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FUNCTIONAL RELATIONS FOR ZETA-FUNCTIONS OF WEIGHT LATTICES OF LIE GROUPS OF TYPE A_3

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ABSTRACT

We study zeta-functions of weight lattices of compact connected semisimple Lie groups of type A_3. Actually we consider zeta-functions of $SU(4)$, $SO(6)$ and $PU(4)$, and give some functional relations and new classes of evaluation formulas for them.

1. INTRODUCTION

For any semisimple Lie algebra $\mathfrak{g}$, the Witten zeta-function $\zeta_W(s;\mathfrak{g})$ is defined by

$$\zeta_W(s;\mathfrak{g}) = \sum_{\rho} (\dim \rho)^{-s}\,, \tag{1.1}$$

where $s \in \mathbb{C}$ and ρ runs over all finite dimensional irreducible representations of $\mathfrak{g}$. This was formulated by Zagier (see (26, Section 7)), who was inspired by Witten's work (25). Witten's motivation of introducing the above zeta-functions is to express the volumes of certain moduli spaces in terms of special values of $\zeta_W(s;\mathfrak{g})$. The result is called Witten's volume formula, and from which it can be shown that

$$\zeta_W(2k;\mathfrak{g}) = C_W(2k,\mathfrak{g})\pi^{2kn} \tag{1.2}$$

for any $k \in \mathbb{N}$, where $C_W(2k,\mathfrak{g}) \in \mathbb{Q}$ (see (26, Theorem, p.506)). The explicit value of $C_W(2k,\mathfrak{g})$ was not determined in their work.

Gunnells and Sczech introduced a method to compute $C_W(2k,\mathfrak{g})$ explicitly (see (3)). The theory of Szenes ((21), (22)) also gives a different algorithm of computing $C_W(2k,\mathfrak{g})$.

Let r be the rank of $\mathfrak{g}$, $\Delta(\mathfrak{g})$ the root system corresponding to $\mathfrak{g}$, and n the number of positive roots belonging to $\Delta(\mathfrak{g})$. In (6; 7; 11; 19), the authors defined the zeta-function $\zeta_r(\mathbf{s};\Delta(\mathfrak{g}))$ of the root system $\Delta(\mathfrak{g})$, where $\mathbf{s}=(s_i)\in\mathbb{C}^n$ (see Section 3). This may be regarded as a multi-variable version of $\zeta_W(s;\mathfrak{g})$ (see also survey papers (10), (16)). The authors further introduced a generalization of Bernoulli polynomials associated with root systems in (7; 9; 13). Using these tools, we can generalize (1.2) ((13, Theorem 4.6)) and express $C_W(2k,\mathfrak{g})$ in terms of Bernoulli polynomials of root systems (see also (10)), hence gives another algorithm for computing $C_W(2k,\mathfrak{g})$. Moreover we can give various functional relations for zeta-functions of root systems (see (6; 7; 8; 9; 13; 14; 19); we will discuss this matter further in the next section).

More recently, the authors defined zeta-functions of weight lattices of compact connected semisimple Lie groups (see (17)). If the group is simply-connected, these zeta-functions coincide with ordinary zeta-functions of root systems of associated Lie algebras. We considered the general connected (but not necessarily simply-connected) case and proved a result analogous to (1.2) for these zeta-functions, and further prove functional relations among them. The present paper is a continuation of (17), and we study zeta-functions of lattices of Lie groups whose associated Lie algebras are of type A_3. The reason why we treat the case A_3 will be mentioned in Section 3.

Throughout this paper, let $\mathbb{N}$ be the set of positive integers, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\mathbb{Z}$ the ring of rational integers, $\mathbb{Q}$ the rational number field, $\mathbb{R}$ the real number field, and $\mathbb{C}$ the complex number field.

2. FUNCTIONAL RELATIONS: A MOTIVATION

In this section we comment our motivation on the study of functional relations.[1] The Euler-Zagier r-ple sum is defined by

$$\zeta_{EZ,r}(\mathbf{s})=\sum_{m_1,\ldots,m_r=1}^{\infty}\frac{1}{m_1^{s_1}(m_1+m_2)^{s_2}\cdots(m_1+\cdots+m_r)^{s_r}}, \tag{2.1}$$

where $\mathbf{s}=(s_1,\ldots,s_r)\in\mathbb{C}^r$ (see (4), (26)). Special values of (2.1) for $\mathbf{s}=\mathbf{k}$, where $\mathbf{k}=(k_1,\ldots,k_r)\in\mathbb{N}^r$ with $k_r\geq 2$, are known to be important in various fields of mathematics. Euler already obtained the following relations among those values in the case $r=2$: The harmonic product relation

$$\zeta(k_1)\zeta(k_2)=\zeta_{EZ,2}(k_1,k_2)+\zeta_{EZ,2}(k_2,k_1)+\zeta(k_1+k_2), \tag{2.2}$$

[1]The contents of this section was given in the talk of the second-named author on the problem session of the conference.

where $\zeta(s) = \zeta_{EZ,1}(s)$ is the Riemann zeta-function, and the sum formula

$$\sum_{j=2}^{k-1} \zeta_{EZ,2}(k-j,j) = \zeta(k) \tag{2.3}$$

for $k \in \mathbb{N}$, $k \geq 3$. After the discovery of the importance of Euler-Zagier sums (around 1990, according to the work of Drinfel'd, Goncharov, Kontsevich, Hoffman and Zagier), many people began to search for various relations among special values of (2.1), and indeed a lot of relations have been discovered.

Around 2000, the second-named author raised a question: are those relations valid only at positive integers, or valid also continuously at other values?

In fact, it is easy to see that (2.2) is valid for any complex numbers s_1, s_2 except for singularities, that is

$$\zeta(s_1)\zeta(s_2) = \zeta_{EZ,2}(s_1,s_2) + \zeta_{EZ,2}(s_2,s_1) + \zeta(s_1+s_2). \tag{2.4}$$

Therefore the harmonic product relation is actually a "functional relation". So far, except for (2.4) and its relatives, no other such functional relations among Euler-Zagier sums has been discovered. (In the double zeta case, the functional equation (12) is known, but it is not a formula which interpolates some known value-relation.)

However, when we consider more extended classes of multiple series, we can find a lot of functional relations! The first examples were reported by the third-named author (23), in which functional relations among $\zeta(s)$ and the Tornheim double sum

$$\zeta_{MT,2}(s_1,s_2,s_3) = \sum_{m,n=1}^{\infty} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3}} \tag{2.5}$$

are proved. Those relations can be regarded as functional relations among zeta-functions of root systems, because $\zeta_{MT,2}(s_1,s_2,s_3)$ coincides with the zeta-function of the root system $\Delta(\mathfrak{s}u(3))$.

It is known that irreducible root systems are classified as types X_r (where X is one of A, B, C, D, E, F, G) by the Killing-Cartan theory. When $\Delta(\mathfrak{g})$ is of type X_r, we denote the corresponding zeta-function as $\zeta_r(\mathbf{s}; X_r)$. Using this notation, we see that (2.5) is $\zeta_2(\mathbf{s}; A_2)$.

Various other functional relations have then been proved in several articles of the authors: on A_2 ((13)), on A_3 ((19), (6), (8), (15)), on $B_2 = C_2$ ((8), (9)), on B_3 and C_3 ((8)), and on G_2 ((14)). A general treatment on the theory of functional relations is given in (13), (9). Those functional relations in fact include various known value-relations among special values of Euler-Zagier sums, and also include Witten's formula (1.2) in several cases.

Moreover, functional relations also exist among zeta-functions of lattices of (not necessarily simply-connected) Lie groups in the sense of (17). In (17), we proved functional relations among zeta-functions whose associated root systems are of type A_2 or C_2.

In this paper we study zeta-functions of weight lattices of compact connected semisimple Lie groups of type A_3. More precisely, we consider zeta-functions of $SU(4)$, $SO(6)$ and $PU(4)$.

In Section 3, we recall the definition of those zeta-functions. In Section 4, we prepare some lemmas which will be necessary later. Then in the remaining sections we prove some functional relations for those zeta-functions which are the main results in this paper (see Theorems 8, 9, 12, 17 and 18). Moreover we give new classes of evaluation formulas for these zeta-functions in terms of the Riemann zeta-function (see Propositions 10 and 13, and Examples 11 and 14) and the Dirichlet L-function with the primitive character of conductor 4 (see Proposition 19 and Examples 20 and 21).

3. ZETA-FUNCTIONS OF WEIGHT LATTICES

In this section, we recall the definition and some properties of zeta-functions of weight lattices of compact connected semisimple Lie groups which we considered in our previous paper (17, Section 3).

We first prepare the same notation as in (9; 11; 13) (see also (6; 7; 10; 14)). Let V be an r-dimensional real vector space equipped with an inner product $\langle \cdot, \cdot \rangle$. The norm $\|\cdot\|$ is defined by $\|v\| = \langle v, v\rangle^{1/2}$. The dual space V^* is identified with V via the inner product of V. Let Δ be a finite reduced root system which may not be irreducible, and $\Psi = \{\alpha_1, \ldots, \alpha_r\}$ its fundamental system. We fix Δ_+ and Δ_- as the set of all positive roots and negative roots respectively. Then we have a decomposition of the root system $\Delta = \Delta_+ \coprod \Delta_-$. Let $Q = Q(\Delta)$ be the root lattice, $Q^\vee$ the coroot lattice, $P = P(\Delta)$ the weight lattice, $P^\vee$ the coweight lattice, and P_+ the set of integral dominant weights defined by

$$Q = \bigoplus_{i=1}^{r} \mathbb{Z}\, \alpha_i, \qquad Q^\vee = \bigoplus_{i=1}^{r} \mathbb{Z}\, \alpha_i^\vee, \tag{3.1}$$

$$P = \bigoplus_{i=1}^{r} \mathbb{Z}\, \lambda_i, \qquad P^\vee = \bigoplus_{i=1}^{r} \mathbb{Z}\, \lambda_i^\vee, \tag{3.2}$$

$$P_+ = \bigoplus_{i=1}^{r} \mathbb{N}_0\, \lambda_i, \tag{3.3}$$

respectively, where the fundamental weights $\{\lambda_j\}_{j=1}^r$ and the fundamental coweights $\{\lambda_j^\vee\}_{j=1}^r$ are the dual bases of $\Psi^\vee$ and Ψ satisfying $\langle \alpha_i^\vee, \lambda_j\rangle = \delta_{ij}$ and $\langle \lambda_i^\vee, \alpha_j\rangle = \delta_{ij}$ respectively. Let

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha = \sum_{j=1}^{r} \lambda_j \tag{3.4}$$

be the lowest strongly dominant weight. Let σ_α be the reflection with respect to a root $\alpha \in \Delta$ defined as

$$\sigma_\alpha : V \to V, \qquad \sigma_\alpha : v \mapsto v - \langle \alpha^\vee, v \rangle \alpha. \tag{3.5}$$

For a subset $A \subset \Delta$, let $W(A)$ be the group generated by reflections σ_α for all $\alpha \in A$. In particular, $W = W(\Delta)$ is the Weyl group, and $\{\sigma_j = \sigma_{\alpha_j} \mid 1 \le j \le r\}$ generates W.

Let $\widetilde{G}$ be a simply-connected compact connected semisimple Lie group, and $\mathfrak{g} = \mathrm{Lie}(\widetilde{G})$. There is a one-to-one correspondence between a compact connected semisimple Lie group G whose universal covering group is $\widetilde{G}$, and a lattice L with $Q(\Delta(\mathfrak{g})) \subset L \subset P(\Delta(\mathfrak{g}))$ up to automorphisms by taking $L = L(G)$ as the weight lattice of G. Let $L_+ = P_+ \cap L$.

