laurinčikas [47]),

• Several Dirichlet L-functions and several periodic Hurwitz zeta-functions (Janulis, Laurinčikas, Macaitienė and Šiaučiūnas [48]),

• An automorphic L-function and several periodic Hurwitz zeta-functions (Laurinčikas, Macaitienė and Šiaučiūnas [109], Macaitienė [131], Pocevičienė and Šiaučiūnas [186], Laurinčikas and Šiaučiūnas [123]).

9. The strong recurrence

In Section 5 we mentioned that the strong universality implies the existence of many zeros in the region where universality is valid (Theorem 11). This immediately gives the following corollary:

Corollary 2. *The Riemann zeta-function $\zeta(s)$ cannot be strongly universal in the region $D(1/2, 1)$.*

Because if $\zeta(s)$ is strongly universal, then by Theorem 11 we have $N(T; \sigma_1, \sigma_2; \zeta) \geq CT$ for $1/2 < \sigma_1 < \sigma_2 < 1$, which contradicts with the known zero-density estimate $N(T; \sigma_1, \sigma_2; \zeta) = o(T)$.

The same conclusion can be shown for many other zeta or L-functions, for which some suitable zero-density estimate is known; or, under the assumption of the analogue of the Riemann hypothesis.

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[5]This paper was published in 2011, but was already completed in 2007.

On the other hand, if the Riemann hypothesis is true, then $\zeta(s)$ has no zero in the region $D(1/2, 1)$. Therefore we can choose $f(s) = \zeta(s)$ in Theorem 1 to obtain

$$(9.1) \quad \liminf_{T \to \infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] \; \middle| \; \sup_{s \in K} |\zeta(s + i\tau) - \zeta(s)| < \varepsilon \right\} > 0.$$

This is called the *strong recurrence* property of $\zeta(s)$. Bagchi discovered that the converse implication is also true.

Theorem 20. (Bagchi [3]) *The Riemann hypothesis for $\zeta(s)$ is true if and only if* (9.1) *holds in the region $D(1/2, 1)$.*

Bagchi himself extended this result to the case of Dirichlet L-functions in [4] [5]. The same type of result in terms of Beurling zeta-functions was given by R. Steuding [203].

Remark 8. It is obvious that the notion of the strong recurrence is closely connected with the notion of almost periodicity. Bohr [12] proved that the Riemann hypothesis for $L(s, \chi)$ with a non-principal character χ is equivalent to the almost periodicity of $L(s, \chi)$ in the region $\Re s > 1/2$. Recently Mauclaire [138] [139] studied the universality in a general framework from the viewpoint of almost periodicity.

Nakamura [171] proved that, if $d_1 = 1, d_2, \ldots, d_r$ are algebraic real numbers which are linearly independent over $\mathbb{Q}$, then the joint universality of the form

$$(9.2) \quad \liminf_{T \to \infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] \; \middle| \; \sup_{s \in K_j} |\zeta(s + id_j\tau) - f_j(s)| < \varepsilon \right. $$
$$ \left. (1 \le j \le r) \right\} > 0$$

holds, where K_j are compact subsets of $D(1/2, 1)$ with connected complement and $f_j \in H_0^c(K_j)$. A key point of Nakamura's proof is the fact that the elements of the set $\{\log p^{d_j} \mid p : \text{prime}, 1 \le j \le r\}$ are linearly independent over $\mathbb{Q}$, which follows from Baker's theorem in transcendental number theory. From the above result it is immediate that

$$(9.3)$$
$$\liminf_{T \to \infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] \; \middle| \; \sup_{s \in K} |\zeta(s + i\tau) - \zeta(s + id\tau)| < \varepsilon \right\} > 0$$

holds if d is algebraic irrational. Nakamura also proved in the same paper that (9.3) is valid for almost all $d \in \mathbb{R}$. (Note that (9.3) for $d = 0$ is (9.1), hence the Riemann hypothesis.)

Nakamura's paper sparked off the interest in this direction of research; Pańkowski [182] proved that (9.3) holds for all (algebraic and transcendental) irrational d, using the six exponentials theorem in transcendental number theory. On the other hand, Garunkštis [34] and Nakamura [172], independently, claimed that (9.3) holds for all non-zero rational. However their arguments included a gap, which was partially filled by Nakamura and Pańkowski [176]. The present situation is:

Theorem 21. (Garunkštis, Nakamura, Pańkowski) *The inequality* (9.3) *holds if d is irrational, or $d = a/b$ is non-zero rational with $(a,b) = 1$, $|a - b| \neq 1$.*

See also Mauclaire [140], and Nakamura and Pańkowski [178]. It is to be noted that the argument of Garunkštis [34] and Nakamura [172] is correct for $\log \zeta(s)$, so

(9.4)
$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0,T] \ \middle| \ \sup_{s\in K} |\log \zeta(s + i\tau) - \log \zeta(s + id\tau)| < \varepsilon \right\} > 0$$

can be shown for any non-zero $d \in \mathbb{R}$. If (9.4) would be valid for $d = 0$, it would imply the Riemann hypothesis.

The strong recurrence property can be shown for more general zeta and L-functions. Some of the aforementioned papers actually consider not only the Riemann zeta-function, but also Dirichlet L-functions. A generalization to a subclass of the Selberg class was discussed by Nakamura [174]. The case of Hurwitz zeta-functions has been studied by Garunkštis and Karikovas [35] and Karikovas and Pańkowski [65].

