# JOINT VALUE DISTRIBUTION THEOREMS ON LERCH ZETA-FUNCTIONS. III

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## ABSTRACT

A joint limit theorem, universality and a theorem on the functional independence for a collection of series of Lerch zeta-functions are proved. They generalize the results of (Laurinčikas and Matsumoto, 2006).

## 1. INTRODUCTION

The Lerch zeta-function $L(\lambda, \alpha, s)$, $s = \sigma + it$, with fixed real parameters $\alpha$ and $\lambda$, $0 < \alpha \leq 1$, is defined, for $\sigma > 1$, by

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i \lambda m}}{(m + \alpha)^s}.$$

For $\lambda \notin \mathbb{Z}$, the function $L(\lambda, \alpha, s)$ is analytically continuable to an entire function. If $\lambda \in \mathbb{Z}$, then the Lerch zeta-function reduces to the Hurwitz zeta-function

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s}, \quad \sigma > 1,$$

and $L(k, 1, s) = \zeta(s)$, $k \in \mathbb{Z}$, where $\zeta(s)$ is the Riemann zeta-function.

In (Laurinčikas and Matsumoto, 1998, 2000) and (Laurinčikas and Matsumoto, 2006) we considered the joint value-distribution of Lerch zeta-functions. We proved joint limit theorems in the sense of weak convergence of probability measures on the complex plane and in the space of analytic functions. We obtained, for some values of the parameters $\lambda$ and $\alpha$, the joint universality as well as the functional independence for Lerch zeta- functions. The aim of this note is to prove a generalization of the mentioned results for series of Lerch zeta-functions. Let, for $r > 1$, $\underline{L}(s) = (L(\lambda_{11}, \alpha_1, s), ..., L(\lambda_{1k_1}, \alpha_1, s), L(\lambda_{21}, \alpha_2, s), ..., L(\lambda_{2k_2}, \alpha_2, s), ..., L(\lambda_{r1}, \alpha_r, s), ..., L(\lambda_{rk_r}, \alpha_r, s))$ be a collection of series of Lerch zeta-functions. Here $k_j$, $j = 1, ..., r$, are positive integers. We suppose that $\lambda_{lj} \notin \mathbb{Z}$, $l = 1, ..., r$, $j = 1, ..., k_l$. For this collection, we will obtain a limit theorem in the space of analytic functions, a joint universality theorem and its consequence on the functional independence. In the aforementioned papers (Laurinčikas and Matsumoto, 1998, 2000, 2006), we studied the case $k_1 = ... = k_r = 1$. To state the results, we need some notation and definitions.

Let, for $T > 0$,

$$\nu_T(...) = \frac{1}{T}\text{meas}\{\tau \in [0, T] : ...\},$$

where meas$\{A\}$ is the Lebesgue measure of a measurable set $A \subset \mathbb{R}$ and in place of dots a condition satisfied by $\tau$ is to be written. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space $S$. For $D = \{s \in \mathbb{C} : \sigma > \frac{1}{2}\}$, let $H(D)$ be the space of analytic on $D$ functions equipped with the topology of uniform convergence on compacta. Moreover, let $H^d(D) = \underbrace{H(D) \times ... \times H(D)}_{d}$, where $d = \sum\limits_{j=1}^{r} k_j$. Denote $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and define

$$\Omega = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$. The Tikhonov theorem shows that $\Omega$ is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measure $m_H$ exists, and this gives a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(m)$ the projection of $\omega \in \Omega$ to the coordinate space $\gamma_m$. Furthermore, let $\Omega^r = \Omega_1 \times ... \times \Omega_r$, where $\Omega_j = \Omega$ for $j = 1, ..., r$. By the Tikhonov theorem again, $\Omega^r$ is a compact topological group, and the probability Haar measure $m_{Hr}$ on $(\Omega^r, \mathcal{B}(\Omega^r))$ can be defined. Now on the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_{Hr})$ define an $H^d(D)$-valued random element $\underline{L}(s, \underline{\omega})$ by

$$\underline{L}(s, \underline{\omega}) = (L(\lambda_{11}, \alpha_1, s, \omega_1), ..., L(\lambda_{1k_1}, \alpha_1, s, \omega_1), L(\lambda_{21}, \alpha_2, s, \omega_2), ...,$$

$$L(\lambda_{2k_2}, \alpha_2, s, \omega_2), ..., L(\lambda_{r1}, \alpha_r, s, \omega_r), ..., L(\lambda_{rk_r}, \alpha_r, s, \omega_r)),$$

where

$$L(\lambda_{lj}, \alpha_l, s, \omega_l) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i \lambda_{lj} m} \omega_l(m)}{(m+\alpha_l)^s}, \quad l = 1, ..., r; j = 1, ..., k_l.$$

Here, for $l = 1, ..., r$, $\omega_l \in \Omega$, and $\underline{\omega} = (\omega_1, ..., \omega_r) \in \Omega^r$. Denote by $P_{\underline{L}}$ the distribution of the random element $\underline{L}(s, \underline{\omega})$, that is

$$P_{\underline{L}}(A) = m_{Hr} \left(\underline{\omega} \in \Omega^r : \underline{L}(s, \underline{\omega}) \in A\right), \quad A \in \mathcal{B}(H^d(D)).$$

