AN INTRODUCTION TO THE THEORY OF ZETA-FUNCTIONS OF ROOT SYSTEMS

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Contents

1. Introduction

The theory of multiple zeta-functions has a long history, from the work of Barnes and Mellin at the beginning of the 20th century, or even from the days of Euler. A new stream of research of multiple zeta-functions began in 1990s, when some fascinating connections between the theory of multiple zeta-functions and various branches of mathematics and mathematical physics were discovered. An epoch-making paper is Zagier [33], in which two types of multiple zeta-functions are discussed. One is the r-fold zeta-function of the form

$$\begin{aligned}\zeta_{EZ,r}(s_1,\ldots,s_r)=\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}m_1^{-s_1}(m_1+m_2)^{-s_2}\\ \times\cdots\times(m_1+\cdots+m_r)^{-s_r},\end{aligned} \tag{1.1}$$

which is now sometimes called the Euler-Zagier r-ple zeta-function. Zagier [33] considered the values of (1.1) when $s_1,\ldots,s_r$ are positive integers and $s_r \geq 2$. Note that Hoffman [5] independently studied the same values at about the same time.

Another type of multiple zeta-functions discussed in Zagier's paper is the class of Witten's zeta-functions. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. The Witten zeta-function associated with $\mathfrak{g}$ is defined as

$$\zeta_W(s;\mathfrak{g}) = \sum_{\varphi} (\dim \varphi)^{-s}, \tag{1.2}$$

where φ runs over all finite dimensional irreducible representations of $\mathfrak{g}$. Special values of this series were first studied by Witten [32] in connection with a problem in quantum gauge theory. As we will see later, we can write down a more explicit form of $\zeta_W(s;\mathfrak{g})$ by using Weyl's dimension formula. We will find that the explicit form of $\zeta_W(s;\mathfrak{g})$ is an r-fold sum, where r is the rank of $\mathfrak{g}$. Therefore $\zeta_W(s;\mathfrak{g})$ is a kind of multiple zeta-functions.

The original form of $\zeta_W(s;\mathfrak{g})$ is a function in one variable, though the sum in the definition is multiple. However it has been noticed recently that, for deeper investigations of $\zeta_W(s;\mathfrak{g})$, it is convenient to introduce the multi-variable generalization of $\zeta_W(s;\mathfrak{g})$ and discuss its properties. This is the main theme of the present article.

Since the present lecture is of introductory nature, we begin with the explanation of the basic theory of Lie algebras. It is impossible to give the full account of the theory here; for the details, see, for example, [2], [7], or [20].

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In what follows, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the set of positive integers, non-negative integers, integers, rational numbers, real numbers, and complex numbers, respectively.

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2. Fundamentals of the theory of Lie algebras

In this article by Lie algebra we mean a finite dimensional vector space $\mathfrak{g}$ over $\mathbb{C}$, with a bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, satisfying the skew-symmetry $[X, X] = 0$ for any $X \in \mathfrak{g}$ and the Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

for any $X, Y, Z \in \mathfrak{g}$. The skew-symmetry implies

$$0 = [X+Y, X+Y] = [X,Y] + [Y,X],$$

hence $[X, Y] = -[Y, X]$. In particular, we call $\mathfrak{g}$ Abelian if $[X, Y] = 0$ for any $X, Y \in \mathfrak{g}$.

A subspace $\mathfrak{a}$ of $\mathfrak{g}$ is called a Lie subalgebra if it is closed under the above bracket operation. The normalizer $N(\mathfrak{a})$ of $\mathfrak{a}$ is the set of all $X \in \mathfrak{g}$ for which

$$[X, \mathfrak{a}] = \{[X, Y] \mid Y \in \mathfrak{a}\} \subset \mathfrak{a}$$

holds. If $N(\mathfrak{a}) = \mathfrak{g}$, we call $\mathfrak{a}$ an ideal of $\mathfrak{g}$.