Now we recall the definition of the zeta-function of the weight lattice $L = L(G)$ of the semisimple Lie group G, that is,

$$\zeta_r(\mathbf{s}, \mathbf{y}; G) = \zeta_r(\mathbf{s}, \mathbf{y}; L; \Delta) := \sum_{\lambda \in L_+ + \rho} e^{2\pi i \langle \mathbf{y}, \lambda \rangle} \prod_{\alpha \in \Delta_+} \frac{1}{\langle \alpha^\vee, \lambda \rangle^{s_\alpha}}, \tag{3.6}$$

where $\mathbf{y} \in V$. Note that this zeta-function can be continued meromorphically to $\mathbb{C}^n$. When $\mathbf{y} = \mathbf{0}$, we sometimes write this zeta-function as $\zeta_r(\mathbf{s}; G)$ or $\zeta_r(\mathbf{s}; L; \Delta)$ for brevity. It is to be noted that if $G = \widetilde{G}$, then $L = P$ and $\zeta_r(\mathbf{s}; \widetilde{G})$ coincides with $\zeta_r(\mathbf{s}; \mathfrak{g})$, also written as $\zeta_r(\mathbf{s}; \Delta(\mathfrak{g}))$ and $\zeta_r(\mathbf{s}; X_r)$ when $\Delta(\mathfrak{g})$ is of type X_r, which is called the zeta-function of the root system of type X_r studied in our previous papers (see, for example, (6; 10; 11; 13)).

In the present paper we concentrate our attention on the case of type A_3, so we write down the explicit form of zeta-functions in this case. Let $\Delta = \Delta(A_3)$ with $\Psi = \{\alpha_1, \alpha_2, \alpha_3\}$, $\Delta_+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3\}$, $P = \sum_{j=1}^3 \mathbb{Z}\lambda_j$ and $Q = \sum_{j=1}^3 \mathbb{Z}\alpha_j$. It is known that $P/Q \simeq \mathbb{Z}/4\mathbb{Z}$. Therefore there is a unique intermediate lattice L_1 with $P \supsetneq L_1 \supsetneq Q$, satisfying $(L_1 : Q) = 2$. The group corresponding to P (resp. Q) is $SU(4)$ (resp. $PU(4)$). The group $G = G(L_1)$ is $SU(4)/\{\pm 1\}$, which is known to be isomorphic to $SO(6)$. We know (for the details, see (17, Example 4.3)) that

$$\begin{aligned} &\zeta_3(\mathbf{s}, \mathbf{y}; SU(4)) = \zeta_3(\mathbf{s}, \mathbf{y}; P; A_3) \\ &= \sum_{m_1, m_2, m_3 = 1}^{\infty} \frac{e^{2\pi i \langle \mathbf{y}, m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3 \rangle}}{m_1^{s_1} m_2^{s_2} m_3^{s_3} (m_1+m_2)^{s_4} (m_2+m_3)^{s_5} (m_1+m_2+m_3)^{s_6}}, \end{aligned} \tag{3.7}$$

with

$$\lambda_1 = \frac{3}{4}\alpha_1 + \frac{1}{2}\alpha_2 + \frac{1}{4}\alpha_3, \quad \lambda_2 = \frac{1}{2}\alpha_1 + \alpha_2 + \frac{1}{2}\alpha_3, \quad \lambda_3 = \frac{1}{4}\alpha_1 + \frac{1}{2}\alpha_2 + \frac{3}{4}\alpha_3.$$

Note that $\zeta_3(\mathbf{s}, \mathbf{0}; SU(4)) = \zeta_3(\mathbf{s}; A_3)$. Further we have

$$\zeta_3(\mathbf{s}, \mathbf{y}; SO(6)) = \zeta_3(\mathbf{s}, \mathbf{y}; L_1; A_3) \tag{3.8}$$

$$= \sum_{\substack{m_1,m_2,m_3=1 \\ m_1 \equiv m_3 \pmod 2}}^{\infty} \frac{e^{2\pi i \langle \mathbf{y}, m_1\lambda_1+m_2\lambda_2+m_3\lambda_3\rangle}}{m_1^{s_1}m_2^{s_2}m_3^{s_3}(m_1+m_2)^{s_4}(m_2+m_3)^{s_5}(m_1+m_2+m_3)^{s_6}},$$

$$\zeta_3(\mathbf{s},\mathbf{y};PU(4)) = \zeta_3(\mathbf{s},\mathbf{y};Q;A_3) \tag{3.9}$$

$$= \sum_{\substack{m_1,m_2,m_3=1 \\ m_1+2m_2+3m_3 \equiv 2 \pmod 4}}^{\infty} \frac{e^{2\pi i \langle \mathbf{y}, m_1\lambda_1+m_2\lambda_2+m_3\lambda_3\rangle}}{m_1^{s_1}m_2^{s_2}m_3^{s_3}(m_1+m_2)^{s_4}(m_2+m_3)^{s_5}(m_1+m_2+m_3)^{s_6}}.$$

In particular,

$$\zeta_3(\mathbf{s},\lambda_1^\vee;SU(4)) = \zeta_3(\mathbf{s},\lambda_1^\vee;P;A_3) \tag{3.10}$$

$$= \sum_{l,m,n=1}^{\infty} \frac{i^{3l+2m+n}}{l^{s_1}m^{s_2}n^{s_3}(l+m)^{s_4}(m+n)^{s_5}(l+m+n)^{s_6}},$$

$$\zeta_3(\mathbf{s},\lambda_2^\vee;SU(4)) = \zeta_3(\mathbf{s},\lambda_2^\vee;P;A_3) \tag{3.11}$$

$$= \sum_{l,m,n=1}^{\infty} \frac{(-1)^{l+n}}{l^{s_1}m^{s_2}n^{s_3}(l+m)^{s_4}(m+n)^{s_5}(l+m+n)^{s_6}},$$

$$\zeta_3(\mathbf{s},\lambda_3^\vee;SU(4)) = \zeta_3(\mathbf{s},\lambda_3^\vee;P;A_3) \tag{3.12}$$

$$= \sum_{l,m,n=1}^{\infty} \frac{i^{l+2m+3n}}{l^{s_1}m^{s_2}n^{s_3}(l+m)^{s_4}(m+n)^{s_5}(l+m+n)^{s_6}}.$$

Note that

$$\zeta_3((s_1,s_2,s_3,s_4,s_5,s_6),\lambda_1^\vee;SU(4)) = \zeta_3((s_3,s_2,s_1,s_5,s_4,s_6),\lambda_3^\vee;SU(4)), \tag{3.13}$$

$$\zeta_3((s_1,s_2,s_3,s_4,s_5,s_6),\lambda_2^\vee;SU(4)) = \zeta_3((s_3,s_2,s_1,s_5,s_4,s_6),\lambda_2^\vee;SU(4)), \tag{3.14}$$

as well as

$$\zeta_3((s_1,s_2,s_3,s_4,s_5,s_6);A_3) = \zeta_3((s_3,s_2,s_1,s_5,s_4,s_6);A_3). \tag{3.15}$$

There are several reasons why the study on the case A_3 deserves one paper. First, since the case A_2 was studied in (17), it is a natural continuation. Second, since the functional relation for $\zeta_3(\mathbf{s};A_3)$ given in (8) is restricted to the case of even integers, we supply a more general result (Theorem 9) here. The third reason is that Lie algebras of type A_r are the most interesting in view of the theory of (17), because $|P(\Delta(A_r))/Q(\Delta(A_r))| \to \infty$ as $r \to \infty$ (hence there are many intermediate lattices between P and Q), while $|P/Q|$ remains small for any Lie algebras of other types (see Bourbaki (2)).

Lastly in this section we quote here one more general result, which is a generalization and a refinement of (1.2).

THEOREM 1. (17, Theorem 3.2) *For a compact connected semisimple Lie group G, let $\Delta = \Delta(G)$ be its root system, and $L = L(G)$ be its weight lattice. Let $\mathbf{k} = (k_\alpha)_{\alpha\in\Delta_+} \in \mathbb{N}^n$*

($n=|\Delta_+|$) satisfying $k_\alpha = k_\beta$ whenever $\|\alpha\| = \|\beta\|$. Let $\kappa = \sum_{\alpha\in\Delta_+} 2k_\alpha$. Then we have for $\nu \in P^\vee/Q^\vee$,

$$\begin{aligned}\zeta_r(2\mathbf{k},\nu;G) &= \zeta_r(2\mathbf{k},\nu;L;\Delta)\\ &= \frac{(-1)^n}{|W|}\Bigg(\prod_{\alpha\in\Delta_+}\frac{(2\pi\sqrt{-1})^{2k_\alpha}}{(2k_\alpha)!}\Bigg)\mathcal{P}(2\mathbf{k},\nu;L;\Delta) \in \mathbb{Q}\cdot\pi^\kappa,\end{aligned} \tag{3.16}$$

where $\mathcal{P}(2\mathbf{k},\nu;L;\Delta)$ is the Bernoulli function associated with L, defined in (17).

Note that when $L = P$, (3.16) coincides with our previous result in (13, Theorem 4.6).

As an example, here we apply this Theorem to the case of $PU(4)$.

EXAMPLE 2. The generating function of $\mathcal{P}(\mathbf{k},\mathbf{y};A_3)$ has been given in (9, Example 2). Therefore, by (17, (3.8)), we have

$$\begin{aligned}&\mathcal{P}((2,2,2,2,2,2),\mathbf{0};Q;A_3)\\ &\quad= \frac{1}{4}(\mathcal{P}((2,2,2,2,2,2),\mathbf{0};A_3) - \mathcal{P}((2,2,2,2,2,2),\lambda_1^\vee;A_3)\\ &\qquad + \mathcal{P}((2,2,2,2,2,2),\lambda_2^\vee;A_3) - \mathcal{P}((2,2,2,2,2,2),\lambda_3^\vee;A_3))\\ &\quad= \frac{1103}{96888422400},\end{aligned} \tag{3.17}$$

because $2\rho = 3\alpha_1+4\alpha_2+3\alpha_3$ and hence $\langle\mathbf{0},2\rho\rangle = 0$, $\langle\lambda_1^\vee,2\rho\rangle = 3$, $\langle\lambda_2^\vee,2\rho\rangle = 4$, $\langle\lambda_3^\vee,2\rho\rangle = 3$. Therefore by Theorem 1, we obtain

$$\zeta_3((2,2,2,2,2,2),\mathbf{0};PU(4)) = \frac{1103\pi^{12}}{145332633600}. \tag{3.18}$$

4. SOME PREPARATORY LEMMAS

In this section, we give explicit functional relations for double polylogarithms (see Lemma 5 and Corollary 6). By use of these results, we give certain functional relations for triple zeta-functions of weight lattices in the next section.

First we quote the following two lemmas. Let $\phi(s) = \sum_{n\geq1}(-1)^n n^{-s} = \left(2^{1-s}-1\right)\zeta(s)$ and $\varepsilon_m = \{1+(-1)^m\}/2$ for $m\in\mathbb{Z}$. Let $\{B_m(X)\}$ be the Bernoulli polynomials defined by

$$\frac{te^{Xt}}{e^t-1} = \sum_{m=0}^{\infty} B_m(X)\frac{t^m}{m!}.$$

LEMMA 3. ((9, Lemma 9.1), (18, Lemma 2.1)) *Let $c\in[0,2\pi)\subset\mathbb{R}$, and $h:\mathbb{N}_0\to\mathbb{C}$ be a* *function (which may depend on c). Then, for $p \in \mathbb{N}$,*

$$\sum_{j=0}^{p} \phi(p-j)\varepsilon_{p-j} \sum_{\xi=0}^{j} h(j-\xi)\frac{(i(c-\pi))^{\xi}}{\xi!} = -\frac{1}{2}\sum_{\xi=0}^{p} h(p-\xi)\frac{(2\pi i)^{\xi}}{\xi!}B_{\xi}\left(\left\{\frac{c}{2\pi}\right\}\right), \quad (4.1)$$

and

$$\sum_{j=0}^{p} \phi(p-j)\varepsilon_{p-j} \sum_{\xi=0}^{j} h(j-\xi)\frac{(i\pi)^{\xi}}{\xi!} = \sum_{\nu=0}^{[p/2]} \zeta(2\nu)h(p-2\nu) - \frac{i\pi}{2}h(p-1), \quad (4.2)$$

where $[x]$ is the integer part of $x \in \mathbb{R}$ and $\{x\} = x - [x]$.