Define, for $A \in \mathcal{B}(H^d(D))$,

$$P_T(A) = \nu_T \left(\underline{L}(s + i\tau) \in A\right).$$

THEOREM 1. *Suppose that the numbers $\alpha_1, ..., \alpha_r$ are algebraically independent over the field of rational numbers $\mathbb{Q}$. Then the probability measure $P_T$ converges weakly to $P_{\underline{L}}$ as $T \to \infty$.*

Theorem 1 is the principal tool for the proof of the joint universality for the collection $\underline{L}(s)$. Let $\lambda_{lj}$, $l = 1, ..., r$, $j = 1, ..., k_l$, be arbitrary rational numbers with denominators $q_{lj}$, $l = 1, ..., r$, $j = 1, ..., k_l$, respectively. Denote by $k = [q_{11}, ..., q_{1k_1}, ..., q_{r1}, ..., q_{rk_r}]$ the least common multiple, and define

$$A = \begin{pmatrix} \mathrm{e}^{2\pi i \lambda_{11}} & \mathrm{e}^{2\pi i \lambda_{12}} & ... & \mathrm{e}^{2\pi i \lambda_{1k_1}} & ... & \mathrm{e}^{2\pi i \lambda_{r1}} & \mathrm{e}^{2\pi i \lambda_{r2}} & ... & \mathrm{e}^{2\pi i \lambda_{rk_r}} \\ \mathrm{e}^{4\pi i \lambda_{11}} & \mathrm{e}^{4\pi i \lambda_{12}} & ... & \mathrm{e}^{4\pi i \lambda_{1k_1}} & ... & \mathrm{e}^{4\pi i \lambda_{r1}} & \mathrm{e}^{4\pi i \lambda_{r2}} & ... & \mathrm{e}^{4\pi i \lambda_{rk_r}} \\ ......... & ......... & ... & ......... & ... & ......... & ......... & ......... & \\ \mathrm{e}^{2\pi i k \lambda_{11}} & \mathrm{e}^{2\pi i k \lambda_{12}} & ... & \mathrm{e}^{2\pi i k \lambda_{1k_1}} & ... & \mathrm{e}^{2\pi i k \lambda_{r1}} & \mathrm{e}^{2\pi i k \lambda_{r2}} & ... & \mathrm{e}^{2\pi i k \lambda_{rk_r}} \end{pmatrix}$$

THEOREM 2. *Suppose that $\alpha_1, ..., \alpha_r$ are algebraically independent over $\mathbb{Q}$ and that rank$(A) = d$. Let $K_{lj}$ be a compact subset of the strip $D_0 =$*

$\left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$ *with connected complement, and let* $f_{lj}(s)$ *be a continuous on* $K_{lj}$ *function which is analytic in the interior of* $K_{lj}$, $l = 1, ..., r$, $j = 1, ..., k_l$. *Then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \nu_T \left( \sup_{1\le l\le r} \sup_{1\le j\le k_l} \sup_{s\in K_{lj}} |L(\lambda_{lj}, \alpha_l, s + i\tau) - f_{lj}(s)| < \varepsilon \right) > 0.$$

Theorem 2 implies the functional independence of functions from the collection $\underline{L}(s)$.

THEOREM 3. *Suppose that* $\alpha_1, ..., \alpha_r$ *are algebraically independent over* $\mathbb{Q}$ *and that rank* $(A) = d$. *For* $j = 0, 1, ..., M$, *let* $F_j$ *be a continuous on* $\mathbb{C}^{Nd}$ *function, and*

$$\sum_{j=0}^{M} s^j F_j(L(\lambda_{11}, \alpha_1, s), L(\lambda_{12}, \alpha_1, s), ..., L(\lambda_{1k_1}, \alpha_1, s), ..., L(\lambda_{r1}, \alpha_r, s), ...,$$

$$L(\lambda_{rk_r}, \alpha_r, s), L'(\lambda_{11}, \alpha_1, s), L'(\lambda_{12}, \alpha_1, s), ..., L'(\lambda_{1k_1}, \alpha_1, s), ..., L'(\lambda_{r1}, \alpha_r, s),$$
$$..., L'(\lambda_{rk_r}, \alpha_r, s), ..., L^{(N-1)}(\lambda_{11}, \alpha_1, s), L^{(N-1)}(\lambda_{12}, \alpha_1, s), ..., L^{(N-1)}(\lambda_{1k_1}, \alpha_1, s),$$
$$..., L^{(N-1)}(\lambda_{r1}, \alpha_r, s), L^{(N-1)}(\lambda_{r2}, \alpha_r, s), ..., L^{(N-1)}(\lambda_{rk_r}, \alpha_r, s)) = 0$$

*identically for* $s \in \mathbb{C}$. *Then* $F_j \equiv 0$, $j = 0, 1, ..., M$.

## 2. PROOF OF THEOREM 1

We will follow the proof of Theorem 1 from (Laurinčikas and Matsumoto, 2002), therefore we will omit some details.

We begin with joint limit theorems for Dirichlet polynomials. For this, we will apply a limit theorem on torus $\Omega^r$ which was proved in (Laurinčikas and Matsumoto, 2006), Lemma 4. Let

$$Q_{T,r}(A) = \nu_T \left( \left( \left( (m+\alpha_1)^{i\tau} : m \in \mathbb{N}_0 \right), ..., \left( (m+\alpha_r)^{i\tau} : m \in \mathbb{N}_0) \right) \in A \right), \ A \in \mathcal{B}(\Omega^r) \right).$$

LEMMA 4. *The probability measure* $Q_{T,r}$ *converges weakly to the Haar measure* $m_{Hr}$ *on* $(\Omega^r, \mathcal{B}(\Omega^r))$ *as* $T \to \infty$.