The derived Lie subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ is the ideal $[\mathfrak{g}, \mathfrak{g}]$, spanned by all $[X, Y]$ $(X, Y \in \mathfrak{g})$. Define the series

$$\mathfrak{g} \supset \mathfrak{g}' \supset \mathfrak{g}'' \supset \cdots \supset \mathfrak{g}^{(n)} \supset \cdots$$

by $\mathfrak{g}^{(n)} = (\mathfrak{g}^{(n-1)})'$. If this series (the derived series) goes down to zero for some finite n, we call $\mathfrak{g}$ solvable. We also define the lower central series, by replacing $\mathfrak{g}^{(n)}$ in the derived series by $\mathfrak{g}^n = [\mathfrak{g}, \mathfrak{g}^{n-1}]$, and call $\mathfrak{g}$ nilpotent if the lower central series goes down to zero.

A Cartan subalgebra of a Lie algebra $\mathfrak{g}$ is a nilpotent Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, with the property $N(\mathfrak{h}) = \mathfrak{h}$. (It follows that $\mathfrak{h}$ is maximal nilpotent.) This is not uniquely determined, but its dimension does not depend on the choice of $\mathfrak{h}$. We call this dimension the rank of $\mathfrak{g}$.

A representation of a Lie algebra $\mathfrak{g}$ on a complex vector space U is a homomorphism φ of $\mathfrak{g}$ into the general linear algebra $GL(U)$. We denote by $\dim \varphi$ the dimension of the representation space U. The most fundamental representation is the adjoint representation $\mathrm{ad} : \mathfrak{g} \to GL(\mathfrak{g})$ defined by $(\mathrm{ad}X)Y = [X, Y]$ for any $X, Y \in \mathfrak{g}$. By using the adjoint representation we define the symmetric bilinear form

$$\langle X, Y \rangle = \kappa(X, Y) = \mathrm{Tr}(\mathrm{ad}X \circ \mathrm{ad}Y),$$

which is called the Killing form.

It is known that any $\mathfrak{g}$ contains the unique maximal solvable ideal $\mathfrak{r}$, the radical of $\mathfrak{g}$. We call a Lie algebra $\mathfrak{g}$ $(\neq \{0\})$ semisimple if its radical is zero. Since an Abelian ideal is solvable, semisimplicity implies that $\mathfrak{g}$ has no non-zero Abelian ideal. If a non-Abelian Lie algebra $\mathfrak{g}$ has no non-trivial ideals, $\mathfrak{g}$ is called simple. A simple Lie algebra is semisimple. It is known that any semisimple Lie algebra can be written as a direct sum of simple Lie algebras. It is also known that $\mathfrak{g}$ is semisimple if and only if the Killing form is non-degenerate.

From now on we assume that $\mathfrak{g}$ is semisimple. Then any Cartan subalgebra is Abelian, and hence is maximal Abelian. We fix one Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and write $r = \mathrm{rank}\ \mathfrak{g} = \dim \mathfrak{h}$. Let α be a non-zero element of the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$. We call α a root of $\mathfrak{g}$ (with respect to $\mathfrak{h}$) if there exists a non-zero $X \in \mathfrak{g}$ such that $\mathrm{ad}(H)X = \alpha(H)X$

for any $H \in \mathfrak{h}$. Denote the set of all such X by $\mathfrak{g}_\alpha$. There are only finitely many roots, and they span $\mathfrak{h}^*$. We denote the set of all roots by $\Delta = \Delta(\mathfrak{g})$. The decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha \tag{2.1}$$

holds. Note that $\dim \mathfrak{g}_\alpha = 1$ for any $\alpha \in \Delta$.

Let $\alpha \in \Delta$. Then there exists a unique element $\alpha' \in \mathfrak{h}$ satisfying $\langle \alpha', H\rangle = \alpha(H)$ for any $H \in \mathfrak{h}$. By this correspondence $\alpha \leftrightarrow \alpha'$ we can identify $\mathfrak{h}^*$ with $\mathfrak{h}$, and transfer the Killing form to $\mathfrak{h}^*$ by putting $\langle \alpha, \alpha\rangle = \langle \alpha', \alpha'\rangle$. Let $\mathfrak{h}_0$ be the real subspace of $\mathfrak{h}$ formed by all real linear combinations of α', for $\alpha \in \Delta$. Then $\mathfrak{h}_0$ and its dual space $\mathfrak{h}_0^*$ can be identified by the above way.