The next lemma is the key to proving functional relations for zeta-functions. For $h \in \mathbb{N}$, let

$$\mathfrak{C} := \{C(l) \in \mathbb{C} \mid l \in \mathbb{Z},\ l \neq 0\},$$
$$\mathfrak{D} := \{D(N;m;\eta) \in \mathbb{R} \mid N, m, \eta \in \mathbb{Z},\ N \neq 0,\ m \geq 0,\ 1 \leq \eta \leq h\},$$
$$\mathfrak{A} := \{a_\eta \in \mathbb{N} \mid 1 \leq \eta \leq h\}$$

be sets of numbers indexed by integers, and let

$$\binom{x}{k} := \begin{cases} \frac{x(x-1)\cdots(x-k+1)}{k!} & (k \in \mathbb{N}), \\ 1 & (k = 0). \end{cases}$$

LEMMA 4. (8, Lemma 6.2) *With the above notation, we assume that the infinite series appearing in*

$$\sum_{\substack{N\in\mathbb{Z}\\N\neq 0}} (-1)^N C(N)e^{iN\theta} - 2\sum_{\eta=1}^{h}\sum_{k=0}^{a_\eta} \phi(a_\eta - k)\varepsilon_{a_\eta - k} \times \sum_{\xi=0}^{k}\left\{\sum_{\substack{N\in\mathbb{Z}\\N\neq 0}} (-1)^N D(N;k-\xi;\eta)e^{iN\theta}\right\}\frac{(i\theta)^{\xi}}{\xi!} \quad (4.3)$$

are absolutely convergent for $\theta \in [-\pi, \pi]$, and that (4.3) is a constant function for $\theta \in (-\pi, \pi)$. Then, for $d \in \mathbb{N}$,

$$\sum_{\substack{N\in\mathbb{Z}\\N\neq 0}} \frac{(-1)^N C(N)e^{iN\theta}}{N^d} = 2\sum_{\eta=1}^{h}\sum_{k=0}^{a_\eta} \phi(a_\eta - k)\varepsilon_{a_\eta - k} \quad (4.4)$$
$$\times \sum_{\xi=0}^{k}\left\{\sum_{\omega=0}^{k-\xi}\binom{\omega+d-1}{\omega}(-1)^{\omega}\sum_{\substack{m\in\mathbb{Z}\\m\neq 0}} \frac{(-1)^m D(m;k-\xi-\omega;\eta)e^{im\theta}}{m^{d+\omega}}\right\}\frac{(i\theta)^{\xi}}{\xi!}$$
$$-2\sum_{k=0}^{d}\phi(d-k)\varepsilon_{d-k}\sum_{\xi=0}^{k}\left\{\sum_{\eta=1}^{h}\sum_{\omega=0}^{a_\eta - 1}\binom{\omega+k-\xi}{\omega}(-1)^{\omega}\right.$$

$$\times \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \frac{D(m;a_\eta-1-\omega;\eta)}{m^{k-\xi+\omega+1}}\Bigg\}\frac{(i\theta)^\xi}{\xi!}$$

holds for $\theta\in[-\pi,\pi]$*, where the infinite series appearing on both sides of (4.4) are absolutely convergent for* $\theta\in[-\pi,\pi]$.

For $p\in\mathbb{N}$, it is known that (see, for example, (8, (4.31), (4.32)))

$$\lim_{L\to\infty}\sum_{\substack{-L\leq l\leq L\\ l\neq 0}}\frac{(-1)^l e^{il\theta}}{l^p}=2\sum_{j=0}^{p}\phi(p-j)\varepsilon_{p-j}\frac{(i\theta)^j}{j!}\quad(\theta\in(-\pi,\pi)). \tag{4.5}$$

Note that the left-hand side is uniformly convergent for $\theta\in(-\pi,\pi)$ (see (24, § 3.35)), and is also absolutely convergent for $p\geq 2$. We prove the following lemma. Note that the case when p and q are even has been already proved in (8, (7.55)).

LEMMA 5. *For* $p\in\mathbb{N}$, $s\in\mathbb{R}$ *with* $s>1$ *and* $x\in\mathbb{C}$ *with* $|x|=1$,

$$\begin{aligned}&\sum_{\substack{l\neq 0,\ m\geq 1\\ l+m\neq 0}}\frac{(-1)^{l+m}x^m e^{i(l+m)\theta}}{l^p m^s (l+m)^q}\\ &-2\sum_{j=0}^{p}\phi(p-j)\varepsilon_{p-j}\sum_{\xi=0}^{j}\binom{q-1+j-\xi}{q-1}(-1)^{j-\xi}\sum_{m=1}^{\infty}\frac{(-1)^m x^m e^{im\theta}}{m^{s+q+j-\xi}}\frac{(i\theta)^\xi}{\xi!}\\ &+2\sum_{j=0}^{q}\phi(q-j)\varepsilon_{q-j}\sum_{\xi=0}^{j}\binom{p-1+j-\xi}{p-1}(-1)^{p-1}\sum_{m=1}^{\infty}\frac{x^m}{m^{s+p+j-\xi}}\frac{(i\theta)^\xi}{\xi!}=0\end{aligned} \tag{4.6}$$

for $\theta\in[-\pi,\pi]$.

Proof. First we assume $p\geq 2$. Then, for $\theta\in(-\pi,\pi)$, it follows from (4.5) that

$$\left(\sum_{\substack{l\in\mathbb{Z}\\ l\neq 0}}\frac{(-1)^l e^{il\theta}}{l^p}-2\sum_{j=0}^{p}\phi(p-j)\varepsilon_{p-j}\frac{(i\theta)^j}{j}\right)\sum_{m=1}^{\infty}\frac{(-1)^m x^m e^{im\theta}}{m^s}=0, \tag{4.7}$$

where the left-hand side is absolutely and uniformly convergent for $\theta\in(-\pi,\pi)$. Therefore we have

$$\begin{aligned}&\sum_{\substack{l\in\mathbb{Z},\ l\neq 0\\ m\geq 1\\ l+m\neq 0}}\frac{(-1)^{l+m}x^m e^{i(l+m)\theta}}{l^p m^s}-2\sum_{j=0}^{p}\phi(p-j)\varepsilon_{p-j}\left\{\sum_{m=1}^{\infty}\frac{(-1)^m x^m e^{im\theta}}{m^s}\right\}\frac{(i\theta)^j}{j!}\\ &=(-1)^{p+1}\sum_{m=1}^{\infty}\frac{x^m}{m^{s+p}}\end{aligned} \tag{4.8}$$

for $\theta \in (-\pi,\pi)$. Now we use Lemma 4 with $h=1$, $a_1=p$,

$$C(N)=\sum_{\substack{l\neq 0,\ m\geq 1\\ l+m=N}}\frac{x^m}{l^p m^s}\ (N\in\mathbb{Z},\ N\neq 0)$$

and $D(N;\mu;1)=x^N N^{-s}$ (if $\mu=0$ and $N\geq 1$), or $=0$ (otherwise). Under these settings, we see that the left-hand side of (4.8) is of the form (4.3). Furthermore the right-hand side of (4.8) is a constant as a function in θ. Therefore we can apply Lemma 4 with $d=q\in\mathbb{N}$. Then (4.4) gives (4.6) for $p\geq 2$.

Next we prove the case $p=1$. As we proved above, (4.6) in the case $p=2$ holds. Replacing x by $-xe^{i\theta}$ in this case, we have

$$\begin{aligned}&\sum_{\substack{l\neq 0,\ m\geq 1\\ l+m\neq 0}}\frac{(-1)^l x^m e^{il\theta}}{l^2 m^s (l+m)^q}\\ &-2\sum_{j=0}^{2}\phi(2-j)\varepsilon_{2-j}\sum_{\xi=0}^{j}\binom{q-1+j-\xi}{q-1}(-1)^{j-\xi}\sum_{m=1}^{\infty}\frac{x^m}{m^{s+q+j-\xi}}\frac{(i\theta)^\xi}{\xi!}\\ &+2\sum_{j=0}^{q}\phi(q-j)\varepsilon_{q-j}\sum_{\xi=0}^{j}\binom{1+j-\xi}{1}(-1)^1\sum_{m=1}^{\infty}\frac{(-1)^m x^m e^{-im\theta}}{m^{s+2+j-\xi}}\frac{(i\theta)^\xi}{\xi!}=0\end{aligned}\tag{4.9}$$

for $\theta\in[-\pi,\pi]$. We denote the first, the second and the third term on the left hand side of (4.9) by $I_1(\theta)$, $I_2(\theta)$ and $I_3(\theta)$, respectively. We differentiate these terms in θ. We can easily compute $I_1'(\theta)$ and $I_2'(\theta)$. As for $I_3'(\theta)$, we have

$$\begin{aligned}I_3'(\theta)=2\sum_{j=0}^{q}\phi(q-j)\varepsilon_{q-j}\Bigg\{&-i\sum_{\xi=0}^{j}(1+j-\xi)(-1)\sum_{m=1}^{\infty}\frac{(-1)^m x^m e^{-im\theta}}{m^{s+1+j-\xi}}\frac{(i\theta)^\xi}{\xi!}\\ &+i\sum_{\xi=1}^{j}(1+j-\xi)(-1)\sum_{m=1}^{\infty}\frac{(-1)^m x^m e^{-im\theta}}{m^{s+2+j-\xi}}\frac{(i\theta)^{\xi-1}}{(\xi-1)!}\Bigg\}.\end{aligned}$$

Note that as for the second member in the curly brackets on the right-hand side, ξ may also run from 1 to $j+1$ because $1+j-(j+1)=0$ in the summand. Hence, by replacing $\xi-1$ by ξ, we have

$$I_3'(\theta)=2i\sum_{j=0}^{q}\phi(q-j)\varepsilon_{q-j}\sum_{\xi=0}^{j}\sum_{m=1}^{\infty}\frac{(-1)^m x^m e^{-im\theta}}{m^{s+1+j-\xi}}\frac{(i\theta)^\xi}{\xi!}.$$

Thus, we see that $(I_1'(\theta)+I_2'(\theta)+I_3'(\theta))/i$, replacing x by $-xe^{i\theta}$, gives (4.6) in the case $p=1$. This completes the proof.

The following special cases of (4.6) will be necessary in the next section.