Let $\sigma_1 > \frac{1}{2}$, and, for $m, n \in \mathbb{N}$,

$$v_l(m,n) = \exp\left\{ -\left( \frac{m+\alpha_l}{n+\alpha_l} \right)^{\sigma_1} \right\}, \quad l = 1, ..., r.$$

Define, for $N_{lj} \in \mathbb{N}_0$, $\hat{\omega}_l \in \Omega$ and $s \in D$,

$$L_{N_{lj},l,j,n}(\lambda_{lj}, \alpha_l, s) = \sum_{m=0}^{N_{lj}} \frac{\mathrm{e}^{2\pi i \lambda_{lj} m} v_l(m,n)}{(m+\alpha_l)^s},$$

$$L_{N_{lj},l,j,n}(\lambda_{lj}, \alpha_l, s, \hat{\omega}_l) = \sum_{m=0}^{N_{lj}} \frac{\mathrm{e}^{2\pi i \lambda_{lj} m} v_l(m,n) \hat{\omega}_l(m)}{(m+\alpha_l)^s},$$

$l = 1, ..., r$, $j = 1, ..., k_l$. Let $\underline{N} = (N_{11}, ..., N_{1k_1}, ..., N_{r1}, ..., N_{rk_r})$, $\underline{\hat{\omega}} = (\hat{\omega}_1, ..., \hat{\omega}_r)$, and

$$\underline{L}_{\underline{N},n}(s) = (L_{N_{11},1,1,n}(\lambda_{11}, \alpha_1, s), ..., L_{N_{1k_1},1,k_1,n}(\lambda_{1k_1}, \alpha_1, s), ...$$
$$L_{N_{r1},r,1,n}(\lambda_{r1}, \alpha_r, s), ..., L_{N_{rk_r},r,k_r,n}(\lambda_{rk_r}, \alpha_r, s)),$$

$$\underline{L}_{\underline{N},n}(s, \underline{\hat{\omega}}) = (L_{N_{11},1,1,n}(\lambda_{11}, \alpha_1, s, \hat{\omega}_1), ..., L_{N_{1k_1},1,k_1,n}(\lambda_{1k_1}, \alpha_1, s, \hat{\omega}_1), ...$$
$$L_{N_{r1},r,1,n}(\lambda_{r1}, \alpha_r, s, \hat{\omega}_r), ..., L_{N_{rk_r},r,k_r,n}(\lambda_{rk_r}, \alpha_r, s, \hat{\omega}_r)).$$

Now, for $A \in \mathcal{B}(H^d(D))$, define the probability measures

$$P_{T,\underline{N},n}(A) = \nu_T\left( \underline{L}_{\underline{N},n}(s+i\tau) \in A \right)$$

and

$$\hat{P}_{T,\underline{N},n}(A) = \nu_T\left( \underline{L}_{\underline{N},n}(s+i\tau, \underline{\hat{\omega}}) \in A \right).$$

THEOREM 5. *The probability measures $P_{T,\underline{N},n}$ and $\hat{P}_{T,\underline{N},n}$ both converge weakly to the same probability measure on $(H^d(D), \mathcal{B}(H^d(D)))$ as $T \to \infty$.*

*Proof.* The theorem is a generalization of Theorem 5 from (Laurinčikas and Matsumoto, 2006). Define a function $h : \Omega^r \to H^d(D)$ by

$$h(\omega_1, ..., \omega_r) = \underline{L}_{\underline{N},n}(s, \underline{\omega}^{-1}),$$

where $\underline{\omega}^{-1} = (\omega_1^{-1}, ..., \omega_r^{-1})$, $(\omega_1, ..., \omega_r) \in \Omega^r$. Then the function $h$ is continuous, moreover,

$$h\left(((m+\alpha_1)^{i\tau} : m \in \mathbb{N}_0), ..., ((m+\alpha_r)^{i\tau} : m \in \mathbb{N}_0)\right) = \underline{L}_{\underline{N},n}(s+i\tau).$$

Therefore $P_{T,\underline{N},n} = Q_{T,r}h^{-1}$, and Lemma 4 together with Theorem 5.1 of (Billingsley, 1968) shows that the measure $P_{T,\underline{N},n}$ converges weakly to $m_{Hr}h^{-1}$ as $T \to \infty$.

Now define $h_1 : \Omega^r \to \Omega^r$ by

$$h_1(\omega_1, ..., \omega_r) = (\omega_1 \hat{\omega}_1^{-1}, ..., \omega_r \hat{\omega}_r^{-1}).$$

Then we have that

$$\underline{L}_{\underline{N},n}(s+i\tau, \underline{\hat{\omega}}) = h\left(h_1\left(((m+\alpha_1)^{i\tau} : m \in \mathbb{N}_0), ..., ((m+\alpha_r)^{i\tau} : m \in \mathbb{N}_0)\right)\right).$$