10. The weighted universality

Laurinčikas [75] considered a weighted version of the universality for $\zeta(s)$. Let $w(t)$ be a positive-valued function of bounded variation defined on $[T_0, \infty)$ (where $T_0 > 0$), satisfying that the variation on $[a, b]$ does not exceed $cw(a)$ with a certain $c > 0$ for any subinterval $[a, b] \subset [T_0, \infty)$. Define

$$U(T, w) = \int_{T_0}^{T} w(t)dt,$$

and assume that $U(T, w) \to \infty$ as $T \to \infty$.

We further assume the following property of $w(t)$, connected with ergodic theory. Let $X(\tau, \omega)$ be any ergodic process defined on a certain probability space Ω, $\tau \in \mathbb{R}$, $\omega \in \Omega$, $E(|X(\tau, \omega)|) < \infty$, and sample

paths are Riemann integrable almost surely on any finite interval. Assume that

$$(10.1) \quad \frac{1}{U(T,w)} \int_{T_0}^{T} w(\tau) X(t+\tau,\omega) d\tau = E(X(0,\omega)) + o((1+|t|)^\alpha)$$

almost surely for any $t \in \mathbb{R}$, with an $\alpha > 0$, as $T \to \infty$. Denote by $I(A)$ the indicator function of the set A.

Theorem 22. (Laurinčikas [75]) *Suppose that $w(t)$ satisfies all the above conditions. Let K be a compact subset of $D(1/2, 1)$ with connected complement, $f \in H_0^c(K)$. Then*

$$(10.2) \quad \liminf_{T\to\infty} \frac{1}{U(T,w)} \int_{T_0}^{T} w(\tau)$$

$$\times I\left(\left\{\tau \in [T_0, T] \;\middle|\; \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon\right\}\right) d\tau > 0$$

holds for any $\varepsilon > 0$.

In the course of Bagchi's proof of the universality theorem, there is a point where the Birkhoff-Khinchin theorem

$$(10.3) \quad \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T} X(\tau,\omega) d\tau = E(X(0,\omega))$$

in ergodic theory is used. This is the motivation of Laurinčikas [75]; clearly (10.1) is a generalization of (10.3).

Laurinčikas [78] generalized Theorem 22 to the case of Matsumoto zeta-functions. Weighted universality theorems for L-functions $L(s, E)$ of elliptic curves over $\mathbb{Q}$ were reported by Garbaliauskienė [26] [27].

11. THE DISCRETE UNIVERSALITY

In the previous sections, we discussed the behaviour of zeta or L-functions when the imaginary part τ of the variable is moving continuously. However, we can also obtain a kind of universality theorems when τ only moves discretely. We already mentioned in Section 1 that Voronin's multi-dimensional denseness theorem is valid in this sense (see Remark 3).

The first *discrete universality* theorem is due to Reich [189] on Dedekind zeta-functions. Let F be a number field and $\zeta_F(s)$ the associated Dedekind zeta-function.

Theorem 23. (Reich [189]) *Let K be a compact subset of the region $D(1 - \max\{2, d_F\}^{-1}, 1)$ with connected complement, and $f \in H_0^c(K)$.*

Then, for any real $h \neq 0$ and any $\varepsilon > 0$,

$$(11.1) \quad \liminf_{N \to \infty} \frac{1}{N} \# \left\{ n \leq N \;\middle|\; \sup_{s \in K} |\zeta_F(s + ihn) - f(s)| < \varepsilon \right\} > 0.$$

The joint discrete universality theorem for Dirichlet L-functions was given in Bagchi [3]. He also obtained the discrete universality for $\zeta(s, \alpha)$ when α is rational, for which Sander and J. Steuding [191] gave a different approach.

Kačinskaitė [54] (see also [56]) proved a discrete universality theorem for Matsumoto zeta-functions under the condition that $\exp(2\pi k/h)$ is irrational for any non-zero integer k. Ignatavičiūtė [46] reported that certain discrete universality and certain joint discrete universality hold for Lerch zeta-functions, provided $\exp(2\pi/h)$ is rational.

As can be seen in the above examples, an interesting point on discrete universality is that the arithmetic nature of the parameter h plays a role.

The discrete universality for L-functions $L(s, E)$ of elliptic curves was first studied in [30] under the assumption that $\exp(2\pi k/h)$ is irrational for any non-zero integer k. However this condition was then removed:

Theorem 24. (Garbaliauskienė, Genys and Laurinčikas [28]) *The discrete universality holds for $L(s, E)$ for any real $h \neq 0$ in the region $D(1, 3/2)$.*

The same type of result can be shown, more generally, for L-functions attached to new forms. When $\exp(2\pi k/h)$ is irrational for any non-zero integer k, this was done in Laurinčikas, Matsumoto and J. Steuding [117].

The discrete universality for periodic zeta-functions was studied in Kačinskaitė, Javtokas and Šiaučiūnas [57] and Laurinčikas, Macaitienė and Šiaučiūnas [108], while the case of periodic Hurwitz zeta-functions was discussed by Laurinčikas and Macaitienė [104]. The result in [104] especially includes the discrete universality of the Hurwitz zeta-function $\zeta(s, \alpha)$ when α is transcendental. Laurinčikas [100] further proved that if the set

$$\{\log(m + \alpha) \mid m \in \mathbb{N}_0\} \cup \{2\pi/h\}$$

is linearly independent over $\mathbb{Q}$, then the discrete universality holds for $\zeta(s, \alpha)$. See also [18]. A joint version is studied in Laurinčikas [101].