COROLLARY 6. *For $p, q \in \mathbb{N}$, $s > 1$ and $x, y \in \mathbb{C}$ with $|x| \le 1$ and $|y| \le 1$, let*

$$\mathfrak{T}(p, s, q; x, y) = \sum_{\substack{l \neq 0,\, m \geq 1 \\ l+m \neq 0}} \frac{x^l y^m}{l^p m^s (l+m)^q}. \tag{4.10}$$

Then

$$\begin{aligned} \mathfrak{T}(p, s, q; 1, 1) &= 2(-1)^p \sum_{k=0}^{[p/2]} \zeta(2k) \binom{p+q-1-2k}{q-1} \zeta(s+p+q-2k) \qquad (4.11) \\ &\quad + 2(-1)^p \sum_{k=0}^{[q/2]} \zeta(2k) \binom{p+q-1-2k}{p-1} \zeta(s+p+q-2k), \\ \mathfrak{T}(p, s, q; -1, 1) &= 2(-1)^p \sum_{k=0}^{[p/2]} \phi(2k) \binom{p+q-1-2k}{q-1} \zeta(s+p+q-2k) \qquad (4.12) \\ &\quad + 2(-1)^p \sum_{k=0}^{[q/2]} \phi(2k) \binom{p+q-1-2k}{p-1} \phi(s+p+q-2k), \\ \mathfrak{T}(p, s, q; 1, -1) &= 2(-1)^p \sum_{k=0}^{[p/2]} \zeta(2k) \binom{p+q-1-2k}{q-1} \phi(s+p+q-2k) \qquad (4.13) \\ &\quad + 2(-1)^p \sum_{k=0}^{[q/2]} \zeta(2k) \binom{p+q-1-2k}{p-1} \phi(s+p+q-2k), \\ \mathfrak{T}(p, s, q; -1, -1) &= 2(-1)^p \sum_{k=0}^{[p/2]} \phi(2k) \binom{p+q-1-2k}{q-1} \phi(s+p+q-2k) \qquad (4.14) \\ &\quad + 2(-1)^p \sum_{k=0}^{[q/2]} \phi(2k) \binom{p+q-1-2k}{p-1} \zeta(s+p+q-2k). \end{aligned}$$

Proof. We can directly obtain (4.14) and (4.12) by letting $(x, \theta) = (1, 0)$, $(-1, 0)$, respectively in (4.6). (Since $\theta = 0$, all the terms corresponding to $\xi \geq 1$ vanish on the right-hand side of (4.6).) As for (4.13) and (4.11), we let $(x, \theta) = (-1, \pi)$, $(1, \pi)$, respectively in (4.6) and use Lemma 3.

REMARK 7. The results of the above corollary are not new. The formula (4.11) was first proved in (23), and then, by a different method, Nakamura (20, Theorem 3.1) has shown all of the above (see also a survey given in (8, Section 3)).

5. THE ZETA-FUNCTION OF $SU(4)$

Now we start to prove functional relations for zeta-functions for lattices of type A_3. We use the same technique as in our previous paper (8, Section 7). Hence the details of their proofs will be omitted.

In this section we study the zeta-function associated with the group $SU(4)$. Our starting point is similar to (4.7), or (8, Equation (7.58)). We begin by considering

$$\left(\sum_{\substack{l\in\mathbb{Z}\\ l\neq 0}}\frac{(-1)^l e^{il\theta}}{l^p}-2\sum_{j=0}^{p}\phi(p-j)\varepsilon_{p-j}\frac{(i\theta)^j}{j}\right)\sum_{\substack{m\in\mathbb{Z},\ m\neq 0\\ n\geq 1\\ m+n\neq 0}}\frac{(-1)^{m+n}x^m y^n e^{i(m+n)\theta}}{m^q n^s (m+n)^b}=0 \quad (5.1)$$

for $\theta\in[-\pi,\pi]$, where $p,q,b\in\mathbb{N}$ with $p\geq 2$, $s\in\mathbb{R}$ with $s>1$ and $x,y\in\mathbb{C}$ with $|x|=|y|=1$. Then, by the (almost) same argument as in (8, pp. 158-160), we obtain

$$\sum_{\substack{l,m\neq 0,\ n\geq 1\\ l+m\neq 0,\ m+n\neq 0\\ l+m+n\neq 0}}\frac{(-1)^{l+m}x^m y^n e^{i(l+m)\theta}}{l^p m^q n^s (l+m)^a (m+n)^b (l+m+n)^c} \quad (5.2)$$

$$\begin{aligned}
&=2\sum_{k=0}^{p}\phi(p-k)\varepsilon_{p-k}\sum_{\xi=0}^{k}\sum_{\omega=0}^{k-\xi}\binom{\omega+a-1}{\omega}(-1)^{\omega}\binom{k-\xi-\omega+c-1}{k-\xi-\omega}\\
&\quad\times(-1)^{k-\xi-\omega}\sum_{\substack{m\neq 0\\ n\geq 1\\ m+n\neq 0}}\frac{(-1)^m x^m y^n e^{im\theta}}{m^{q+a+\omega}n^s(m+n)^{b+c+k-\xi-\omega}}\frac{(i\theta)^{\xi}}{\xi!}\\
&-2\sum_{j=0}^{c}\phi(c-k)\varepsilon_{c-k}\sum_{\xi=0}^{k}\sum_{\omega=0}^{k-\xi}\binom{\omega+a-1}{\omega}(-1)^{\omega}\binom{k-\xi-\omega+p-1}{p-1}\\
&\quad\times(-1)^{p-1+a+\omega}\sum_{\substack{m\neq 0\\ n\geq 1\\ m+n\neq 0}}\frac{(-1)^n x^m y^n e^{-in\theta}}{m^q n^{s+a+\omega}(m+n)^{p+b+k-\xi-\omega}}\frac{(i\theta)^{\xi}}{\xi!}\\
&-2\sum_{k=0}^{a}\phi(a-k)\varepsilon_{a-k}\sum_{\xi=0}^{k}\sum_{\omega=0}^{p-1}\binom{\omega+k-\xi}{\omega}(-1)^{\omega}\binom{p+c-2-\omega}{p-1-\omega}\\
&\quad\times(-1)^{p-1-\omega}\sum_{\substack{m\neq 0\\ n\geq 1\\ m+n\neq 0}}\frac{x^m y^n}{m^{q+k-\xi+\omega+1}n^s(m+n)^{p+b+c-1-\omega}}\frac{(i\theta)^{\xi}}{\xi!}\\
&+2\sum_{k=0}^{a}\phi(a-k)\varepsilon_{a-k}\sum_{\xi=0}^{k}\sum_{\omega=0}^{c-1}\binom{\omega+k-\xi}{\omega}(-1)^{\omega}\binom{p+c-2-\omega}{p-1}\\
&\quad\times(-1)^{p+k-\xi+\omega}\sum_{\substack{m\neq 0\\ n\geq 1\\ m+n\neq 0}}\frac{x^m y^n}{m^q n^{s+k-\xi+\omega+1}(m+n)^{p+b+c-1-\omega}}\frac{(i\theta)^{\xi}}{\xi!}
\end{aligned}$$

for $\theta \in [-\pi, \pi]$ and $p, q, a, b, c \in \mathbb{N}$. (A small difference is that, in the course of the argument, we replaced x by $-e^{-i\theta}$ in (8), while this time we replace y by $-ye^{-i\theta}$.) Note that (5.2) in the case $p = 1$ can be proved similarly to Lemma 5.

Now we put $(x, y, \theta) = (-1, -1, 0)$ in (5.2), namely we take notice of the constant term of (5.2). We proceed similarly to the argument in (8); that is, we decompose the left-hand side of (5.2) by the method written in (8, p. 160), while apply (4.2) to the right-hand side. Then we obtain the following theorem.

THEOREM 8. *For* $p, q, a, b, c \in \mathbb{N}$,

$$\begin{aligned}
&\zeta_3((p,q,s,a,b,c),\lambda_2^\vee;SU(4)) + (-1)^p\zeta_3((p,a,s,q,c,b),\lambda_2^\vee;SU(4)) \qquad (5.3)\\
&+(-1)^{p+a}\zeta_3((q,a,c,p,s,b),\lambda_2^\vee;SU(4)) + (-1)^{p+a+c}\zeta_3((q,s,c,b,a,p),\lambda_2^\vee;SU(4))\\
&+(-1)^{q}\zeta_3((a,q,b,p,s,c),\lambda_2^\vee;SU(4)) + (-1)^{q+b}\zeta_3((a,s,b,c,q,p),\lambda_2^\vee;SU(4))\\
&+(-1)^{q+a}\zeta_3((a,p,b,q,c,s),\lambda_2^\vee;SU(4)) + (-1)^{q+a+b}\zeta_3((a,c,b,s,p,q),\lambda_2^\vee;SU(4))\\
&+(-1)^{q+a+b+c}\zeta_3((s,c,p,a,b,q),\lambda_2^\vee;SU(4)) + (-1)^{p+q+a}\zeta_3((q,p,c,a,b,s),\lambda_2^\vee;SU(4))\\
&+(-1)^{p+q+a+c}\zeta_3((q,b,c,s,p,a),\lambda_2^\vee;SU(4))\\
&+(-1)^{p+q+a+b+c}\zeta_3((s,b,p,q,c,a),\lambda_2^\vee;SU(4))\\
&\quad = 2(-1)^p\sum_{j=0}^{[p/2]}\phi(2j)\sum_{\omega=0}^{p-2j}\binom{\omega+a-1}{\omega}\binom{p+c-2j-\omega-1}{c-1}\\
&\qquad\quad \times\mathfrak{T}(q+a+\omega,s,p+b+c-2j-\omega;1,-1)\\
&\quad + 2(-1)^{p+a}\sum_{j=0}^{[c/2]}\phi(2j)\sum_{\omega=0}^{c-2j}\binom{\omega+a-1}{\omega}\binom{p+c-2j-\omega-1}{p-1}\\
&\qquad\quad \times\mathfrak{T}(q,s+a+\omega,p+b+c-2j-\omega;-1,1)\\
&\quad + 2(-1)^{p}\sum_{j=0}^{[a/2]}\phi(2j)\sum_{\omega=0}^{p-1}\binom{\omega+a-2j}{\omega}\binom{p+c-2-\omega}{c-1}\\
&\qquad\quad \times\mathfrak{T}(q+a-2j+\omega+1,s,p+b+c-1-\omega;-1,-1)\\
&\quad + 2(-1)^{p+a}\sum_{j=0}^{[a/2]}\phi(2j)\sum_{\omega=0}^{c-1}\binom{\omega+a-2j}{\omega}\binom{p+c-2-\omega}{p-1}\\
&\qquad\quad \times\mathfrak{T}(q,s+a-2j+\omega+1,p+b+c-1-\omega;-1,-1)
\end{aligned}$$

holds for $s \in \mathbb{C}$ *except for singularities of functions on both sides, where* $\mathfrak{T}(p, s, q; x, y)$ *is defined by* (4.10)*. Moreover, from* (4.11)-(4.14) *we see that the right-hand side of the above can be written in terms of the Riemann zeta-function.*

Setting $(a,b,c,p,q,s)=(2k,2k,2k,2k,2k,2k)$ for $k\in\mathbb{N}$, we obtain

$$\zeta_3((2k,2k,2k,2k,2k,2k),\lambda_2^\vee;SU(4))\in\mathbb{Q}\cdot\pi^{12k},$$

and the rational coefficients can be determined explicitly. This gives an example of (17, Theorem 3.2).

Similarly, by putting $(x,y,\theta)=(1,1,\pi)$ in (5.2) and using (4.2) in Lemma 3, we obtain the following theorem for $\zeta_3(\mathbf{s};A_3)=\zeta_3(\mathbf{s},\mathbf{0};SU(4))$ (see (3.7)).