Since α' is clearly non-zero, we can define

$$\alpha^\vee = \frac{2}{\langle \alpha', \alpha'\rangle}\alpha', \tag{2.2}$$

which we call the coroot associated with α. Clearly $\alpha(\alpha^\vee) = 2$. More generally, the values

$$a(\beta, \alpha) = \beta(\alpha^\vee) = \frac{2\langle \beta, \alpha\rangle}{\langle \alpha, \alpha\rangle} \tag{2.3}$$

for any $\alpha, \beta \in \Delta$ are integers, which we call the Cartan integers of $\mathfrak{g}$. It can be shown that $\beta - a(\beta, \alpha)\alpha$ is again a root. Another important property is that if both α and $c\alpha$ ($c \in \mathbb{C}$) are roots, then $c = \pm 1$.

For any $\alpha, \beta \in \Delta$, there exist $p, q \in \mathbb{N}_0$, such that $\beta + t\alpha \in \Delta$ if and only if $-q \le t \le p$. The sequence $\beta - q\alpha, \ldots, \beta + p\alpha$ is called the α-string through β. It is known that

$$a(\beta, \alpha) = q - p. \tag{2.4}$$

Now we define the notion of (abstract) reduced root systems. Let V be an r-dimensional real vector space with an inner product $\langle\ ,\ \rangle$. For any $\alpha \in V \setminus \{0\}$, define $\sigma_\alpha : V \to V$ by $\sigma_\alpha(\beta) = \beta - a(\beta, \alpha)\alpha$. A finite non-empty subset R of V, not containing 0, is called a reduced root system if it spans V and satisfies

(i) for $\alpha, \beta \in R$, $a(\beta, \alpha) = 2\langle \beta, \alpha\rangle/\langle \alpha, \alpha\rangle \in \mathbb{Z}$,

(ii) for $\alpha, \beta \in R$, the vector $\sigma_\alpha(\beta)$ is also in R,

(iii) if both α and $c\alpha$ ($c \in \mathbb{R}$) are in R, then $c = \pm 1$.

We call r the rank of the root system R. Obviously Δ is a reduced root system with the vector space $V = \mathfrak{h}_0^*$.

For $\alpha \in V \setminus \{0\}$, let $P_\alpha = \{\beta \in V \mid \langle \beta, \alpha\rangle = 0\}$. This is the hyperplane orthogonal to α. Since R is a finite set, it is obvious that

$V \setminus \bigcup_{\alpha\in R} P_\alpha$ is non-empty. The elements of this set are called regular. If $\gamma \in V$ is regular, then $R = R_+(\gamma) \cup (-R_+(\gamma))$, where

$$R_+(\gamma) = \{\alpha \in R \mid \langle \gamma, \alpha\rangle > 0\}.$$

We call $\alpha \in R_+(\gamma)$ decomposable if $\alpha = \beta_1 + \beta_2$ for some $\beta_1, \beta_2 \in R_+(\gamma)$, and indecomposable otherwise. Denote by $\Psi = \Psi_\gamma$ the set of all indecomposable elements of $R_+(\gamma)$. Then it is known that Ψ is a basis of V, and each root β can be written as $\beta = \sum_{\alpha\in\Psi} k_\alpha \alpha$ with integral coefficients k_α, all non-negative or all non-positive. We call β positive (resp. negative) if all $k_\alpha \geq 0$ (resp. ≤ 0). The elements of Ψ are called simple or fundamental, and $|\Psi| = r$. We call Ψ a base, or a fundamental system, of R. When $R = \Delta = \Delta(\mathfrak{g})$, we write $\Psi = \Psi(\Delta) = \Psi(\mathfrak{g})$.