Macaitienė [130] obtained a discrete universality theorem for general Dirichlet series.

Theorem 25. (Macaitienė [130]) *Let $f(s)$ be general Dirichlet series as in Theorem 12, and further suppose that λ_n are algebraic numbers*

and $\exp(2\pi/h) \in \mathbb{Q}$. *Then the discrete universality holds for* $f(s)$ *in the region* $D(\sigma_1, \sigma_a)$.

The discrete analogue of mixed universality can also be considered. This direction was first studied by Kačinskaitė [55]. Consider the case $\exp(2\pi/h) \in \mathbb{Q}$. Write $\exp(2\pi/h) = a/b$ with $a, b \in \mathbb{Z}$ and $(a, b) = 1$. Denote by P_h the set of all prime numbers appearing as a prime factor of a or b. Define the modified Dirichlet L-function $L_h(s, \chi)$ by removing all Euler factors corresponding to primes in P_h, that is

$$L_h(s, \chi) = \prod_{p \notin P_h} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Theorem 26. (Kačinskaitė [55]) *Let* K_1, K_2 *be compact subsets of* $D(1/2, 1)$ *with connected complements, and* $f_1 \in H_0^c(K_1)$, $f_2 \in H^c(K_2)$. *If* α *is transcendental,* $\mathfrak{B}$ *is a periodic sequence and* $\exp(2\pi/h) \in \mathbb{Q}$ *as above, then*

$$(11.2) \qquad \liminf_{N \to \infty} \frac{1}{N} \# \left\{ n \le N \;\middle|\; \sup_{s \in K_1} |L_h(s + ihn, \chi) - f_1(s)| < \varepsilon, \right.$$

$$\left. \sup_{s \in K_2} |\zeta(s + ihn, \alpha, \mathfrak{B}) - f_2(s)| < \varepsilon \right\} > 0$$

for any $\varepsilon > 0$ [6] .

Buivytas and Laurinčikas [16] proved that if the set

$$\{\log p \mid p : \text{prime}\} \cup \{\log(m + \alpha) \mid m \in \mathbb{N}_0\} \cup \{2\pi/h\}$$

is linearly independent over $\mathbb{Q}$, then the discrete mixed universality holds for $\zeta(s)$ and $\zeta(s, \alpha)$, that is,

$$(11.3) \qquad \liminf_{N \to \infty} \frac{1}{N} \# \left\{ n \le N \;\middle|\; \sup_{s \in K_1} |\zeta(s + ihn) - f_1(s)| < \varepsilon, \right.$$

$$\left. \sup_{s \in K_2} |\zeta(s + ihn, \alpha) - f_2(s)| < \varepsilon \right\} > 0.$$

In (11.2) and (11.3), the shifting parameter h is common to the both of relevant zeta (or L)-functions. Buivytas and Laurinčikas [17] studied the case when the parameter for $\zeta(s)$ is different from the parameter for $\zeta(s, \alpha)$.

We will encounter a rather different type of discrete universality theorems in Section 18.

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[6]The statement in [55] is given for $L(s, \chi)$, but her argument is valid not for $L(s, \chi)$, but for $L_h(s, \chi)$.

12. The χ-universality

The main point of Voronin's universality theorem is the existence of τ, the imaginary part of the complex variable s, satisfying a certain approximation condition. This is the common feature of all universality theorems mentioned in the previous sections. However there is another type of universality, which is concerned with the existence of a character satisfying certain approximation conditions.

The first theorem in this direction is as follows.

Theorem 27. (Bagchi [3], Gonek [40], Eminyan [25]) *Let K be a compact subset of $D(1/2, 1)$ with connected complement, and $f \in H_0^c(K)$. Let Q be an infinite set of positive integers. Then for any $\varepsilon > 0$,*

$$(12.1) \qquad \liminf_{\substack{q \to \infty \\ q \in Q}} \frac{1}{\varphi(q)} \# \left\{ \chi(\bmod\ q) \ \middle|\ \sup_{s \in K} |L(s, \chi) - f(s)| < \varepsilon \right\} > 0$$

holds, provided Q is one of the following[7]:

(i) Q is the set of all prime numbers;

(ii) Q is the set of positive integers of the form $q = p_1^{a_1} \cdots p_r^{a_r}$ $(a_1, \ldots, a_r \in \mathbb{N} \cup \{0\})$, where $\{p_1, \ldots, p_r\}$ is a fixed finite set of prime numbers.

This type of results is called the χ-*universality*, or *the universality in χ-aspect*. The universality for Hecke L-functions of number fields in χ-aspect was discussed by Mishou and Koyama [157], and by Mishou [147] [148].

Let χ_d be a real Dirichlet character with discriminant d. Another interesting direction of research is the universality for $L(s, \chi_d)$ in d-aspect. This direction was studied in a series of papers by Mishou and Nagoshi [158] [159] [160] [161]. Let Λ^+ (resp. Λ^-) be the set of all positive (resp. negative) discriminants, and $\Lambda^+(X)$ (resp. $\Lambda^-(X)$) be the set of discriminants d satisfying $0 < d \leq X$ (resp. $-X \leq d < 0$).