Therefore, similarly to the case of the measure $P_{T,\underline{N},n}$, we find that the measure $\hat{P}_{T,\underline{N},n}$ converges weakly to the measure $m_{Hr}(hh_1)^{-1}$ as $T \to \infty$, and the theorem follows in virtue of the equality

$$m_{Hr}(hh_1)^{-1} = (m_{Hr}h_1^{-1})h^{-1} = m_{Hr}h^{-1}.$$

The next step is limit theorems for absolutely convergent Dirichlet series. For $l = 1, ..., r$, $j = 1, ..., k_l$, define

$$L_{l,j,n}(\lambda_{lj}, \alpha_l, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda_{lj} m} v_l(m,n)}{(m+\alpha_l)^s} \tag{1}$$

and

$$L_{l,j,n}(\lambda_{lj}, \alpha_l, s, \omega_l) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda_{lj} m} v_l(m,n)\omega_l(m)}{(m+\alpha_l)^s}, \tag{2}$$

where $(\omega_1, ..., \omega_r) \in \Omega^r$, and put

$$\begin{aligned}\underline{L}_n(s) = (&L_{1,1,n}(\lambda_{11}, \alpha_1, s), ..., L_{1,k_1,n}(\lambda_{1k_1}, \alpha_1, s), ...\\ &L_{r,1,n}(\lambda_{r1}, \alpha_r, s), ..., L_{r,k_r,n}(\lambda_{rk_r}, \alpha_r, s))\end{aligned}$$

and

$$\underline{L}_n(s, \underline{\omega}) = (L_{1,1,n}(\lambda_{11}, \alpha_1, s, \omega_1), ..., L_{1,k_1,n}(\lambda_{1k_1}, \alpha_1, s, \omega_1), ...$$

$$L_{r,1,n}(\lambda_{r1}, \alpha_r, s, \omega_r), ..., L_{r,k_r,n}(\lambda_{rk_r}, \alpha_r, s, \omega_r)).$$

Note that series (1) and (2) both converge absolutely for $\sigma > \frac{1}{2}$. Our next aim is to show limit theorems for the probability measures

$$P_{T,n}(A) = \nu_T\left(\underline{L}_n(s+i\tau) \in A\right), \quad A \in \mathcal{B}(H^d(D)),$$

and

$$\hat{P}_{T,n}(A) = \nu_T\left(\underline{L}_n(s+i\tau, \underline{\omega}) \in A\right), \quad A \in \mathcal{B}(H^d(D)).$$

THEOREM 6. *On $(H^d(D), \mathcal{B}(H^d(D)))$ there exists a probability measure $P_n$ such that the measures $P_{T,n}$ and $\hat{P}_{T,n}$ both converge weakly to $P_n$ as $T \to \infty$.*

*Proof.* We argue similarly to the proof of Theorem 6 from (Laurinčikas and Matsumoto, 2006). For simplicity, let $N_{lj} = N$ for all $l = 1, ..., r$, $j = 1, ..., k_l$. Then by Theorem 5 we have that the probability measures $P_{T,N,n} = P_{T,\underline{N},n}$ and $\hat{P}_{T,N,n} = \hat{P}_{T,\underline{N},n}$ both converge weakly to the same measure $P_{N,n}$ as $T \to \infty$. Note that in the definition of $\hat{P}_{T,N,n}$ we write $\underline{\omega}$ in place of $\hat{\underline{\omega}}$.

Let $\eta$ be a random variable defined on a certain probability space $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}), \mathbb{P})$ and uniformly distributed on $[0,1]$. Define, for $l = 1, ..., r$, $j = 1, ..., k_l$,

$$X_{T,N,l,j,n} = X_{T,N,l,j,n}(s) = L_{N,l,j,n}(\lambda_{lj}, \alpha_j, s + iT\eta).$$

Then $X_{T,N,l,j,n}$ is an $H(D)$-valued random element defined on $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}), \mathbb{P})$, and by Theorem 5 we have that

$$\underline{X}_{T,N,n} = \underline{X}_{T,N,n}(s) \stackrel{def}{=} (X_{T,N,1,1,n}, ..., X_{T,N,1,k_1,n},$$

$$..., X_{T,N,r,1,n}, ..., X_{T,N,r,k_r,n}) \xrightarrow[T\to\infty]{\mathcal{D}} \underline{X}_{N,n}, \tag{3}$$

where $\underline{X}_{N,n} = (X_{N,1,1,n}, ..., X_{N,1,k_1,n}, ..., X_{N,r,1,n}, ..., X_{N,r,k_r,n})$ is an $H^d(D)$-valued random element with the distribution $P_{N,n}$, and $\xrightarrow{\mathcal{D}}$ means the convergence in distribution.

Let $\{K_m : m \in \mathbb{N}\}$ be a sequence of compact subsets of $D$ such that $\bigcup\limits_{m=1}^{\infty} K_m = D$, $K_m \subset K_{m+1}$, and, for any compact subset $K$ of $D$, we find an $m$ for which $K \subseteq K_m$ holds. Then

$$\rho(f,g)=\sum_{m=1}^{\infty} 2^{-m}\frac{\sup\limits_{s\in K_m}|f(s)-g(s)|}{1+\sup\limits_{s\in K_m}|f(s)-g(s)|}, \quad f,g\in H(D),$$

is a metric on $H(D)$ which induces its topology. Then we can define a metric on $H^d(D)$ by

$$\rho_d(\underline{f},\underline{g})=\max_{1\le l\le r}\max_{1\le j\le k_l}\rho(f_{lj},g_{lj}),$$

where $\underline{f}=(f_{11},...,f_{1k_1},...,f_{r1},...,f_{rk_r})$, $\underline{g}=(g_{11},...,g_{1k_1},...,g_{r1},...,g_{rk_r})\in H^d(D)$.