THEOREM 9. *For* $p,q,a,b,c\in\mathbb{N}$,

$$\begin{aligned}
&\zeta_3((p,q,s,a,b,c);A_3)+(-1)^p\zeta_3((p,a,s,q,c,b);A_3)\\
&+(-1)^{p+a}\zeta_3((q,a,c,p,s,b);A_3)+(-1)^{p+a+c}\zeta_3((q,s,c,b,a,p);A_3)\\
&+(-1)^{q}\zeta_3((a,q,b,p,s,c);A_3)+(-1)^{q+b}\zeta_3((a,s,b,c,q,p);A_3)\\
&+(-1)^{q+a}\zeta_3((a,p,b,q,c,s);A_3)+(-1)^{q+a+b}\zeta_3((a,c,b,s,p,q);A_3)\\
&+(-1)^{q+a+b+c}\zeta_3((s,c,p,a,b,q);A_3)+(-1)^{p+q+a}\zeta_3((q,p,c,a,b,s);A_3)\\
&+(-1)^{p+q+a+c}\zeta_3((q,b,c,s,p,a);A_3)+(-1)^{p+q+a+b+c}\zeta_3((s,b,p,q,c,a);A_3)\\
&\quad=2(-1)^p\sum_{j=0}^{[p/2]}\zeta(2j)\sum_{\omega=0}^{p-2j}\binom{\omega+a-1}{\omega}\binom{p+c-2j-\omega-1}{c-1}\\
&\qquad\times\mathfrak{T}(q+a+\omega,s,p+b+c-2j-\omega;1,1)\\
&\quad+2(-1)^{p+a}\sum_{j=0}^{[c/2]}\zeta(2j)\sum_{\omega=0}^{c-2j}\binom{\omega+a-1}{\omega}\binom{p+c-2j-\omega-1}{p-1}\\
&\qquad\times\mathfrak{T}(q,s+a+\omega,p+b+c-2j-\omega;1,1)\\
&\quad+2(-1)^p\sum_{j=0}^{[a/2]}\zeta(2j)\sum_{\omega=0}^{p-1}\binom{\omega+a-2j}{\omega}\binom{p+c-2-\omega}{c-1}\\
&\qquad\times\mathfrak{T}(q+a-2j+\omega+1,s,p+b+c-1-\omega;1,1)\\
&\quad+2(-1)^{p+a}\sum_{j=0}^{[a/2]}\zeta(2j)\sum_{\omega=0}^{c-1}\binom{\omega+a-2j}{\omega}\binom{p+2-2-\omega}{p-1}\\
&\qquad\times\mathfrak{T}(q,s+a-2j+\omega+1,p+b+c-\omega-1;1,1)
\end{aligned}\tag{5.4}$$

holds for $s\in\mathbb{C}$ *except for singularities of functions on both sides.*

When p,q,a,b,c are all even, this theorem has already been proved in (8, Theorem 7.1). The expression of the left-hand side in (8) is a little different from the above, but we can easily check that those two expressions are equal, using (3.15).

Setting $(a,b,c,p,q,s)=(2k,2k,2k,2k,2k,2k)$ for $k\in\mathbb{N}$, we obtain (1.2) for A_3 with the explicit value of the coefficient. On the other hand, when $p = q = a = b = c$ which is an odd integer, the left-hand sides of the above two theorems are equal to 0. This is because $\zeta_3((p,p,p,s,p,p);A_3) = \zeta_3((p,p,p,p,s,p);A_3)$ and $\zeta_3((p,p,p,s,p,p),\lambda_2^\vee;SU(4)) = \zeta_3((p,p,p,p,s,p),\lambda_2^\vee;SU(4))$, by (3.14), (3.15). Hence, unfortunately we can obtain no information about, for example, $\zeta_3((2k+1,2k+1,2k+1,2k+1,2k+1,2k+1);A_3)$ and $\zeta_3(((2k+1,2k+1,2k+1,2k+1,2k+1,2k+1)),\lambda_2^\vee;SU(4))$ $(k \in \mathbb{N}_0)$ from the above theorems.

However, choosing (a,b,c,p,q,s) suitably, we can obtain some classes of evaluation formulas for them. For example, set

$$(a,b,c,p,q,s) = (2k+1,2k+1,2k+1,2k+1,2k,2k+1) \qquad (k \in \mathbb{N})$$

in (5.3) and (5.4). Then the left-hand sides of them are

$$\begin{aligned}
&2\zeta_3((2k+1,2k,2k+1,2k+1,2k+1,2k+1),\lambda_2^\vee;SU(4)) \\
&\qquad -2\zeta_3((2k+1,2k+1,2k+1,2k,2k+1,2k+1),\lambda_2^\vee;SU(4)) \\
&\quad = -2\zeta_3((2k,2k+1,2k+1,2k+1,2k+1,2k+1),\lambda_2^\vee;SU(4)), \\
&2\zeta_3((2k+1,2k,2k+1,2k+1,2k+1,2k+1);A_3) \\
&\qquad -2\zeta_3((2k+1,2k+1,2k+1,2k,2k+1,2k+1);A_3) \\
&\quad = -2\zeta_3((2k,2k+1,2k+1,2k+1,2k+1,2k+1);A_3),
\end{aligned}$$

respectively, by using the relation

$$\frac{1}{l^{2k+1}m^{2k}(l+m)^{2k+1}} - \frac{1}{l^{2k+1}m^{2k+1}(l+m)^{2k}} = -\frac{1}{l^{2k}m^{2k+1}(l+m)^{2k+1}}.$$

Therefore we obtain the following.

PROPOSITION 10. *For $k \in \mathbb{N}$,*

$$\begin{aligned}
&\zeta_3((2k,2k+1,2k+1,2k+1,2k+1,2k+1),\lambda_2^\vee;SU(4)) \in \mathbb{Q}[\{\zeta(j)\,|\,j \in \mathbb{N}_{>1}\}], \\
&\zeta_3((2k,2k+1,2k+1,2k+1,2k+1,2k+1);A_3) \in \mathbb{Q}[\{\zeta(j)\,|\,j \in \mathbb{N}_{>1}\}],
\end{aligned}$$

and the rational coefficients can be determined explicitly.

EXAMPLE 11. Setting $(a,b,c,p,q,s) = (2k+1,2k+1,2k+1,2k+1,2k,2k+1)$ in (5.4), we can obtain

$$\begin{aligned}
&\zeta_3(2,3,3,3,3,3;A_3) = \frac{\pi^6}{63}\,\zeta(11) + \frac{199\pi^4}{30}\,\zeta(13) - 365\pi^2\,\zeta(15) + 2941\,\zeta(17), \\
&\zeta_3(4,5,5,5,5,5;A_3)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{152\pi^{12}}{18243225}\,\zeta(17) + \frac{17\pi^{10}}{6237}\,\zeta(19) + \frac{29\pi^8}{54}\,\zeta(21) + \frac{979\pi^6}{9}\,\zeta(23)\\
&\quad + \frac{15585\pi^4}{2}\,\zeta(25) - 660975\pi^2\,\zeta(27) + 5654565\,\zeta(29),
\end{aligned}
$$

$$
\begin{aligned}
&\zeta_3(6,7,7,7,7,7;A_3)\\
&= \frac{2062\pi^{18}}{506224616625}\,\zeta(23) + \frac{11776\pi^{16}}{5367718125}\,\zeta(25) + \frac{8\pi^{14}}{13365}\,\zeta(27)\\
&\quad + \frac{10223594\pi^{12}}{91216125}\,\zeta(29) + \frac{103486\pi^{10}}{6075}\,\zeta(31) + \frac{5459978\pi^8}{2025}\,\zeta(33)\\
&\quad + \frac{3464974\pi^6}{15}\,\zeta(35) + \frac{41963621\pi^4}{3}\,\zeta(37) - 1456076440\pi^2\,\zeta(39)\\
&\quad + 12758984832\,\zeta(41),
\end{aligned}
$$

$$
\begin{aligned}
&\zeta_3(8,9,9,9,9,9;A_3)\\
&= \frac{64586\pi^{24}}{37355158168453125}\,\zeta(29) + \frac{422704\pi^{22}}{298841265347625}\,\zeta(31)\\
&\quad + \frac{10664\pi^{20}}{19088409375}\,\zeta(33) + \frac{663259\pi^{18}}{4632120675}\,\zeta(35) + \frac{2307883\pi^{16}}{84341250}\,\zeta(37)\\
&\quad + \frac{6327646\pi^{14}}{1488375}\,\zeta(39) + \frac{860790601\pi^{12}}{1488375}\,\zeta(41) + \frac{380997529\pi^{10}}{4725}\,\zeta(43)\\
&\quad + \frac{7867619353\pi^8}{1050}\,\zeta(45) + \frac{164035120733\pi^6}{315}\,\zeta(47) + \frac{59740238129\pi^4}{2}\,\zeta(49)\\
&\quad - 3514635376395\pi^2\,\zeta(51) + 31198575194215\,\zeta(53),
\end{aligned}
$$

$$
\begin{aligned}
&\zeta_3(10,11,11,11,11,11,11;A_3)\\
&= \frac{221912776\pi^{30}}{332660210652234981140625}\,\zeta(35) + \frac{10705232\pi^{28}}{13854831558583640625}\,\zeta(37)\\
&\quad + \frac{5135896\pi^{26}}{12250072111921875}\,\zeta(39) + \frac{4767865562\pi^{24}}{33250195732359375}\,\zeta(41) + \frac{222974564\pi^{22}}{6269397175125}\,\zeta(43)\\
&\quad + \frac{24806393774\pi^{20}}{3569532553125}\,\zeta(45) + \frac{2589565814\pi^{18}}{2290609125}\,\zeta(47) + \frac{1188339011\pi^{16}}{7441875}\,\zeta(49)\\
&\quad + \frac{3650193872\pi^{14}}{178605}\,\zeta(51) + \frac{11782765221344\pi^{12}}{4465125}\,\zeta(53) + \frac{35232949154\pi^{10}}{135}\,\zeta(55)\\
&\quad + \frac{18601660627979\pi^8}{945}\,\zeta(57) + \frac{393366314952754\pi^6}{315}\,\zeta(59)\\
&\quad + \frac{1050680447134747\pi^4}{15}\,\zeta(61) - 8947964548486678\pi^2\,\zeta(63)\\
&\quad + 80075393000830422\,\zeta(65).
\end{aligned}
$$

Also, setting $(a,b,c,p,q,s) = (2k+1, 2k+1, 2k+1, 2k+1, 2k, 2k+1)$ in (5.3), we can obtain

$$
\begin{aligned}
&\zeta_3((2,3,3,3,3,3),\lambda_2^\vee;SU(4))\\
&= \frac{17\pi^8}{344064}\zeta(9) + \frac{22847\pi^6}{1720320}\zeta(11) + \frac{49005\pi^4}{16384}\zeta(13) + \frac{3768307\pi^2}{98304}\zeta(15) - \frac{11189819}{16384}\zeta(17),\\
&\zeta_3((4,5,5,5,5,5),\lambda_2^\vee;SU(4))
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{693547\pi^{14}}{51011754393600}\zeta(15) + \frac{714624223\pi^{12}}{81618807029760}\zeta(17) + \frac{28726157\pi^{10}}{11072962560}\zeta(19) \\
&+ \frac{25906094783\pi^{8}}{54358179840}\zeta(21) + \frac{9177921545\pi^{6}}{113246208}\zeta(23) + \frac{2422120970909\pi^{4}}{671088640}\zeta(25) \\
&+ \frac{7798050014825\pi^{2}}{134217728}\zeta(27) - \frac{270498379148235}{268435456}\zeta(29),
\end{aligned}
$$

$\zeta_3((6,7,7,7,7,7),\lambda_2^\vee;SU(4))$

$$
\begin{aligned}
&= \frac{2752145869\pi^{20}}{773055350341160140800}\zeta(21) + \frac{1098434242057681\pi^{18}}{255108265612582846464000}\zeta(23) \\
&+ \frac{150866953637\pi^{16}}{68882685493248000}\zeta(25) + \frac{20612241204619\pi^{14}}{34824024332697600}\zeta(27) \\
&+ \frac{19614225808011463\pi^{12}}{179094982282444800}\zeta(29) + \frac{6776217678200971\pi^{10}}{417470821171200}\zeta(31) \\
&+ \frac{337234670875566533\pi^{8}}{139156940390400}\zeta(33) + \frac{7289362333395816433\pi^{6}}{43293270343680}\zeta(35) \\
&+ \frac{3412540143011100899\pi^{4}}{515396075520}\zeta(37) + \frac{32450037853433343325\pi^{2}}{274877906944}\zeta(39) \\
&- \frac{2744091345586219901 25}{137438953472}\zeta(41).
\end{aligned}
$$

The authors also checked, by using Mathematica 8, that the above formulas agree with numerical computation, based on the definitions of zeta-functions.