Theorem 28. (Mishou and Nagoshi [158]) *Let Ω be a simply connected domain in $D(1/2, 1)$ which is symmetric with respect to the real axis. Let $f(s)$ be holomorphic and non-vanishing on Ω, and positive-valued*

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[7]Bagchi [3, Theorem 5.3.11] proved the case (i). In the statement of Gonek [40, Theorem 5.1] there is no restriction on Q, but Mishou [146] pointed out that condition (ii) is necessary to verify Gonek's proof. Eminyan [25] studied the special case $r = 1$ of (ii).

on $\Omega \cap \mathbb{R}$. Let K be a compact subset of Ω. Then, for any $\varepsilon > 0$,

(12.2)
$$\liminf_{X \to \infty} \frac{1}{\#\Lambda^{\pm}(X)} \# \left\{ d \in \Lambda^{\pm}(X) \; \middle| \; \sup_{s \in K} |L(s, \chi_d) - f(s)| < \varepsilon \right\} > 0$$

holds.

This theorem especially implies that for any $s \in D(1/2, 1) \setminus \mathbb{R}$, the set $\{L(s, \chi_d) \mid d \in \Lambda^{\pm}\}$ is dense in $\mathbb{C}$, and for any real number σ with $1/2 < \sigma < 1$, the set $\{L(\sigma, \chi_d) \mid d \in \Lambda^{\pm}\}$ is dense in the set of positive real numbers $\mathbb{R}_+$.

In the same paper Mishou and Nagoshi also studied the situation on the line $\Re s = 1$, and proved that the set $\{L(1, \chi_d) \mid d \in \Lambda^{\pm}\}$ is dense in $\mathbb{R}_+$. Therefore we can deduce denseness results on class numbers of quadratic fields. Let $h(d)$ be the class number of $\mathbb{Q}(\sqrt{d})$, and when $d > 0$, let $\varepsilon(d)$ be the fundamental unit of $\mathbb{Q}(\sqrt{d})$. Then the above result implies that both the sets

$$\left\{ \frac{h(d) \log \varepsilon(d)}{\sqrt{d}} \; \middle| \; d \in \Lambda^+ \right\}, \qquad \left\{ \frac{h(d)}{\sqrt{d}} \; \middle| \; d \in \Lambda^- \right\}$$

are dense in $\mathbb{R}_+$.

In [159] [161], the same type of problem for prime discriminants are studied. As an application, in [161] Mishou and Nagoshi gave a quantitative result on a problem of Ayoub, Chowla and Walum on certain character sums [2]. In [160] Mishou and Nagoshi gave some conditions equivalent to the Riemann hypothesis from their viewpoint.

The universality in d-aspect for Hecke L-functions of class group characters for imaginary quadratic fields are studied by Mishou [151]. In [150], Mishou considered cubic characters associated with the field $\mathbb{Q}(\sqrt{-3})$, and proved a universality theorem for associated Hecke L-functions in cubic character aspect.

13. Other applications

So far we mentioned applications of universality to the theory of distribution of zeros (Section 5), to the Riemann hypothesis (Section 9), and to algebraic number theory (Section 12). Those applications, however, do not exhaust the potentiality of universality theory. In this section we discuss other applications of universality.

In Section 1 we mentioned that Voronin, before proving his universality theorem, obtained the multi-dimensional denseness theorem (Theorem 2) of $\zeta(s)$ and its derivatives. However, now, we can say that Theorem 2 is just an immediate consequence of the universality

theorem (see [76, Theorem 6.6.2]). Moreover, from Theorem 2 it is easy to obtain the following functional-independence property of $\zeta(s)$.

Theorem 29. *Let $f_l : \mathbb{C}^m \to \mathbb{C}$ $(0 \leq l \leq n)$ be continuous functions, and assume that the equality*

$$(13.1) \qquad \sum_{l=0}^{n} s^l f_l(\zeta(s), \zeta'(s), \ldots, \zeta^{(m-1)}(s)) = 0$$

holds for all s. Then $f_l \equiv 0$ for $0 \leq l \leq n$.

This type of result was first noticed by Voronin himself ([206] [209]); see also [76, Theorem 6.6.3].

When f_l's are polynomials, then (13.1) is an algebraic differential equation. Therefore Theorem 29 in this case implies that $\zeta(s)$ does not satisfy any non-trivial algebraic differential equation. This property was already noticed by Hilbert in his famous address [45] at the 2nd International Congress of Mathematicians (Paris, 1900). Theorem 29 is a generalization of this algebraic-independence property.

Similarly to the case of $\zeta(s)$, if a Dirichlet series $\varphi(s)$ is universal, it is easy to prove the theorems analogous to Theorem 2 and Theorem 29 for $\varphi(s)$.

An application of the universality to the problem on Dirichlet polynomials was done by Andersson [1]. He used the universality theorem to show that several conjectures proposed by Ramachandra [187] and Balasubramanian and Ramachandra [6], on lower bounds of certain integrals of Dirichlet polynomials, are false.

The universality property was applied even in physics; see Gutzwiller [44], Bitar, Khuri and Ren [10].

14. The general notion of universality

The main theme of the present article is the universality for zeta and L-functions. However, the notion of universality was first introduced in mathematics, under a very different motivation.