Two root systems R_1 and R_2 (with the underlying vector spaces V_1 and V_2, respectively) are equivalent when there is a similarity (constant multiple of an isometry) $V_1 \to V_2$ which sends R_1 onto R_2. A very important fact is that there is a bijection between the set of equivalence classes of reduced root systems and the set of isomorphism classes of semisimple Lie algebras. Therefore, hereafter, we assume that the root system R is corresponding to a semisimple Lie algebra $\mathfrak{g}$, the inner product is given by the Killing form, and write $R = \Delta = \Delta(\mathfrak{g})$. We also write $\Delta_+ = R_+(\gamma)$, $\Delta_- = -(R_+(\gamma))$. Therefore $\Delta = \Delta_+ \cup \Delta_-$. Note that Ψ, Δ_+ and Δ_- depend on the choice of γ.

Let γ' be the element of $\mathfrak{h}_0$ corresponding to $\gamma \in \mathfrak{h}_0^*$. Then $\alpha \in \Delta_+$ if and only if $\alpha(\gamma') > 0$. This suggests the definition of the following partial order in $\mathfrak{h}_0^*$; for $\lambda, \mu \in \mathfrak{h}_0^*$, we define $\lambda > \mu$ (resp. $\lambda \geq \mu$) if $\lambda(\gamma') > \mu(\gamma')$ (resp. $\lambda(\gamma') \geq \mu(\gamma')$). The definition of this order also depends on γ.

The mapping σ_α is the reflection with respect to P_α. All σ_α, $\alpha \in \Delta$, generates a group $W = W(\Delta)$ of isometries of V, which is called the Weyl group of Δ. This group is generated by all the elements of Ψ. If Ψ' is another fundamental system of Δ, then there exists an element $w \in W$ for which $\Psi' = w(\Psi)$ holds.

3. Examples of simple Lie algebras

A root system Δ is called irreducible if it cannot be written as the union of two proper subsets, each root in one of them is orthogonal to each root in the other. Irreducible root systems have been completely classified by the Cartan-Killing theory. The result can be written as the list of all irreducible root systems, that is, A_r $(r \geq 1)$, B_r $(r \geq 2)$, C_r $(r \geq 3)$, D_r $(r \geq 4)$, and the five exceptional systems E_6, E_7, E_8, F_4, and G_2. This list exactly corresponds to the list of possible simple

Lie algebras. Hereafter, when $\mathfrak{g}$ corresponds to the root system of type X_r ($X = A, B, C, D, E, F$ or G), we sometimes write $\Delta(\mathfrak{g}) = \Delta(X_r)$, $\Psi(\mathfrak{g}) = \Psi(X_r)$, $\zeta_r(\mathbf{s};\mathfrak{g}) = \zeta_r(\mathbf{s}; X_r)$, etc. (The subscript r indicates the rank of $\mathfrak{g}$ as in the preceding section.)

The root system of type A_r corresponds to the Lie algebra

$$\mathfrak{sl}(r+1) = \{X \in \mathcal{M}_{r+1}(\mathbb{C}) \mid \mathrm{Tr}X = 0\},$$

where $\mathcal{M}_m(\mathbb{C})$ denotes the set of all $m \times m$ matrices with complex entries. The bracket operation is given by $[X, Y] = X \cdot Y - Y \cdot X$, where the "dot" on the right-hand side is the usual matrix multiplication. By $\mathrm{diag}(a_1, \ldots, a_{r+1})$ we mean the $(r+1) \times (r+1)$ matrix whose diagonal entries are $a_1, \ldots, a_{r+1}$ and all other entries are 0. When $\mathfrak{g} = \mathfrak{g}(A_r) = \mathfrak{sl}(r+1)$, the set $\mathfrak{h}$ of all $\mathrm{diag}(a_1, \ldots, a_{r+1})$ with $a_1 + \cdots + a_{r+1} = 0$ is a Cartan subalgebra of $\mathfrak{g}$. Then

$$\Delta = \Delta(A_r) = \{\varepsilon_i - \varepsilon_j \mid 1 \le i, j \le r+1,\ i \ne j\},$$

where ε_i is defined by $\varepsilon_i(\mathrm{diag}(a_1, \ldots, a_{r+1})) = a_i$.