Since the series for $L_{l,j,n}(\lambda_{lj},\alpha_l,s)$, $l=1,...,r$, $j=1,...,k_l$, is absolutely convergent for $\sigma>\frac{1}{2}$, we have that, for any $M_{mljn}>0$, $m\in\mathbb{N}$, $l=1,...,r$, $j=1,...,k_l$, $n\in\mathbb{N}_0$,

$$\limsup_{T\to\infty}\mathbb{P}\left(\sup_{s\in K_m}|X_{T,N,l,j,n}(s)|>M_{mljn}\quad\text{for at least one pair}(l,j)\right)\le$$

$$\le\sum_{l=1}^{r}\sum_{j=1}^{k_l}\limsup_{T\to\infty}\mathbb{P}\left(\sup_{s\in K_m}|X_{T,N,l,j,n}(s)|>M_{mljn}\right)\le$$

$$\le\sum_{l=1}^{r}\sum_{j=1}^{k_l}\frac{1}{M_{mljn}}\sup_{N\ge1}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\sup_{s\in K_m}|L_{N,l,j,n}(\lambda_{lj},\alpha_l,s+i\tau)|\mathrm{d}\tau=$$

$$=\sum_{l=1}^{r}\sum_{j=1}^{k_l}\frac{R_{mljn}}{M_{mljn}}<\infty, \tag{4}$$

where

$$R_{mljn}=\sup_{N\ge1}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\sup_{s\in K_m}|L_{N,l,j,n}(\lambda_{lj},\alpha_l,s+i\tau)|\mathrm{d}\tau.$$

Now choose $M_{mljn} = R_{mljn} 2^m d/\varepsilon$. Then (4) yields

$$\limsup_{T\to\infty} \mathbb{P}\left( \sup_{s\in K_m} |X_{T,N,l,j,n}(s)| > M_{mljn} \quad \text{for at least one pair } (l,j) \right) \leq \frac{\varepsilon}{2^m}.$$

This and (3) show that, for all $m \in \mathbb{N}$,

$$\mathbb{P}\left( \sup_{s\in K_m} |X_{N,l,j,n}(s)| > M_{mljn} \quad \text{for at least one pair } (l,j) \right) \leq \frac{\varepsilon}{2^m}. \qquad (5)$$

Define

$$H_\varepsilon^d = \{(f_{11}, ..., f_{1k_1}, ..., f_{r1}, ..., f_{rk_r}) \in H^d(D) : \sup_{s\in K_m} |f_{lj}(s)| \leq M_{mljn},$$
$$l = 1, ..., r,\ j = 1, ..., k_l,\ m \in \mathbb{N}\}.$$

Then, by the compactness principle, the set $H_\varepsilon^d$ is compact in $H^d(D)$, and in view of (5), for all $N \in \mathbb{N}_0$ and any fixed $n \in \mathbb{N}_0$,

$$\mathbb{P}\left( \underline{X}_{N,n} \in H_\varepsilon^d \right) \geq 1 - \varepsilon.$$

Taking into account the definition of $\underline{X}_{N,n}$, hence we obtain that

$$P_{N,n}\left( H_\varepsilon^d \right) \geq 1 - \varepsilon$$

for all $N \in \mathbb{N}_0$ and any fixed $n \in \mathbb{N}_0$. This shows that the family of probability measures $\{P_{N,n} : N \in \mathbb{N}_0\}$ is tight, therefore, by the Prokhorov theorem, it is relatively compact.

By the definition, for $l = 1, ..., r,\ j = 1, ..., k_l$,

$$\lim_{N\to\infty} L_{N,l,j,n}(\lambda_{lj}, \alpha_l, s) = L_{l,j,n}(\lambda_{lj}, \alpha_l, s),$$

uniformly on compact subsets of the half-plane $D$. Hence, using the above notation, we have, for every $\varepsilon > 0$,

$$\lim_{N\to\infty} \limsup_{T\to\infty} \nu_T \left( \rho_d(\underline{L}_{N,n}(s+i\tau), \underline{L}_n(s+i\tau)) \geq \varepsilon \right) \leq$$

$$\leq \lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T\varepsilon} \int_0^T \rho_d \left( \underline{L}_{N,n}(s+i\tau), \underline{L}_n(s+i\tau) \right) \mathrm{d}\tau \leq$$

$$\leq \lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T\varepsilon} \sum_{l=1}^{r} \sum_{j=1}^{k_l} \int_0^T \rho\left(L_{N,l,j,n}(\lambda_{lj}, \alpha_l, s+i\tau), L_{l,j,n}(\lambda_{lj}, \alpha_l, s+i\tau)\right) \mathrm{d}\tau = 0. \tag{6}$$

Now, for $l = 1, ..., r,\ j = 1, ..., k_l$, we define

$$X_{T,l,j,n} = X_{T,l,j,n}(s) = L_{l,j,n}(\lambda_{lj}, \alpha_l, s+iT\eta),$$

and let

$$\underline{X}_{T,n} = \left(X_{T,1,1,n}, ..., X_{T,1,k_1,n}, ..., X_{T,r,1,n}, ..., X_{T,r,k_r,n}\right).$$

Then (6) implies

$$\lim_{N\to\infty} \limsup_{T\to\infty} \mathbb{P}\left(\rho_d(\underline{X}_{T,N,n}, \underline{X}_{T,n}) \geq \varepsilon\right) = 0. \tag{7}$$

Moreover, the relative compactness of the family $\{P_{N,n} : N \in \mathbb{N}_0\}$ shows that there exists $\{P_{N_1,n}\} \subset \{P_{N,n}\}$ such that $P_{N_1,n}$ converges weakly to some probability measure $P_n$ as $N_1 \to \infty$. Hence

$$\underline{X}_{N_1,n} \xrightarrow[N_1\to\infty]{\mathcal{D}} P_n. \tag{8}$$

Therefore, relations (3), (7) (8) and Theorem 4.2 of (Billingsley, 1968) yield

$$\underline{X}_{T,n} \xrightarrow[T\to\infty]{\mathcal{D}} P_n, \tag{9}$$

and we have that the probability measure $P_{T,n}$ converges weakly to $P_n$ as $T \to \infty$.