The first discovery of the universality phenomenon is usually attributed to M. Fekete (1914/15, reported in [181]), who proved that there exists a real power series $\sum_{n=1}^{\infty} a_n x^n$ such that, for any continuous $f : [-1, 1] \to \mathbb{R}$ with $f(0) = 0$ we can choose positive integers $m_1, m_2, \ldots$ for which

$$\sum_{n=1}^{m_k} a_n x^n \to f(x) \qquad (k \to \infty)$$

holds uniformly on $[-1, 1]$. The proof is based on Weierstrass' approximation theorem.

G. D. Birkhoff [9] proved that there exists an entire function $\psi(z)$ such that, for any entire function $f(z)$, we can choose complex numbers $a_1, a_2, \ldots$ for which $\psi(z + a_k) \to f(z)$ (as $k \to \infty$) uniformly in any compact subset of $\mathbb{C}$.

The terminology "universality" was first used by Marcinkiewicz [132]. Various functions satisfying some property similar to those discovered by Fekete and Birkhoff are known. However, before the work of Voronin [205], all of those functions were constructed very artificially. So far the class of zeta and L-functions is the only "natural" class of functions for which the universality property can be proved. For the more detailed history of this general notion of universality, see Grosse-Erdmann [43] and Steuding [201, Appendix] [8].

It is to be noted that the real origin of the whole theory is Riemann's theorem that a conditionally convergent series can be convergent (or divergent) to any value after some suitable rearrangement. In fact, Fekete's result may be regarded as an analogue of Riemann's theorem for continuous functions, while Pecherskiĭ's theorem [185] (mentioned in Section 1 as an essential tool in Voronin's proof) gives an analogue of Riemann's theorem in Hilbert spaces.

A very general definition of universality was proposed by Grosse-Erdmann [42] [43].

Definition 1. Let X, Y be topological spaces, W be a non-empty closed subset of Y, and $T_\tau : X \to Y$ $(\tau \in I)$ be a family of mappings with the index set I. We call $x \in X$ *universal with respect to W* if the closure of the set $\{T_\tau(x) \mid \tau \in I\}$ contains W.

Let K be as in Theorem 1, $X = Y = H(K^\circ)$, where K° is the interior of K. Then obviously $H_0^c(K) \subset H(K^\circ)$. Put $W = \overline{H_0^c(K)}$ (the topological closure of $H_0^c(K)$ in the space $H(K^\circ)$). Define T_τ by $T_\tau(f(z)) = f(z + i\tau)$ for $f \in H(K^\circ)$. Then Theorem 1 implies that any element of $H_0^c(K)$ can be approximated by some suitable element of $\{T_\tau(\zeta) \mid \tau \in \mathbb{R}\}$. Therefore Theorem 1 asserts that the Riemann zeta-function $\zeta(s)$ is universal with respect to $\overline{H_0^c(K)}$ in the sense of Definition 1.

The notion of joint universality can also be formulated in this general setting.

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[8] In the same Appendix, Steuding mentioned a p-adic version of Fekete's theorem, which was originally proved in [196].

Definition 2. Let $X, Y_1, \ldots, Y_r$ be topological spaces, and $T_\tau^{(j)} : X \to Y_j$ ($\tau \in I$, $1 \leq j \leq r$) be families of mappings. We call $x_1, \ldots, x_r \in X$ *jointly universal* if the set $\{(T_\tau^{(1)}(x_1), \ldots, T_\tau^{(r)}(x_r)) \mid \tau \in I\}$ is dense in $Y_1 \times \cdots \times Y_r$.

Remark 9. The case $r = 1$ of Definition 2 is the case $W = Y$ of Definition 1.

In Section 1 we mentioned that the Kronecker-Weyl approximation theorem (see Remark 1) is used in the proof of Theorem 1. We can see that the Kronecker-Weyl theorem itself implies a certain universality phenomenon. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, and consider the situation when $X = \mathbb{R}$, $Y_1 = \cdots = Y_r = S^1$ in Definition 2. Define $T_\tau^{(1)} = \cdots = T_\tau^{(r)} = T_\tau : \mathbb{R} \to S^1$ by $T_\tau(x) = e^{2\pi i \tau x}$. Then the Kronecker-Weyl theorem implies that, if $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$ are linearly independent over $\mathbb{Q}$, then the orbit of

$$(T_\tau(\alpha_1), \ldots, T_\tau(\alpha_r)) = (e^{2\pi i \tau \alpha_1}, \ldots, e^{2\pi i \tau \alpha_r})$$

is dense in $S^1 \times \cdots \times S^1$. Therefore $\alpha_1, \ldots, \alpha_r$ are jointly universal.

The above observation shows that both the Voronin theorem and the Kronecker-Weyl theorem express certain universality properties. Is it possible to combine these two universality theorems? The answer is yes, and we will discuss this matter in the next section.

15. The hybrid universality

The first affirmative answer to the question raised at the end of Section 14 was given by Gonek [40], and a slightly general result was later obtained by Kaczorowski and Kulas [61].

Theorem 30. (Gonek [40], Kaczorowski and Kulas [61]) *Let K be a compact subset of $D(1/2, 1)$, $f_1, \ldots, f_r \in H_0^c(K)$, $\chi_1, \ldots, \chi_r$ be pairwise non-equivalent Dirichlet characters, $z > 0$, and $(\theta_p)_{p \leq z}$ be a sequence of real numbers indexed by prime numbers up to z. Then, for any $\varepsilon > 0$,*

(15.1)

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] \ \middle| \ \max_{1 \leq j \leq r} \sup_{s \in K} |L(s + i\tau, \chi_j) - f_j(s)| < \varepsilon, \right.$$
$$\left. \max_{p \leq z} \left\| \tau \frac{\log p}{2\pi} - \theta_p \right\| < \varepsilon \right\} > 0$$

holds.