Choose $\gamma' = \mathrm{diag}(a_1, \ldots, a_{r+1})$ with $a_1 > a_2 > \cdots > a_r > 0$ and $a_{r+1} = -(a_1 + \cdots + a_r)$. Then $\varepsilon_i - \varepsilon_j > 0$ implies $a_i - a_j > 0$, which further implies $i < j$. Therefore

$$\Delta_+(A_r) = \{\varepsilon_i - \varepsilon_j \mid 1 \le i, j \le r+1,\ i < j\}.$$

The fundamental system is $\Psi(A_r) = \{\alpha_i \mid 1 \le i \le r\}$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. It can be shown that the Killing form on $\mathfrak{sl}(r+1)$ is $\langle X, Y\rangle = 2(r+1)\mathrm{Tr}(X \cdot Y)$. Therefore $\alpha_i' = (2(r+1))^{-1}(e_i - e_{i+1})$, where e_i is the matrix whose (i,i)-entry is 1 and all other entries are 0. Hence the coroot corresponding to α_i is $\alpha_i^\vee = e_i - e_{i+1}$. All positive roots and coroots can be written as

$$\varepsilon_i - \varepsilon_j = \sum_{i \le k < j} \alpha_k \tag{3.1}$$

and

$$e_i - e_j = \sum_{i \le k < j} \alpha_k^\vee, \tag{3.2}$$

respectively.

Such explicit descriptions of positive roots can be done for other types of simple Lie algebras. Here we only mention the case of B_r type, which corresponds to the algebra

$$\mathfrak{o}(2r+1) = \{X \in \mathcal{M}_{2r+1}(\mathbb{C}) \mid {}^tX + X = 0\}.$$

A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{o}(2r+1)$ is the set of all matrices of the form

$$\left[\begin{array}{c|cc|c|cc}
0 & & & & & \\ \hline
 & 0 & a_1 & & & \\
 & -a_1 & 0 & & & \\ \hline
 & & & \ddots & & \\ \hline
 & & & & 0 & a_r \\
 & & & & -a_r & 0
\end{array}\right]$$

(where all elements in the empty blocks are 0), but we may identify this $\mathfrak{h}$ with

$$\{\mathrm{diag}(a_1,\ldots,a_r) \mid a_1,\ldots,a_r \in \mathbb{C}\} \subset \mathbb{C}^r.$$

Positive roots are ε_i $(1 \le i \le r)$ and $\varepsilon_i \pm \varepsilon_j$ $(1 \le i < j \le r)$. The fundamental system $\Psi(B_r)$ consists of $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \le i \le r-1)$ and $\alpha_r = \varepsilon_r$. The corresponding coroots are $\alpha_i^\vee = e_i - e_{i+1}$ $(1 \le i \le r-1)$ and $\alpha_r^\vee = 2e_r$. The list of positive coroots is

$$2e_i = 2\sum_{i \le k < r} \alpha_k^\vee + \alpha_r^\vee \qquad (1 \le i \le r),$$

$$e_i - e_j = \sum_{i \le k < j} \alpha_k^\vee \qquad (1 \le i < j \le r),$$

and

$$e_i + e_j = \sum_{i \le k < j} \alpha_k^\vee + 2\sum_{j \le k < r} \alpha_k^\vee + \alpha_r^\vee \qquad (1 \le i < j \le r).$$

4. Weyl's dimension formula

Now we return to the general situation. Let $\varphi : \mathfrak{g} \to GL(U)$ be a representation, where U is a finite dimensional complex vector space. We call $u \in U$ a weight vector if it is a joint eigenvector of all the operators $\varphi(H)$, $H \in \mathfrak{h}$. Hence $\varphi(H)u = \lambda(H)u$, where $\lambda(H) \in \mathbb{C}$. Then $\lambda : H \mapsto \lambda(H)$ is an element of $\mathfrak{h}^*$. We call λ the weight of u.