Relation (9) shows that the measure $P_n$ is independent of the sequence $N_1$. Thus, by Theorem 2.3 of (Billingsley, 1968),

$$X_{N,n} \xrightarrow[N\to\infty]{\mathcal{D}} P_n. \tag{10}$$

Now define

$$\underline{\hat{X}}_{T,N,n}(s) = \underline{L}_{N,n}(s+iT\eta, \omega)$$

and

$$\underline{\hat{X}}_{T,n}(s) = \underline{L}_n(s+iT\eta, \omega).$$

Then, repeating the above arguments for the random elements $\underline{\hat{X}}_{T,N,n}(s)$ and $\underline{\hat{X}}_{T,n}(s)$, recalling that $\hat{P}_{T,N,n}$ also converges weakly to $P_{N,n}$ and using (10), we obtain that the probability measure $\hat{P}_{T,n}$ also converges weakly to $P_n$ as $T \to \infty$. The theorem is proved.

Now we approximate in mean the vectors $\underline{L}(s)$ and $\underline{L}(s, \underline{\omega})$ by the vectors $\underline{L}_n(s)$ and $\underline{L}_n(s, \underline{\omega})$, respectively.

THEOREM 7. *We have*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int\limits_0^T \rho_d \left(\underline{L}(s+i\tau), \underline{L}_n(s+i\tau)\right) \mathrm{d}\tau = 0,$$

and, for almost all $\underline{\omega} \in \Omega^r$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int\limits_0^T \rho_d \left(\underline{L}(s+i\tau, \underline{\omega}), \underline{L}_n(s+i\tau, \underline{\omega})\right) \mathrm{d}\tau = 0.$$

*Proof.* Obviously, the numbers $\alpha_1, ..., \alpha_r$ are transcendental. Therefore, by Lemmas 5.2.11 and 5.2.13 of (Laurinčikas and Garunkštis, 2002), for $l = 1, ..., r$, $j = 1, ..., k_l$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int\limits_0^T \rho \left(L(\lambda_{lj}, \alpha_l, s+i\tau), L_{l,j,n}(\lambda_{lj}, \alpha_l, s+i\tau)\right) \mathrm{d}\tau = 0,$$

and, for almost all $\omega_l \in \Omega$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int\limits_0^T \rho \left(L(\lambda_{lj}, \alpha_l, s+i\tau, \omega_l), L_{l,j,n}(\lambda_{lj}, \alpha_l, s+i\tau, \omega_l)\right) \mathrm{d}\tau = 0.$$

Thus to prove the theorem it suffices to use the definition of metric $\rho_d$.

Now we are ready to prove Theorem 1 without indication of the limit measure. Define

$$\hat{P}_T(A) = \nu_T \left(\underline{L}(s+i\tau, \underline{\omega}) \in A\right), \quad A \in \mathcal{B}(H^d(D)).$$

THEOREM 8. *On* $(H^d(D), \mathcal{B}(H^d(D)))$ *there exists a probability measure* $P$ *such that the measures* $P_T$ *and* $\hat{P}_T$ *(for almost all* $\underline{\omega}$*) both converge weakly to* $P$ *as* $T \to \infty$.

*Proof.* The proof uses Theorems 6 and 7 and differs from that of Theorem 8 from (Laurinčikas and Matsumoto, 2006) only by evident details which are clear from the proof of Theorem 6.

For the identification of the limit measure $P$ in Theorem 8, as usual, we apply some statements of ergodic theory.

Let $a_{\tau,l} = \{(m + \alpha_l)^{-i\tau} : m \in \mathbb{N}_0\}$, $\tau \in \mathbb{R}$, $l = 1, ..., r$. Then $\{a_{\tau,l} : \tau \in \mathbb{R}\}$, for each $l = 1, ..., r$, is a one-parameters group. Define the one-parameter family $\{\Phi_\tau : \tau \in \mathbb{R}\} = \{(\varphi_{\tau,1}, ..., \varphi_{\tau,r}) : \tau \in \mathbb{R}\}$ of transformations on $\Omega^r$ by $\varphi_{\tau,l}(\omega_l) = a_{\tau,l}\omega_l$, $\omega_l \in \Omega_l$, $l = 1, ..., r$. Then we obtain a one-parameter group of measurable measure preserving transformations on $\Omega^r$.

LEMMA 9. *The one-parameter group* $\{\Phi_\tau : \tau \in \mathbb{R}\}$ *is ergodic.*

Proof of the lemma is given in (Laurinčikas and Matsumoto, 2006).

*Proof of Theorem 1.* The proof is based on Theorem 8 and Lemma 9 as well as on the Birkhoff- Khinchine theorem and completely coincides with the proof of Theorem 1 from (Laurinčikas and Matsumoto, 2006).