The combination of the universality of Voronin type and of Kronecker-Weyl type is now called the *hybrid universality*. The above theorem is therefore an example of the hybrid joint universality.

Pańkowski [183] proved that the second inequality in (15.1) can be replaced by $\max_{1 \leq k \leq m} ||\tau \alpha_k - \theta_k|| < \varepsilon$, where $\alpha_1, \ldots, \alpha_m$ are real numbers which are linearly independent over $\mathbb{Q}$, and $\theta_1, \ldots, \theta_m$ are arbitrary real numbers. This is exactly the same inequality as in the Kronecker-Weyl theorem (see Remark 1).

In the same paper Pańkowski remarked that the same statement can be shown for more general L-functions which have Euler products. The hybrid joint universality for some zeta-functions without Euler products was discussed in Pańkowski [184].

Hybrid universality theorems are quite useful in applications. Gonek [40] used his hybrid universality theorem to show the joint universality for Dedekind zeta-functions of Abelian number fields (mentioned in Section 3). The key fact here is that those Dedekind zeta-functions can be written as products of Dirichlet L-functions. On the other hand, the aim of Kaczorowski and Kulas [61] was to study the distribution of zeros of linear combinations of the form $\sum P_j(s)L(s, \chi_j)$, where $P_j(s)$ are Dirichlet polynomials. Kaczorowski and Kulas applied Theorem 30 to show a theorem[9], similar to Theorem 11, for such linear combinations.

Sander and J. Steuding [191] also considered the universality for sums, or products of Dirichlet L-functions. In particular they proved the joint universality for Hurwitz zeta-functions $\zeta(s, a/q)$ $(1 \leq a \leq q)$ under a certain condition on target functions. Hurwitz zeta-functions usually do not have Euler products, but when $a = q$ (and when q is even and $a = q/2$) the corresponding Hurwitz zeta-function is essentially the Riemann zeta-function and hence has the Euler product. Therefore the result of Sander and J. Steuding is an example of mixed universality (see Section 8).

In the paper of Kaczorowski and Kulas [61], the coefficients $P_j(s)$ of linear combinations are Dirichlet polynomials. Nakamura and Pańkowski [177] considered a more general situation when $P_j(s)$ are Dirichlet series. Their general statement is as follows.

Theorem 31. (Nakamura and Pańkowski [177]) *Let $P_1(s), \ldots, P_r(s)$ $(r \geq 2)$ be general Dirichlet series, not identically vanishing, absolutely convergent in $\Re s > 1/2$. Moreover assume that at least two of those are non-vanishing in $D(1/2, 1)$. Let $L_1(s), \ldots, L_r(s)$ be hybridly jointly universal in the above sense. Then $L(s) = \sum_{j=1}^{r} P_j(s)L_j(s)$ is strongly universal in $D(1/2, 1)$.*

As a corollary, by Theorem 11 we find $N(T; \sigma_1, \sigma_2; L) \geq CT$ for the above $L(s)$, for any σ_1, σ_2 satisfying $1/2 < \sigma_1 < \sigma_2 < 1$.

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[9]This theorem was later sharpened by Ki and Lee [66] by using the method of mean motions [52] [15].

In the above theorem $L(s)$ is a linear form of L_j's, but in [175] [180], Nakamura and Pańkowski obtained more general statements; they considered polynomials of L_j's whose coefficients are general Dirichlet series, and proved results similar to Theorem 31. Note that when coefficients of polynomials are constants, such a result was already given in Kačinskaitė, J. Steuding, Šiaučiūnas and Šleževičienė [59].

Many important zeta and L-functions have such polynomial expressions. Consequently, Nakamura and Pańkowski succeeded in proving the inequalities like (5.3) on the distribution of zeros of those zeta or L-functions, such as zeta-functions attached to symmetric matrices (in the theory of prehomogeneous vector spaces), Estermann zeta-functions, Igusa zeta-functions associated with local Diophantine problems, spectral zeta-functions associated with Laplacians on Riemannian manifolds, Epstein zeta-functions (see [179]), and also various multiple zeta-functions (of Euler-Zagier, of Barnes, of Shintani, of Witten and so on).

16. Quantitative results

It is an important question how to obtain quantitative information related with universality. For example, let

(16.1)
$$d(\zeta, f, K, \varepsilon) = \liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] \ \middle| \ \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\}.$$

Theorem 1 asserts that $d(\zeta, f, K, \varepsilon)$ is positive; but how to evaluate this value? Or, how to find the smallest value of τ (which we denote by $\tau(\zeta, f, K, \varepsilon)$) satisfying the inequality $\sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon$? Voronin's proof gives no information on these questions, because in the course of the proof Voronin used Pecherskiĭ's rearrangement theorem, which is ineffective.

The first attempt to get a quantitative version of the universality theorem is due to Good [41]. The fundamental idea of Good is to combine the argument of Voronin with the method of Montgomery [162], in which Montgomery studied large values of $\log \zeta(s)$. Instead of Pecherskiĭ's theorem, Good used convexity arguments (Hadamard's three circles theorem and the Hahn-Banach theorem). Also Koksma's quantitative version of Weyl's criterion is invoked. The statements of Good's main results are quite complicated, but it includes a quantitative version of the discrete universality theorem for $\zeta(s)$.