For each $\lambda \in \mathfrak{h}^*$, let U_λ be the subspace of U consisting of 0 and all weight vectors u with weight λ. When U_λ is non-zero, we call λ a weight of φ. There is only a finite number of weights. A weight λ is called dominant if $\lambda(\alpha^\vee) \ge 0$ for all $\alpha \in \Psi$. If the strict inequality holds for all $\alpha \in \Psi$, then we call strongly dominant.

Write $\Psi = \{\alpha_1,\ldots,\alpha_r\}$, and define $\lambda_i \in \mathfrak{h}^*$ by $\lambda_i(\alpha_j^\vee) = \delta_{ij}$ (Kronecker's delta). Then clearly λ_i $(1 \le i \le r)$ are dominant. We call them fundamental weights, and write $\Lambda = \{\lambda_1,\ldots,\lambda_r\}$. The weight

$\rho = \lambda_1 + \cdots + \lambda_r$ is called the lowest strongly dominant form. It is known that any dominant weight λ can be written as

$$\lambda = n_1\lambda_1 + \cdots + n_r\lambda_r \qquad (n_1, \ldots, n_r \in \mathbb{N}_0). \tag{4.1}$$

A representation (φ, U) is irreducible if there is no non-trivial invariant subspace of U. A principal result is that there is a bijection between the set of (equivalence classes of) irreducible representations and the set of dominant weights. Let φ a finite dimensional irreducible representation of $\mathfrak{g}$, and λ the corresponding dominant weight. Then Weyl's dimension formula asserts

$$\begin{aligned} \dim\varphi &= \prod_{\alpha\in\Delta_+} \frac{\langle\alpha^\vee, \lambda+\rho\rangle}{\langle\alpha^\vee, \rho\rangle} \\ &= \prod_{\alpha\in\Delta_+} \frac{\langle\alpha^\vee, (n_1+1)\lambda_1 + \cdots + (n_r+1)\lambda_r\rangle}{\langle\alpha^\vee, \lambda_1 + \cdots + \lambda_r\rangle}. \end{aligned} \tag{4.2}$$

(Here we use the notation $\langle\alpha^\vee, \rho\rangle = \rho(\alpha)$.) At a first glance it seems that the right-hand side depends on the choice of $\mathfrak{h}$ and γ. It is possible, however, to show directly that the right-hand side is independent of those choices, by using the fact that any two Cartan subalgebras are conjugate, and the aforementioned transitivity of the Weyl group on the set of fundamental systems.

Since each φ corresponds to each $(n_1, \ldots, n_r) \in \mathbb{N}_0^r$, substituting (4.2) into (1.2), we now obtain the following explicit form of Witten zeta-functions:

$$\zeta_W(s;\mathfrak{g}) = K(\mathfrak{g})^s \sum_{m_1=1}^\infty \cdots \sum_{m_r=1}^\infty \prod_{\alpha\in\Delta_+} \langle\alpha^\vee, m_1\lambda_1 + \cdots + m_r\lambda_r\rangle^{-s}, \tag{4.3}$$

where

$$K(\mathfrak{g}) = \prod_{\alpha\in\Delta_+} \langle\alpha^\vee, \lambda_1 + \cdots + \lambda_r\rangle. \tag{4.4}$$

To investigate the analytic behaviour of the multiple sum part of (4.3), it is convenient to introduce the following multi-variable version. Let $\mathbf{s} = (s_\alpha)_{\alpha\in\Delta_+} \in \mathbb{C}^n$, where $n = |\Delta_+|$. Let

$$\zeta_r(\mathbf{s};\mathfrak{g}) = \sum_{m_1=1}^\infty \cdots \sum_{m_r=1}^\infty \prod_{\alpha\in\Delta_+} \langle\alpha^\vee, m_1\lambda_1 + \cdots + m_r\lambda_r\rangle^{-s_\alpha}. \tag{4.5}$$

Then

$$\zeta_W(s;\mathfrak{g}) = K(\mathfrak{g})^s \zeta_r((s, \ldots, s);\mathfrak{g}). \tag{4.6}$$