6. THE ZETA-FUNCTION OF $SO(6)$

Next we consider $\zeta_3(\mathbf{s};SO(6))$. It follows from (3.7) and (3.8) that

$$
\zeta_3(\mathbf{s};SO(6)) = \frac{1}{2}\left\{\zeta_3(\mathbf{s};A_3) + \zeta_3(\mathbf{s},\lambda_2^\vee;SU(4))\right\}. \tag{6.1}
$$

Combining Theorems 8 and 9 and using (6.1), we can obtain functional relations among $\zeta_3(\mathbf{s};SO(6))$ and $\zeta(s)$.

THEOREM 12. *For* $p,q,a,b,c\in\mathbb{N}$,

$$
\begin{aligned}
&\zeta_3((p,q,s,a,b,c);SO(6)) + (-1)^p\zeta_3((p,a,s,q,c,b);SO(6)) \\
&+ (-1)^{p+a}\zeta_3((q,a,c,p,s,b);SO(6)) + (-1)^{p+a+c}\zeta_3((q,s,c,b,a,p);SO(6)) \\
&+ (-1)^{q}\zeta_3((a,q,b,p,s,c);SO(6)) + (-1)^{q+b}\zeta_3((a,s,b,c,q,p);SO(6)) \\
&+ (-1)^{q+a}\zeta_3((a,p,b,q,c,s);SO(6)) + (-1)^{q+a+b}\zeta_3((a,c,b,s,p,q);SO(6)) \\
&+ (-1)^{q+a+b+c}\zeta_3((s,c,p,a,b,q);SO(6)) + (-1)^{p+q+a}\zeta_3((q,p,c,a,b,s);SO(6)) \\
&+ (-1)^{p+q+a+c}\zeta_3((q,b,c,s,p,a);SO(6)) + (-1)^{p+q+a+b+c}\zeta_3((s,b,p,q,c,a);SO(6)) \\
&= \frac{1}{2}\left(J_0 + J_2\right)
\end{aligned} \tag{6.2}
$$

holds for $s \in \mathbb{C}$ *except for singularities of functions on both sides, where* J_0 *and* J_2 *are the right-hand sides of* (5.4) *and* (5.3)*, respectively.*

Similarly to Proposition 10 and Example 11, we can obtain the following.

PROPOSITION 13. *For* $k \in \mathbb{N}$*,*

$$\zeta_3((2k,2k,2k,2k,2k,2k);SO(6)) \in \mathbb{Q}\cdot\pi^{12k},$$
$$\zeta_3((2k,2k+1,2k+1,2k+1,2k+1,2k+1);SO(6)) \in \mathbb{Q}[\{\zeta(j)\,|\,j\in\mathbb{N}_{>1}\}],$$

and the rational coefficients can be determined explicitly.

EXAMPLE 14. Setting $(a,b,c,p,q,s)=(2,2,2,2,2,s)$ in (6.2), we obtain

$$\begin{aligned} &2\zeta_3((2,s,2,2,2,2);SO(6))+4\zeta_3((2,2,s,2,2,2);SO(6)) \\ &+4\zeta_3((2,2,2,s,2,2);SO(6))+2\zeta_3((2,2,2,2,2,s);SO(6)) \\ &=\left(93\cdot 2^{-s-8}+306\right)\zeta(s+10)+\left(3\cdot 2^{-s-4}-260\right)\zeta(2)\zeta(s+8) \\ &-\left(67\cdot 2^{-s-6}-110\right)\zeta(4)\zeta(s+6)-\frac{1}{8}\left(5\cdot 2^{-s-3}-21\right)\zeta(6)\zeta(s+4). \end{aligned} \tag{6.3}$$

In particular, when $s=2$ in (6.3), we have

$$\zeta_3((2,2,2,2,2,2);SO(6))=\frac{10411}{1307674368000}\pi^{12}. \tag{6.4}$$

Also, combining (6.1) and the results in Example 11, we obtain, for example,

$$\begin{aligned} \zeta_3((2,3,3,3,3,3);SO(6))=&\frac{17\pi^8}{688128}\zeta(9)+\frac{150461\pi^6}{10321920}\zeta(11)+\frac{2365283\pi^4}{491520}\zeta(13) \\ &-\frac{32112653\pi^2}{196608}\zeta(15)+\frac{36995525}{32768}\zeta(17). \end{aligned} \tag{6.5}$$

7. THE ZETA-FUNCTION OF $PU(4)$

Finally we consider the case of the group $PU(4)$. An interesting feature in this case is the appearance of a Dirichlet L-function, so we will describe some details of the argument.

First we slightly generalize the results used in the previous sections. Let

$$\phi(s;\alpha)=\sum_{m=1}^{\infty}e^{2m\pi i\alpha}m^{-s}$$

be the Lerch zeta-function for $\alpha\in\mathbb{R}$. We can easily see that $\phi(s;1/2)$ is equal to $\phi(s)=$

$\left(2^{1-s}-1\right)\zeta(s)$ used in Section 4, and

$$\phi(s;1/4)=2^{-s}\left(2^{1-s}-1\right)\zeta(s)+iL(s,\chi_4), \tag{7.1}$$

$$\phi(s;-1/4)=2^{-s}\left(2^{1-s}-1\right)\zeta(s)-iL(s,\chi_4), \tag{7.2}$$

where $L(s,\chi_4)=\sum_{m\geq 0}(-1)^m(2m+1)^{-s}$ be the Dirichlet L-function associated with the primitive Dirichlet character χ_4 of conductor 4. Moreover we let

$$\Lambda(s;i)=\sum_{m\in\mathbb{Z}\setminus\{0\}}\frac{i^m}{m^s}=2^{-s}\left(1+e^{-\pi is}\right)\left(2^{1-s}-1\right)\zeta(s)+i\left(1-e^{-\pi is}\right)L(s,\chi_4),$$

$$\Lambda(s;-i)=\sum_{m\in\mathbb{Z}\setminus\{0\}}\frac{(-i)^m}{m^s}=2^{-s}\left(1+e^{-\pi is}\right)\left(2^{1-s}-1\right)\zeta(s)-i\left(1-e^{-\pi is}\right)L(s,\chi_4),$$

where $m^{-s}=\exp(-s(\log|m|+\pi i))$ for $m<0$. In particular, for $k\in\mathbb{N}$ and $l\in\mathbb{N}_0$, we have

$$\Lambda(2k;i)=2^{1-2k}\left(2^{1-2k}-1\right)\zeta(2k);\quad \Lambda(2l+1;i)=2iL(2l+1,\chi_4), \tag{7.3}$$

$$\Lambda(2k;-i)=2^{1-2k}\left(2^{1-2k}-1\right)\zeta(2k);\quad \Lambda(2l+1;-i)=-2iL(2l+1,\chi_4). \tag{7.4}$$

Also, it is well-known that

$$\lim_{K\to\infty}\sum_{\substack{k=-K\\k\neq 0}}^{K}\frac{e^{2\pi ik\alpha}}{k^j}=-B_j(\alpha)\frac{(2\pi i)^j}{j!}\quad (j\in\mathbb{N};\ \alpha\in[0,1)) \tag{7.5}$$

(see, for example, (1, Theorem 12.19)). Here, setting $c=\pi/2$ and $3\pi/2$ in (4.1) and using (7.5) with $\alpha=\pm 1/4$, we obtain the following.

LEMMA 15. *For any $p\in\mathbb{N}$ and any function $h:\mathbb{N}_0\to\mathbb{C}$,*

$$\sum_{j=0}^{p}\phi(p-j)\varepsilon_{p-j}\sum_{\xi=0}^{j}h(j-\xi)\frac{(-i\pi/2)^\xi}{\xi!}=\frac{1}{2}\sum_{\xi=0}^{p}\Lambda(\xi;i)h(p-\xi), \tag{7.6}$$

$$\sum_{j=0}^{p}\phi(p-j)\varepsilon_{p-j}\sum_{\xi=0}^{j}h(j-\xi)\frac{(i\pi/2)^\xi}{\xi!}=\frac{1}{2}\sum_{\xi=0}^{p}\Lambda(\xi;-i)h(p-\xi). \tag{7.7}$$

Set $(x,\theta)=(-i,0)$, $(-i,\pi/2)$, $(-1,\pi/2)$, $(i,0)$, $(i,-\pi/2)$, $(-1,-\pi/2)$ in (4.6) and use Lemma 15. Then, by the same method as in the proof of Corollary 6, we obtain the following.

LEMMA 16. *For $p,q\in\mathbb{N}$ and $s>1$,*

$$\mathfrak{T}(p,s,q;-1,i)=2(-1)^p\sum_{k=0}^{[p/2]}\phi(2k)\binom{p+q-1-2k}{q-1}\phi(s+p+q-2k;1/4) \tag{7.8}$$

$$+ 2(-1)^p \sum_{k=0}^{[q/2]} \phi(2k)\binom{p+q-1-2k}{p-1}\phi(s+p+q-2k;-1/4),$$

$$\mathfrak{T}(p,s,q;-i,-1) = (-1)^p \sum_{l=0}^{p} \mathit{\Lambda}(l;-i)(-1)^l \binom{p+q-1-l}{q-1}\phi(s+p+q-l) \tag{7.9}$$

$$+ (-1)^p \sum_{l=0}^{q} \mathit{\Lambda}(l;-i)\binom{p+q-1-l}{p-1}\phi(s+p+q-l;-1/4),$$

$$\mathfrak{T}(p,s,q;-i,i) = (-1)^p \sum_{l=0}^{p} \mathit{\Lambda}(l;-i)(-1)^l \binom{p+q-1-l}{q-1}\phi(s+p+q-l;1/4) \tag{7.10}$$

$$+ (-1)^p \sum_{l=0}^{q} \mathit{\Lambda}(l;-i)\binom{p+q-1-l}{p-1}\phi(s+p+q-l),$$

$$\mathfrak{T}(p,s,q;-1,-i) = 2(-1)^p \sum_{k=0}^{[p/2]} \phi(2k)\binom{p+q-1-2k}{q-1}\phi(s+q+j;-1/4) \tag{7.11}$$

$$+ 2(-1)^p \sum_{k=0}^{[q/2]} \phi(2k)\binom{p+q-1-2k}{p-1}\phi(s+p+q-2k;1/4),$$

$$\mathfrak{T}(p,s,q;i,-1) = (-1)^p \sum_{l=0}^{p} \mathit{\Lambda}(l;i)(-1)^l \binom{p+q-1-l}{q-1}\phi(s+p+q-l) \tag{7.12}$$

$$+ (-1)^p \sum_{l=0}^{q} \mathit{\Lambda}(l;i)\binom{p+q-1-l}{p-1}\phi(s+p+q-l;1/4),$$

$$\mathfrak{T}(p,s,q;i,-i) = (-1)^p \sum_{l=0}^{p} \mathit{\Lambda}(l;i)(-1)^l \binom{p+q-1-l}{q-1}\phi(s+p+q-l;-1/4) \tag{7.13}$$

$$+ (-1)^p \sum_{l=0}^{q} \mathit{\Lambda}(l;i)\binom{p+q-1-l}{p-1}\phi(s+p+q-l).$$

Setting $(x,y,\theta) = (-i,i,\pi/2)$ and $(i,-i,-\pi/2)$ in (5.2), and using Lemma 15, we obtain the following.