Remark 10. Actually Good stated the discrete universality for $\log \zeta(s)$. From the universality for $\log \zeta(s)$, the universality for $\zeta(s)$ itself can be

immediately deduced by exponentiation. A proof of the universality for $\log \zeta(s)$ by Voronin's original method is presented in the book of Karatsuba and Voronin [64, Chapter VII].

Good's idea was further pursued by Garunkštis [33], who obtained a more explicit quantitative result when K is small. His main theorem is still rather complicated, but as a typical special case, he showed the following inequalities. Let $K = K(r)$ be as in Section 1.

Theorem 32. (Garunkštis [33]) *Let $0 < \varepsilon \leq 1/2$, $f \in H(K(0.05))$, and assume $\max_{s \in K(0.06)} |f(s)| \leq 1$. Then we have*

$$d(\log \zeta, f, K(0.0001), \varepsilon) \geq \exp(-1/\varepsilon^{13}),$$
$$\tau(\log \zeta, f, K(0.0001), \varepsilon) \leq \exp\exp(10/\varepsilon^{13}).$$

Besides the above work of Good and Garunkštis, there are various different approaches toward quantitative results. Laurinčikas [81] pointed out that quantitative information on the speed of convergence of a certain functional limit theorem would give a quantitative result on the universality for Lerch zeta-functions. J. Steuding [199] [200] considered the quantity defined by replacing liminf on (1.3) by limsup, and discussed its upper bounds.

The author pointed out in [137] that from Theorem 2, by comparing the Taylor expansions of $\zeta(s+i\tau)$ and $f(s)$, it is possible to deduce a certain weaker version of universal approximation. On the other hand, a quantitative version of Theorem 2 was shown by Voronin himself [210]. Combining these two ideas, a quantitative version of weak universal approximation theorem was obtained in Garunkštis, Laurinčikas, Matsumoto, J. & R. Steuding [37].

A nice survey on the effectivization problem is given in Laurinčikas [99].

17. The universality for derived functions

When some function $\varphi(s)$ satisfies the universality property, a natural question is to ask whether functions derived from $\varphi(s)$ by some standard operations, such as $\varphi'(s)$, $\varphi(s)^2$, $\exp(\varphi(s))$ etc, also satisfy the universality property, or not.

We already mentioned the universality of $\log \zeta(s)$ (see Remark 10). Concerning the derivatives, Bagchi [4] proved that mth derivatives of Dirichlet L-functions $L^{(m)}(s, \chi)$ ($m \in \mathbb{N}$) are strongly universal. Laurinčikas [73] studied the universality for $(\zeta'/\zeta)(s)$, and then considered the same problem for $L(s, g)$ (for a cusp form g) in [86] [87]. The universality for derivatives of L-functions of elliptic curves was studied

by Garbaliauskienė and Laurinčikas [32], and its discrete analogue was discussed by Belovas, Garbaliauskienė and Ivanauskaitė [8].

After these early attempts, Laurinčikas [94] (see also [97]) formulated a more general framework of *composite universality*. Let F be an operator $F : H(D(1/2, 1)) \to H(D(1/2, 1))$. Laurinčikas [94] considered when $F(\zeta(s))$ has the universality property.

A simple affirmative case is the Lipschitz class $\mathrm{Lip}(\alpha)$. We call F belongs to $\mathrm{Lip}(\alpha)$ when

1) for any polynomial $q = q(s)$ and any compact $K \subset D(1/2, 1)$, there exists $q_0 \in F^{-1}\{q\}$ such that $q_0(s) \neq 0$ on K, and

2) for any compact $K \subset D(1/2, 1)$ with connected complement, there exist $c > 0$ and a compact $K_1 \subset D(1/2, 1)$ with connected complement, for which

$$\sup_{s \in K} |F(g_1(s)) - F(g_2(s))| \leq c \sup_{s \in K_1} |g_1(s) - g_2(s)|^\alpha$$

holds for all $g_1, g_2 \in H(D(1/2, 1))$.

Then it is pointed out in [94] that if $F \in \mathrm{Lip}(\alpha)$, then $F(\zeta(s))$ has the universality property. This claim especially includes the proof of the universality for $\zeta'(s)$.

To prove other theorems in [94], Laurinčikas applied the method of functional limit theorems (cf. Mauclaire [140]). Those results imply, for example, $\zeta'(s) + \zeta''(s)$, $\zeta(s)^m$ (m-th power), $\exp(\zeta(s))$, and $\sin(\zeta(s))$ are universal.

Later, Laurinčikas and his colleagues generalized the result in [94] to various other situations.

• Hurwitz zeta-functions (Laurinčikas [98] for the single case, Laurinčikas [95] for the joint case, and Laurinčikas and Rašytė [120] for the discrete case),

• Periodic and periodic Hurwitz zeta-functions (Laurinčikas [96], Korsakienė, Pocevičienė and Šiaučiūnas [67]),

• Automorphic L-functions (Laurinčikas, Matsumoto and J. Steuding [118]).

A hybrid version of the joint composite universality for Dirichlet L-functions was given by Laurinčikas, Matsumoto and J. Steuding [119].