If $\mathfrak{g}$ is a direct sum of two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, then

$$\zeta_W(s;\mathfrak{g}) = \zeta_W(s;\mathfrak{g}_1)\zeta_W(s;\mathfrak{g}_2). \tag{4.7}$$

This follows easily from the fact that any irreducible representation φ of $\mathfrak{g}$ is equivalent to the tensor product of two irreducible representations φ_1 of $\mathfrak{g}_1$ and φ_2 of $\mathfrak{g}_2$, and conversely, if φ_i is an irreducible representation of $\mathfrak{g}_i$ ($i = 1, 2$) then $\varphi_1 \otimes \varphi_2$ is an irreducible representation of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. Therefore it is sufficient to study Witten zeta-functions only in the case when $\mathfrak{g}$ is simple. And when $\mathfrak{g}$ is simple, by using the data of the classification theory, we can give a more explicit form of $\zeta_W(s;\mathfrak{g})$ and $\zeta_r(\mathbf{s};\mathfrak{g})$. We will discuss some low-rank cases in the next section.

5. The cases A_1, A_2 and B_2

Let $\mathfrak{g}$ be of A_r type, that is, $\mathfrak{g} = \mathfrak{sl}(r+1)$. Since any positive coroot $\alpha^\vee$ of $\mathfrak{sl}(r+1)$ can be written as (3.2), we have

$$\begin{aligned}&\langle \alpha^\vee, m_1\lambda_1 + \cdots + m_r\lambda_r\rangle \\ &\quad = \sum_{i\le k<j} \langle \alpha_k^\vee, m_1\lambda_1 + \cdots + m_r\lambda_r\rangle = m_i + \cdots + m_{j-1}.\end{aligned} \tag{5.1}$$

Therefore

$$\zeta_r(\mathbf{s};A_r) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{1\le i<j\le r+1} (m_i + \cdots + m_{j-1})^{-s_{ij}} \tag{5.2}$$

where s_{ij} corresponds to the coroot $e_i - e_j$.

Remark 1. If we put $s_{ij} = 0$ for all (i, j) with $i \ge 2$, then (5.2) is reduced to (1.1). Therefore $\zeta_r(\mathbf{s};\mathfrak{g})$ is not only a multi-variable version of Witten zeta-functions, but also a generalization of Euler-Zagier sums.

When $r = 1$, it is clear from (5.2) that $\zeta_1(s;A_1)$ is nothing but the Riemann zeta-function $\zeta(s)$. The case $r = 2$ is also a classical object. In this case we find that

$$\zeta_2(s_1, s_2, s_3;A_2) = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} m_1^{-s_1} m_2^{-s_2} (m_1 + m_2)^{-s_3} \tag{5.3}$$

($\mathbf{s} = (s_1, s_2, s_3)$, here we use a suffix system different from that used in (5.2)), which is sometimes called the Mordell-Tornheim (or simply Tornheim) double zeta-function and denoted by $\zeta_{MT,2}(s_1, s_2, s_3)$. It is Tornheim [24] who first introduced the double series (5.3) and studied its values when s_1, s_2, s_3 are positive integers. He proved various evaluation formulas, which express special values of (5.3) in terms of Bernoulli numbers. Mordell [19] considered the case when $s_1 = s_2 = s_3 = k \in \mathbb{N}$, and proved that if k is even then

$$\zeta_2(k,k,k;A_2) = C(k,k,k;A_2)\pi^{3k}, \qquad C(k,k,k;A_2) \in \mathbb{Q}. \tag{5.4}$$

In particular, when $k = 2$, Mordell obtained

$$\zeta_2(2,2,2;A_2) = \frac{1}{2835}\pi^6. \tag{5.5}$$

The explicit value of $C(k,k,k;A_2)$ for even $k \geq 4$ was obtained by Subbarao and Sitaramachandrarao [21], and by Zagier [33]. An evaluation formula for the value $\zeta_2(k,k,k;A_2)$ when k is odd was obtained by Huard, Williams and Zhang [6].