THEOREM 17. *For* $p,q,a,b,c \in \mathbb{N}$,

$$\begin{aligned}
&\zeta_3((p,q,s,a,b,c),\lambda_1^\vee;SU(4)) + (-1)^p\zeta_3((p,a,s,q,c,b),\lambda_1^\vee;SU(4)) \\
&+ (-1)^{p+a}\zeta_3((q,a,c,p,s,b),\lambda_1^\vee;SU(4)) + (-1)^{p+a+c}\zeta_3((q,s,c,b,a,p),\lambda_1^\vee;SU(4)) \\
&+ (-1)^{q}\zeta_3((a,q,b,p,s,c),\lambda_1^\vee;SU(4)) + (-1)^{q+b}\zeta_3((a,s,b,c,q,p),\lambda_1^\vee;SU(4)) \\
&+ (-1)^{q+a}\zeta_3((a,p,b,q,c,s),\lambda_1^\vee;SU(4)) + (-1)^{q+a+b}\zeta_3((a,c,b,s,p,q),\lambda_1^\vee;SU(4)) \\
&+ (-1)^{q+a+b+c}\zeta_3((s,c,p,a,b,q),\lambda_1^\vee;SU(4)) + (-1)^{p+q+a}\zeta_3((q,p,c,a,b,s),\lambda_1^\vee;SU(4)) \\
&+ (-1)^{p+q+a+c}\zeta_3((q,b,c,s,p,a),\lambda_1^\vee;SU(4))
\end{aligned} \tag{7.14}$$

$$
\begin{aligned}
&+(-1)^{p+q+a+b+c}\zeta_3((s,b,p,q,c,a),\lambda_1^\vee;SU(4))\\
&\quad=(-1)^p\sum_{j=0}^{p}\Lambda(j;-i)(-1)^j\sum_{\omega=0}^{p-j}\binom{\omega+a-1}{\omega}\binom{p+c-j-\omega-1}{c-1}\\
&\qquad\times\mathfrak{T}(q+a+\omega,s,p+b+c-j-\omega;-1,i)\\
&\quad+(-1)^{p+a}\sum_{j=0}^{c}\Lambda(j;-i)\sum_{\omega=0}^{c-j}\binom{\omega+a-1}{\omega}\binom{p+c-j-\omega-1}{p-1}\\
&\qquad\times\mathfrak{T}(q,s+a+\omega,p+b+c-j-\omega;-i,-1)\\
&\quad+(-1)^p\sum_{j=0}^{a}\Lambda(j;-i)\sum_{\omega=0}^{p-1}\binom{\omega+a-j}{\omega}\binom{p+c-2-\omega}{c-1}\\
&\qquad\times\mathfrak{T}(q+a-j+\omega+1,s,p+b+c-1-\omega;-i,i)\\
&\quad+(-1)^{p+a}\sum_{j=0}^{a}\Lambda(j;-i)(-1)^j\sum_{\omega=0}^{c-1}\binom{\omega+a-j}{\omega}\binom{p+c-2-\omega}{p-1}\\
&\qquad\times\mathfrak{T}(q,s+a-j+\omega+1,p+b+c-1-\omega;-i,i)
\end{aligned}
$$

and

$$
\begin{aligned}
&\zeta_3((p,q,s,a,b,c),\lambda_3^\vee;SU(4))+(-1)^p\zeta_3((p,a,s,q,c,b),\lambda_3^\vee;SU(4)) \qquad (7.15)\\
&+(-1)^{p+a}\zeta_3((q,a,c,p,s,b),\lambda_3^\vee;SU(4))+(-1)^{p+a+c}\zeta_3((q,s,c,b,a,p),\lambda_3^\vee;SU(4))\\
&+(-1)^{q}\zeta_3((a,q,b,p,s,c),\lambda_3^\vee;SU(4))+(-1)^{q+b}\zeta_3((a,s,b,c,q,p),\lambda_3^\vee;SU(4))\\
&+(-1)^{q+a}\zeta_3((a,p,b,q,c,s),\lambda_3^\vee;SU(4))+(-1)^{q+a+b}\zeta_3((a,c,b,s,p,q),\lambda_3^\vee;SU(4))\\
&+(-1)^{q+a+b+c}\zeta_3((s,c,p,a,b,q),\lambda_3^\vee;SU(4))+(-1)^{p+q+a}\zeta_3((q,p,c,a,b,s),\lambda_3^\vee;SU(4))\\
&+(-1)^{p+q+a+c}\zeta_3((q,b,c,s,p,a),\lambda_3^\vee;SU(4))\\
&+(-1)^{p+q+a+b+c}\zeta_3((s,b,p,q,c,a),\lambda_3^\vee;SU(4))\\
&\quad=(-1)^p\sum_{j=0}^{p}\Lambda(j;i)(-1)^j\sum_{\omega=0}^{p-j}\binom{\omega+a-1}{\omega}\binom{p+c-j-\omega-1}{c-1}\\
&\qquad\times\mathfrak{T}(q+a+\omega,s,p+b+c-j-\omega;-1,-i)\\
&\quad+(-1)^{p+a}\sum_{j=0}^{c}\Lambda(j;i)\sum_{\omega=0}^{c-j}\binom{\omega+a-1}{\omega}\binom{p+c-j-\omega-1}{p-1}\\
&\qquad\times\mathfrak{T}(q,s+a+\omega,p+b+c-j-\omega;i,-1)\\
&\quad+(-1)^p\sum_{j=0}^{a}\Lambda(j;i)\sum_{\omega=0}^{p-1}\binom{\omega+a-j}{\omega}\binom{p+c-2-\omega}{c-1}\\
&\qquad\times\mathfrak{T}(q+a-j+\omega+1,s,p+b+c-1-\omega;i,-i)
\end{aligned}
$$

$$+(-1)^{p+a}\sum_{j=0}^{a} \Lambda(j;i)(-1)^j \sum_{\omega=0}^{c-1} \binom{\omega+a-j}{\omega}\binom{p+c-2-\omega}{p-1}$$
$$\times \mathfrak{T}(q, s+a-j+\omega+1, p+b+c-1-\omega; i, -i)$$

hold for $s \in \mathbb{C}$ except for singularities of functions on both sides. Moreover, since $\Lambda(j;\pm i)$ and $\mathfrak{T}(p,s,q;x,y)$ satisfy (7.3)-(7.4) *and* (7.8)-(7.13)*, respectively, we find that the right-hand sides of* (7.14) *and* (7.15) *can be written in terms of $\zeta(s)$ and $L(s,\chi_4)$.*

Setting $(a,b,c,p,q,s) = (2k,2k,2k,2k,2k,2k)$ for $k \in \mathbb{N}$, we obtain

$$\zeta_3((2k,2k,2k,2k,2k,2k), \lambda_j^\vee; SU(4)) \in \mathbb{Q}\cdot\pi^{12k} \quad (j=1,3),$$

because, since χ_4 is an odd character, $L(2l+1,\chi_4) \in \mathbb{Q}\cdot\pi^{2l+1}$ (see, for example, (5, p.12)). This is again an example of (17, Theorem 3.2).

Now we note that

$$1 - i^{3l+2m+n} + (-1)^{l+n} - i^{l+2m+3n} = \begin{cases} 4 & \text{if } l+2m+3n \equiv 2 \pmod 4, \\ 0 & \text{otherwise.} \end{cases} \tag{7.16}$$

This is because $(-1)^{l+n} = 1$ and $i^{3l+2m+n} = i^{l+2m+3n}$ when l and n are both even or both odd, while $(-1)^{l+n} = -1$ and $i^{3l+2m+n} = -i^{l+2m+3n}$ otherwise. Therefore

$$\begin{aligned} &\zeta_3(\mathbf{s}, \{\mathbf{0}\}; PU(4)) \qquad (7.17)\\ &= \frac{1}{4}\Big(\zeta_3(\mathbf{s}, A_3) - \zeta_3(\mathbf{s}, \lambda_1^\vee; SU(4)) + \zeta_3(\mathbf{s}, \lambda_2^\vee; SU(4)) - \zeta_3(\mathbf{s}, \lambda_3^\vee; SU(4))\Big), \end{aligned}$$

which is further equal to

$$\frac{1}{4}\Big(2\zeta_3(\mathbf{s}, SO(6)) - \zeta_3(\mathbf{s}, \lambda_1^\vee; SU(4)) - \zeta_3(\mathbf{s}, \lambda_3^\vee; SU(4))\Big)$$

by (6.1). Hence it follows from (3.13) and (3.14) that

$$\zeta_3((s_1,s_2,s_3,s_4,s_5,s_6), \{\mathbf{0}\}; PU(4)) = \zeta_3((s_3,s_2,s_1,s_5,s_4,s_6), \{\mathbf{0}\}; PU(4)). \tag{7.18}$$

Using (7.17) and combining Theorem 12 and Theorem 17, we obtain the following functional relation among $\zeta_3(\mathbf{s}, \{\mathbf{0}\}; PU(4))$, $\zeta(s)$ and $L(s,\chi_4)$.

THEOREM 18. *For $p,q,a,b,c \in \mathbb{N}$,*

$$\begin{aligned} &\zeta_3((p,q,s,a,b,c), \mathbf{0}; PU(4)) + (-1)^p \zeta_3((p,a,s,q,c,b), \mathbf{0}; PU(4)) \qquad (7.19)\\ &+ (-1)^{p+a}\zeta_3((q,a,c,p,s,b), \mathbf{0}; PU(4)) + (-1)^{p+a+c}\zeta_3((q,s,c,b,a,p), \mathbf{0}; PU(4))\\ &+ (-1)^q \zeta_3((a,q,b,p,s,c), \mathbf{0}; PU(4)) + (-1)^{q+b}\zeta_3((a,s,b,c,q,p), \mathbf{0}; PU(4)) \end{aligned}$$

$$
\begin{aligned}
&+(-1)^{q+a}\zeta_3((a,p,b,q,c,s),\mathbf{0};PU(4))+(-1)^{q+a+b}\zeta_3((a,c,b,s,p,q),\mathbf{0};PU(4))\\
&+(-1)^{q+a+b+c}\zeta_3((s,c,p,a,b,q),\mathbf{0};PU(4))+(-1)^{p+q+a}\zeta_3((q,p,c,a,b,s),\mathbf{0};PU(4))\\
&+(-1)^{p+q+a+c}\zeta_3((q,b,c,s,p,a),\mathbf{0};PU(4))\\
&+(-1)^{p+q+a+b+c}\zeta_3((s,b,p,q,c,a),\mathbf{0};PU(4))\\
&=\frac{1}{4}\left(J_0-J_1+J_2-J_3\right),
\end{aligned}
$$

where J_0, J_1, J_2, J_3 *are the right-hand sides of* (5.4), (7.14), (5.3), (7.15), *respectively.*

Setting $(a,b,c,p,q,s)=(2k,2k,2k,2k,2k,2k)$ for $k\in\mathbb{N}$, we obtain

$$\zeta_3((2k,2k,2k,2k,2k,2k),\mathbf{0};PU(4))\in\mathbb{Q}\cdot\pi^{12k}.$$

Also, set $(a,b,c,p,q,s)=(2k+1,2k+1,2k+1,2k+1,2k,2k+1)$ in (7.14), (7.15) and (7.19). Then, by the same method as in the proof of Proposition 10, we obtain the following.

PROPOSITION 19. *For* $k\in\mathbb{N}$ *and* $j=1,3$,

$$
\begin{aligned}
&\zeta_3((2k,2k+1,2k+1,2k+1,2k+1,2k+1),\lambda_j^\vee;SU(4))\\
&\qquad\in\mathbb{Q}[\{\zeta(j+1),\ L(j,\chi_4)\,|\,j\in\mathbb{N}\}],
\end{aligned}
$$

and so

$$
\begin{aligned}
&\zeta_3((2k,2k+1,2k+1,2k+1,2k+1,2k+1),\{\mathbf{0}\};PU(4))\\
&\qquad\in\mathbb{Q}[\{\zeta(j+1),\ L(j;\chi_4)\,|\,j\in\mathbb{N}\}],
\end{aligned}
$$

where the rational coefficients can be determined explicitly.