## 3. PROOF OF THEOREMS 2 AND 3

We recall that $D_0 = \left\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\right\}$. For the proof of Theorem 2 we need a limit theorem on $H^d(D_0)$. Let $P_{\underline{L}_0}$ be the restriction of the distribution $P_{\underline{L}}$ of the random element $\underline{L}(s, \underline{\omega})$ to $H^d(D_0)$, that is

$$P_{\underline{L}_0}(A) = m_{Hr}\left(\underline{\omega} \in \Omega^r : \underline{L}(s, \underline{\omega}) \in A\right), \quad A \in \mathcal{B}(H^d(D_0)).$$

LEMMA 10. *Suppose that* $\alpha_1, ..., \alpha_r$ *are algebraically independent over* $\mathbb{Q}$. *Then the probability measure*

$$\nu_T\left(\underline{L}(s + i\tau) \in A\right), \quad A \in \mathcal{B}(H^d(D_0)),$$

*converges weakly to* $P_{\underline{L}_0}$ *as* $T \to \infty$.

*Proof.* The statement of the Lemma is a consequence of Theorem 1, the continuity of the function $h : H^d(D) \to H^d(D_0)$ defined by $h(\underline{g}) = \underline{g}\big|_{s\in D_0}$, $\underline{g} \in H^d(D)$, and of a property of weak convergence of probability measures (see Theorem 5.1 of (Billingsley, 1968)).

Now we deal with the support of the measure $P_{\underline{L}_0}$. The support of $P_{\underline{L}_0}$ is a minimal closed set $S_{P_{\underline{L}_0}} \subset H^d(D_0)$ such that $P_{\underline{L}_0}(S_{P_{\underline{L}_0}}) = 1$, and $S_{P_{\underline{L}_0}}$ consists of all $\underline{g} \in H^d(D_0)$ such that for every neighborhood $G$ of $\underline{g}$ the inequality $P_{\underline{L}_0}(G) > 0$ is satisfied.

THEOREM 11. *Suppose that $\alpha_1, ..., \alpha_r$ are algebraically independent over $\mathbb{Q}$ and that rank$(A) = d$. Then the support of $P_{\underline{L}_0}$ is the whole of $H^d(D_0)$.*

*Proof.* Define

$$\underline{\hat{L}}_0(s,\omega) = (L(\lambda_{11}, \alpha_1, s, \omega), ..., L(\lambda_{1k_1}, \alpha_1, s, \omega), ..., L(\lambda_{r1}, \alpha_r, s, \omega), \\ ..., L(\lambda_{rk_r}, \alpha_r, s, \omega)), \quad s \in D_0,$$

where, for $\omega \in \Omega$, $l = 1, ..., r$, $j = 1, ..., k_l$,

$$L(\lambda_{lj}, \alpha_l, s, \omega) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i m \lambda_{lj}} \omega(m)}{(m+\alpha_l)^s}.$$

Then we have that $S_{P_{\underline{L}_0}} \supseteq S_{P_{\underline{\hat{L}}_0}}$, and it is sufficient to prove that $S_{P_{\underline{\hat{L}}_0}} = H^d(D_0)$.

The support of each random variable $\omega(m)$, $m \in \mathbb{N}_0$, is the unit circle $\gamma$. Moreover, by construction $\{\omega(m) : m \in \mathbb{N}_0\}$ is a sequence of independent random variables on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. This and Lemma 5 of (Laurinčikas and Matsumoto, 2002) show that the support $S_{P_{\underline{\hat{L}}_0}}$ is the closure of all convergent series

$$\sum_{m=0}^{\infty} \underline{L}_{m0}(s, a),$$

where

$$\underline{L}_{m0}(s,a) = \left( \frac{\mathrm{e}^{2\pi i m \lambda_{11}} a_m}{(m+\alpha_1)^s}, ..., \frac{\mathrm{e}^{2\pi i m \lambda_{1k_1}} a_m}{(m+\alpha_1)^s}, ..., \frac{\mathrm{e}^{2\pi i m \lambda_{r1}} a_m}{(m+\alpha_r)^s}, ..., \frac{\mathrm{e}^{2\pi i m \lambda_{rk_r}} a_m}{(m+\alpha_r)^s} \right)$$

with $a = \{a_m : a_m \in \gamma, m \in \mathbb{N}_0\}$. In order to prove that the latter set of series is dense in $H^d(D_0)$, we will apply Lemma 6 of (Laurinčikas and Matsumoto, 2002).

Since $\underline{\hat{L}}_0(s,\omega)$ is an $H^d(D_0)$-valued random element, there exists a sequence $b = \{b_m : b_m \in \gamma, m \in \mathbb{N}_0\}$ such that the series

$$\sum_{m=0}^{\infty} \underline{L}_{m0}(s,b) \tag{11}$$

is convergent in $H^d(D_0)$. Moreover, for every compact subset $K \subset D_0$,

$$\sum_{m=0}^{\infty} \sum_{l=1}^{r} \sum_{j=1}^{k_l} \sup_{s \in K} \left| \frac{\mathrm{e}^{2\pi i m \lambda_{lj}} b_m}{(m+\alpha_l)^s} \right|^2 < \infty. \tag{12}$$