An alternative approach to composite universality was done by Meyrath [143].

Yet another approach is due to Christ, J. Steuding and Vlachou [20]. Let $\Omega_0 \times \cdots \times \Omega_n$ be an open subset of $\mathbb{C}^{n+1}$, and let $F : \Omega_0 \times \cdots \times \Omega_n \to \mathbb{C}$ be continuous. In [20], the universality of $F(\zeta(s), \zeta'(s), \ldots, \zeta^{(n)}(s))$ was discussed. First, applying the idea in [37], they proved a weaker form of universal approximation. Then, when F is non-constant and

analytic, they obtained a kind of universality theorem on a certain small circle. Their proof relies on the implicit function theorem and the Picard-Lindelöf theorem on certain differential equations.

18. ERGODICITY AND THE UNIVERSALITY

It is quite natural to understand universality from the ergodic viewpoint. In fact, the universality theorem for a certain Dirichlet series $\varphi(s)$ implies that the orbit $\{\varphi(s + i\tau) \mid \tau \in \mathbb{R}\}$ is dense in a certain function space, and this orbit comes back to an arbitrarily small neighbourhood of any target function infinitely often. This is really an ergodic phenomenon.

Therefore, we can expect that there are some explicit connections between universality theory and ergodic theory. We mentioned already in Section 10 that the Birkhoff-Khinchin theorem in ergodic theory is used in Bagchi's proof of the universality.

Recently J. Steuding [202] formulated a kind of universality theorem, whose statement itself is written in terms of ergodic theory.

Let $(X, \mathcal{B}, P)$ be a probability space, and let $T : X \to X$ a measure-preserving transformation. We call T ergodic with respect to P if $A \in \mathcal{B}$ satisfies $T^{-1}(A) = A$, then either $P(A) = 0$ or $P(A) = 1$ holds. In this case we call $(X, \mathcal{B}, P, T)$ an ergodic dynamical system. Here we consider the case $X = \mathbb{R}$ and $\mathcal{B}$ is the standard Borel σ-algebra.

Let $D \subset \mathbb{C}$ be a domain, $K_1, \ldots, K_r$ be compact subsets of D with connected complements, and $f_j \in H_0^c(K_j)$ $(1 \le j \le r)$. We call $\varphi_1, \ldots, \varphi_r \in H(D)$ is *jointly ergodic universal* if for any K_j, f_j, T, $\varepsilon > 0$, and for almost all $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ for which

$$(18.1) \qquad \max_{1 \le j \le r} \sup_{s \in K_j} |\varphi_j(s + iT^n x) - f_j(s)| < \varepsilon$$

holds. Here, $T^n x$ means the T-times iteration of T. If the above statament holds for $f_j \in H^c(K_j)$ $(1 \le j \le r)$, then we call $\varphi_1, \ldots, \varphi_r \in H(D)$ *jointly strongly ergodic universal*.

Theorem 33. (J. Steuding [202]) *Let D, K_j, T be as above. Let $\varphi_1, \ldots, \varphi_r$ be a family of L-functions. Then, there exists a real number τ such that*

$$(18.2) \qquad \max_{1 \le j \le r} \sup_{s \in K_j} |\varphi_j(s + i\tau) - f_j(s)| < \varepsilon$$

for any $f_j \in H_0^c(K_j)$ (resp. $H^c(K_j)$) and any $\varepsilon > 0$, if and only if $\varphi_1, \ldots, \varphi_r$ is jointly (resp. jointly strongly) ergodic universal. And in

this case, we have

(18.3)

$$\liminf_{N\to\infty} \frac{1}{N} \# \left\{ n \in \mathbb{N} \ \middle| \ \max_{1\le j\le r} \sup_{s\in K_j} |\varphi_j(s + iT^n x) - f_j(s)| < \varepsilon \right\} > 0.$$

Srichan, R. & J. Steuding [195] discovered the universality produced by a random walk. They considered a lattice Λ on $\mathbb{C}$ and a random walk $(s_n)_{n=0}^{\infty}$ on this lattice, and proved the following result. Let K be a compact subset of $D(1/2, 1)$ with connected complement (with a condition given in terms of Λ), and $f \in H_0^c(K)$. Then for any $\varepsilon > 0$,

(18.4) $\qquad \liminf_{N\to\infty} \frac{1}{N} \# \left\{ n \in \mathbb{N} \ \middle| \ \sup_{s\in K} |\zeta(s + s_n) - f(s)| < \varepsilon \right\} > 0$

holds almost surely. They also mentioned a result similar to the above for two-dimensional Brownian motions.

The results in this section suggest that universality is a kind of ergodic phenomenon and is to be understood from the viewpoint of dynamical systems. Universality theorems imply that the properties of zeta-functions in the critical strip are quite inaccessible, which is probably the underlying reason of the extreme difficulty of the Riemann hypothesis. Moreover in Section 9 we mentioned that the Riemann hypothesis itself can be reformulated in terms of dynamical systems. Therefore the Riemann hypothesis is perhaps to be understood as a phenomenon with dynamical-system flavour[10]. In order to pursue this viewpoint, it is indispensable to study universality more deeply and extensively.

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[10]This argument might remind us the work of Deninger [21] [22] which is in a different context but also with dynamical-system flavour.

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Graduate School of Mathematics,
Nagoya University,
Chikusa-ku, Nagoya 464-8602,
Japan