EXAMPLE 20. Setting $(a,b,c,p,q,s)=(2,2,2,2,2,s)$ in (7.14) and (7.15), and using the data $L(1,\chi_4)=\pi/4,\ L(3,\chi_4)=\pi^3/32,\ L(5,\chi_4)=(5/1536)\pi^5$, we obtain

$$
\begin{aligned}
&2\Big\{\zeta_3((s,2,2,2,2,2),\lambda_1^\vee;SU(4))+\zeta_3((2,s,2,2,2,2),\lambda_1^\vee;SU(4))\\
&\quad+\zeta_3((2,2,s,2,2,2),\lambda_1^\vee;SU(4))+\zeta_3((2,2,2,s,2,2),\lambda_1^\vee;SU(4))\\
&\quad+\zeta_3((2,2,2,2,s,2),\lambda_1^\vee;SU(4))+\zeta_3((2,2,2,2,2,s),\lambda_1^\vee;SU(4))\Big\}\\
&=\left(372\cdot 2^{-s-10}+306\right)\left(2^{-s-9}-1\right)\zeta(s+10)+100\pi L(s+9,\chi_4)\\
&\quad+\left(7\cdot 2^{-s-8}+\frac{32}{3}\right)\left(2^{-s-7}-1\right)\pi^2\zeta(s+8)+\frac{17}{6}\pi^3L(s+7,\chi_4)\\
&\quad+\left(\frac{113\cdot 2^{-s-6}}{1440}+\frac{1}{288}\right)\left(2^{-s-5}-1\right)\pi^4\zeta(s+6)
\end{aligned}
$$

$$+\frac{1}{32}\pi^5 L(s+5,\chi_4)+\frac{289\cdot 2^{-s-4}}{241920}\left(2^{-s-3}-1\right)\pi^6\zeta(s+4),$$

and

$$\begin{aligned}
&2\Big\{\zeta_3((s,2,2,2,2,2),\lambda_3^\vee;SU(4))+\zeta_3((2,s,2,2,2,2),\lambda_3^\vee;SU(4))\\
&\quad+\zeta_3((2,2,s,2,2,2),\lambda_3^\vee;SU(4))+\zeta_3((2,2,2,s,2,2),\lambda_3^\vee;SU(4))\\
&\quad+\zeta_3((2,2,2,2,s,2),\lambda_3^\vee;SU(4))+\zeta_3((2,2,2,2,2,s),\lambda_3^\vee;SU(4))\Big\}\\
&=\left(372\cdot 2^{-s-10}+306\right)\left(2^{-s-9}-1\right)\zeta(s+10)+100\pi L(s+9,\chi_4)\\
&\quad+\left(7\cdot 2^{-s-8}+\frac{32}{3}\right)\left(2^{-s-7}-1\right)\pi^2\zeta(s+8)+\frac{17}{6}\pi^3 L(s+7,\chi_4)\\
&\quad+\left(\frac{113\cdot 2^{-s-6}}{1440}+\frac{1}{288}\right)\left(2^{-s-5}-1\right)\pi^4\zeta(s+6)\\
&\quad+\frac{1}{32}\pi^5 L(s+5;\chi_4)+\frac{289\cdot 2^{-s-4}}{241920}\left(2^{-s-3}-1\right)\pi^6\zeta(s+4).
\end{aligned}$$

Note that the reason why the right-hand sides of the above two formulas are the same is given by (3.13). Setting $(a,b,c,p,q,s)=(2,2,2,2,2,s)$ in (7.19), we obtain

$$\begin{aligned}
&2\Big\{\zeta_3((s,2,2,2,2,2),\mathbf{0};PU(4))+\zeta_3((2,s,2,2,2,2),\mathbf{0};PU(4))\\
&\quad+\zeta_3((2,2,s,2,2,2),\mathbf{0};PU(4))+\zeta_3((2,2,2,s,2,2),\mathbf{0};PU(4))\\
&\quad+\zeta_3((2,2,2,2,s,2),\mathbf{0};PU(4))+\zeta_3((2,2,2,2,2,s),\mathbf{0};PU(4))\Big\}\\
&=\left(-\frac{93\cdot 2^{-2s}}{262144}+\frac{33\cdot 2^{-s}}{512}+306\right)\zeta(s+10)-50\pi L(s+9,\chi_4)\\
&\quad+\left(-\frac{7\cdot 2^{-2s}}{65536}-\frac{19\cdot 2^{-s}}{1536}-\frac{49}{3}\right)\pi^2\zeta(s+8)-\frac{17}{12}\pi^3 L(s+7,\chi_4)\\
&\quad+\left(-\frac{113\cdot 2^{-2s}}{5898240}-\frac{323\cdot 2^{-s}}{61440}+\frac{353}{576}\right)\pi^4\zeta(s+6)-\frac{\pi^5}{64}L(s+5,\chi_4)\\
&\quad+\left(-\frac{289\cdot 2^{-2s}}{61931520}-\frac{31\cdot 2^{-s}}{7741440}+\frac{1}{720}\right)\pi^6\zeta(s+4).
\end{aligned}$$

In particular, setting $s=2$, we again obtain (3.18).

EXAMPLE 21. Similarly to Example 11, We obtain, for example,

$$\begin{aligned}
&\zeta_3((2,3,3,3,3,3),\lambda_1^\vee;SU(4))=\zeta_3((2,3,3,3,3,3),\lambda_3^\vee;SU(4))\\
&=\frac{2125\pi^8}{45097156608}\zeta(9)+\frac{11\pi^7}{15360}L(10,\chi_4)-\frac{440049247\pi^6}{225485783040}\zeta(11)\\
&\quad+\frac{13\pi^5}{96}L(12,\chi_4)-\frac{1056786549\pi^4}{2147483648}\zeta(13)+11\pi^3 L(14,\chi_4)
\end{aligned}$$

$$
- \frac{199887481225\pi^2}{4294967296}\zeta(15) + 399\pi\, L(16,\chi_4) - \frac{2424501730875}{2147483648}\zeta(17),
$$

and

$$
\begin{aligned}
&\zeta_3((2,3,3,3,3,3),\{\mathbf{0}\};PU(4))\\
&\quad = \frac{1111987\pi^8}{90194313216}\zeta(9) - \frac{11\pi^7}{30720}L(10,\chi_4) + \frac{11180759837\pi^6}{1352914698240}\zeta(11)\\
&\qquad - \frac{13\pi^5}{192}L(12,\chi_4) + \frac{170862984923\pi^4}{64424509440}\zeta(13) - \frac{11\pi^3}{2}L(14,\chi_4)\\
&\qquad - \frac{1504872383333\pi^2}{25769803776}\zeta(15) - \frac{399\pi}{2}L(16,\chi_4) + \frac{4849040457275}{4294967296}\zeta(17).
\end{aligned}
$$

REFERENCES

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, 1976.

[2] N. Bourbaki, *Groupes et Algèbres de Lie, Chapitres 4, 5 et 6*, Hermann, 1968.

[3] P. E. Gunnells and R. Sczech, Evaluation of Dedekind sums, Eisenstein cocycles, and special values of L-functions, *Duke Math. J.* **118** (2003), 229–260.

[4] M. E. Hoffman, Multiple harmonic series, *Pacific J. Math.* **152** (1992), 275-290.

[5] K. Iwasawa, *Lectures on p-adic L-functions*, Princeton Univ. Press, 1972.

[6] Y. Komori, K. Matsumoto and H. Tsumura, Zeta-functions of root systems, in "The Conference on L-functions" (Fukuoka, 2006), L. Weng and M. Kaneko (eds.), World Scientific, 2007, pp. 115–140.

[7] Y. Komori, K. Matsumoto and H. Tsumura, Zeta and L-functions and Bernoulli polynomials of root systems, Proc. Japan Acad., Series A, **84** (2008), 57–62.

[8] Y. Komori, K. Matsumoto and H. Tsumura, Functional relations for zeta-functions of root systems, in "Number Theory: Dreaming in Dreams - Proceedings of the 5th China-Japan Seminar", T. Aoki, S. Kanemitsu and J. -Y. Liu (eds.), World Scientific Publ, 2010, pp. 135–183.

[9] Y. Komori, K. Matsumoto and H. Tsumura, On multiple Bernoulli polynomials and multiple L-functions of root systems, Proc. London Math. Soc. **100** (2010), 303–347.

[10] Y. Komori, K. Matsumoto and H. Tsumura, An introduction to the theory of zeta-functions of root systems, in "Algebraic and Analytic Aspects of Zeta Functions and L-functions", G. Bhowmik, K. Matsumoto and H. Tsumura (eds.), MSJ Memoirs, Vol. 21, Mathematical Society of Japan, 2010, pp. 115–140.

[11] Y. Komori, K. Matsumoto and H. Tsumura, On Witten multiple zeta-functions associated with semisimple Lie algebras II, J. Math. Soc. Japan **62** (2010), 355–394.

[12] Y. Komori, K. Matsumoto and H. Tsumura, Functional equations and functional relations for the Euler double zeta-function and its generalization of Eisenstein type, Publ. Math. Debrecen **77** (2010), 15-31.

[13] Y. Komori, K. Matsumoto and H. Tsumura, On Witten multiple zeta-functions associated with semisimple Lie algebras III, to appear in "Multiple Dirichlet Series, L-functions and Automorphic Forms" (Proc. Edinburgh Conf., 2008), D. Bump et al. (eds.), Progr. Math., Birkhäuser, arXiv:0907.0955.

[14] Y. Komori, K. Matsumoto and H. Tsumura, On Witten multiple zeta-functions associated with semisimple Lie algebras IV, Glasgow Math. J. **53** (2011), 185-206.

[15] Y. Komori, K. Matsumoto and H. Tsumura, Shuffle products for multiple zeta values and partial fraction decompositions of zeta-functions of root systems, Math. Z. **268** (2011), 993-1011.

[16] Y. Komori, K. Matsumoto and H. Tsumura, A survey on the theory of multiple Bernoulli polynomials and multiple L-functions of root systems, in "Infinite Analysis 2010, Developments in Quantum Integrable Systems", A. Kuniba et al. (eds.), RIMS Kôkyûroku Bessatsu **B28** (2011), 99-120.

[17] Y. Komori, K. Matsumoto and H. Tsumura, Zeta-functions of weight lattices of compact connected semisimple Lie groups, arXiv:math/1011.0323, submitted.

[18] K. Matsumoto, T. Nakamura, H. Ochiai and H. Tsumura, On value-relations, functional relations and singularities of Mordell-Tornheim and related triple zeta-functions, Acta Arith. **132** (2008), 99–125.

[19] K. Matsumoto and H. Tsumura, On Witten multiple zeta-functions associated with semisimple Lie algebras I, Ann. Inst. Fourier **56** (2006), 1457–1504.

[20] T. Nakamura, Double Lerch value relations and functional relations for Witten zeta functions, Tokyo J. Math. **31** (2008), 551-574.

[21] A. Szenes, Iterated residues and multiple Bernoulli polynomials, Internat. Math. Res. Notices, **18** (1998), 937–958.

[22] A. Szenes, Residue formula for rational trigonometric sums, Duke Math. J. **118** (2003), 189–228.

[23] H. Tsumura, On functional relations between the Mordell-Tornheim double zeta functions and the Riemann zeta function, Math. Proc. Cambridge Philos. Soc., **142** (2007), 395–405.

[24] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th ed., Cambridge University Press, Cambridge, 1927.

[25] E. Witten, On quantum gauge theories in two dimensions, Comm. Math. Phys. **141** (1991), 153–209.

[26] D. Zagier, Values of zeta functions and their applications, in “First European Congress of Mathematics” Vol. II, A. Joseph et al. (eds.), Progr. Math. 120, Birkhäuser, 1994, pp. 497–512.