Now let $\mu_{lj}$, $l = 1, ..., r$, $j = 1, ..., k_l$, be complex measures on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact supports contained in $D_0$ and such that

$$\sum_{m=0}^{\infty} \left| \sum_{l=1}^{r} \sum_{j=1}^{k_l} \int\limits_{\mathbb{C}} \frac{\mathrm{e}^{2\pi i m \lambda_{lj}}}{(m+\alpha_l)^s} \mathrm{d}\mu_{lj}(s) \right| < \infty.$$

This, in view of the estimate (see(Laurinčikas and Matsumoto, 2000))

$$(m+\alpha_l)^{-s} = m^{-s} + O\left(m^{-1-\sigma}|s|\mathrm{e}^{O(|s|)}\right), \quad l = 1, ..., r,$$

and the definition of $\mu_{lj}$, leads to

$$\sum_{m=0}^{\infty} \left| \sum_{l=1}^{r} \sum_{j=1}^{k_l} \int\limits_{\mathbb{C}} \frac{\mathrm{e}^{2\pi i m \lambda_{lj}}}{m^s} \mathrm{d}\mu_{lj}(s) \right| < \infty. \tag{13}$$

The sequence $\left\{\mathrm{e}^{2\pi i m \lambda_{lj}},\ m \in \mathbb{N}_0\right\}$ (for each $l, j$) is periodic with period $k$. Thus, by (13), for every $h = 1, ..., k$,

$$\sum_{\substack{m=0 \\ m \equiv h (\mathrm{mod} k)}}^{\infty} \left| \sum_{l=1}^{r} \sum_{j=1}^{k_l} \int\limits_{\mathbb{C}} \frac{\mathrm{e}^{2\pi i h \lambda_{lj}}}{m^s} \mathrm{d}\mu_{lj}(s) \right| < \infty. \tag{14}$$

Define, for $A \in \mathcal{B}(\mathbb{C})$ and $h = 1, ..., k$,

$$\nu_h(A) = \sum_{l=1}^{r} \sum_{j=1}^{k_l} \mathrm{e}^{2\pi i h \lambda_{lj}} \mu_{lj}(A).$$

Then $\nu_h$, $h = 1, ..., k$, are also complex measures on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact supports contained in $D_0$, and by (14), for every $h = 1, ..., k$,

$$\sum_{\substack{m=0 \\ m \equiv h (\mathrm{mod} k)}}^{\infty} \left| \int_{\mathbb{C}} m^{-s} \mathrm{d}\nu_h(s) \right| < \infty. \tag{15}$$

Let, for $h = 1, ..., k$ and $z \in \mathbb{C}$,

$$\rho_h(z) = \int_{\mathbb{C}} \mathrm{e}^{-sz} \mathrm{d}\nu_h(s).$$

Then we can rewrite (14) in the form

$$\sum_{\substack{m=0 \\ m \equiv h (\mathrm{mod} k)}}^{\infty} |\rho_h(\log m)| < \infty, \quad h = 1, ..., k. \tag{16}$$

For all $h = 1, ..., k$, $\rho_h(z)$ is an entire function of exponential type. Therefore, taking into account (16) and Lemma 6.4.10 of (Laurinčikas, 1996), we have that, for all $h = 1, ..., k$, either $\rho_h(z) \equiv 0$, or

$$\limsup_{x \to \infty} \frac{\log |\rho_h(x)|}{x} > -1. \tag{17}$$

If (17) takes place, then in view of Lemma 5 of (Laurinčikas and Matsumoto, 2000), which is a case of general positive density method, we obtain that, for $h = 1, ..., k$,

$$\sum_{\substack{m=0 \\ m \equiv h (\mathrm{mod} k)}}^{\infty} |\rho_h(\log m)| = \infty.$$

However, this contradicts (16). Thus, it remains the case $\rho_h(z) \equiv 0$, $h = 1, ..., k$. This shows that, for $h = 1, ..., k$,

$$\sum_{l=1}^{r} \sum_{j=1}^{k_l} \mathrm{e}^{2\pi i h \lambda_{lj}} \int_{\mathbb{C}} \mathrm{e}^{-sz} \mathrm{d}\mu_{lj}(s) \equiv 0. \tag{18}$$

Now we apply the hypothesis that $\mathrm{rank}(A) = d$. Then the system of equations (18) has the unique solution

$$\int_{\mathbb{C}} \mathrm{e}^{-sz} \mathrm{d}\mu_{lj}(s) \equiv 0, \ l = 1, ..., r, \quad j = 1, ..., k_l.$$

Hence, by differentiation we find that

$$\int_{\mathbb{C}} s^m \mathrm{d}\mu_{lj}(s) = 0$$

for all $m \in \mathbb{N}_0$ and $l = 1, ..., r, \quad j = 1, ..., k_l$. This together with convergence of series (11) and (12) show that all hypotheses of Lemma 6 from (Laurinčikas and Matsumoto, 2002) are satisfied, and we obtain that the set of all convergent series

$$\sum_{m=0}^{\infty} \underline{L}_{m0}(s, ab)$$

with $a = \{a_m : a_m \in \gamma, \ m \in \mathbb{N}_0\}$ is dense in $H^d(D_0)$. Then set of all convergent series

$$\sum_{m=0}^{\infty} \underline{L}_{m0}(s, a)$$

is also dense in $H^d(D_0)$, and the theorem is proved.

Now Theorem 2 is deduced by a standard way from Lemma 10, Theorem 11 as well as Mergelyan's theorem, while Theorem 3 is a consequence of Theorem 2. For the details, see (Laurinčikas and Matsumoto, 2000).

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