Subbarao and Sitaramachandrarao [21] discovered a kind of reciprocity relation. They proved that if k_1, k_2, k_3 are positive even integers, then

$$\zeta_2(k_1,k_2,k_3;A_2) + \zeta_2(k_2,k_3,k_1;A_2) + \zeta_2(k_3,k_1,k_2;A_2)$$

can be expressed in terms of Bernoulli numbers. The third-named author [25] proved the following more general result.

Theorem 1 ([25])**.** *Let $k_1, k_2, k_3 \in \mathbb{N}_0$ satisfying $k_1+k_2 \geq 2$ and $k_3 \geq 2$. Then*

$$\zeta_2(k_1,k_2,k_3;A_2) + (-1)^{k_2}\zeta_2(k_2,k_3,k_1;A_2) + (-1)^{k_2+k_3}\zeta_2(k_3,k_1,k_2;A_2)$$

is a polynomial in $\zeta(j)$, $2 \leq j \leq k_1+k_2+k_3$, with rational coefficients. When $k_1+k_2+k_3$ is even, then the above quantity can be expressed in terms of Bernoulli numbers.

The second assertion of Theorem 1 is not explicitly stated in [25], but it can be seen from the expression given in Theorem 3.1 of [15].

The basic idea in [25] ("u-method") is to introduce the parameter $u > 1$, and consider the series

$$\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty} \frac{(-u)^{-m_1-m_2}}{m_1^{s_1} m_2^{s_2} (m_1+m_2)^{s_3}}$$

(or some variant of it). Because of the existence of the factor $(-u)^{-m_1-m_2}$, the above series has nice convergence properties, so we can treat it much easier than the original series (5.3). Then finally take the limit $u \to 1$ carefully to obtain various formulas on (5.3).

A typical example of formulas given by Theorem 1 is as follows:

$$\zeta_2(4,5,3;A_2) - \zeta_2(5,3,4;A_2) + \zeta_2(3,4,5;A_2) = \frac{19}{182432250}\pi^{12}. \tag{5.6}$$

Subbarao and Sitaramachandrarao [21] raised the problem of evaluating the special values of the following alternating analogues of $\zeta_2(\mathbf{s}; A_2)$:

$$\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\frac{(-1)^{m_2}}{m_1^{s_1}m_2^{s_2}(m_1+m_2)^{s_3}}, \qquad \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\frac{(-1)^{m_1+m_2}}{m_1^{s_1}m_2^{s_2}(m_1+m_2)^{s_3}}.$$

This problem was solved in some cases by the third-named author [26], [27], [29], again by the u-method. In [27], [29], he introduced the partial Tornheim series

$$\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}(2m_1+b_1)^{-s_1}(2m_2+b_2)^{-s_2}(2m_1+2m_2+b_1+b_2)^{-s_3} \tag{5.7}$$

where $b_1, b_2 \in \{1,2\}$, reduced the problem to the evaluation of special values of (5.7), and applied the u-method to (5.7).

The above are the results on the special values of the series. Next we consider $\zeta_2(\mathbf{s}; A_2)$ as a function of complex variables. The meromorphic continuation of $\zeta_2(s_1,s_2,s_3;A_2)$ to $\mathbb{C}^3$ was first established by S. Akiyama and also by S. Egami in 1999, but both of their proofs are unpublished. The second-named author proved the following theorem in [16].

Theorem 2 ([16])**.** *The function $\zeta_2(s_1,s_2,s_3;A_2)$ can be continued meromorphically to the whole space $\mathbb{C}^3$, and its singularities are $s_1+s_3=1-l$, $s_2+s_3=1-l$ $(l\in\mathbb{N}_0)$, and $s_1+s_2+s_3=2$.*

The key to the proof of Theorem 2 in [16] is the Mellin-Barnes integral formula

$$(1+\lambda)^{-s} = \frac{1}{2\pi\sqrt{-1}}\int_{(c)}\frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)}\lambda^z dz, \tag{5.8}$$

where $s,\lambda\in\mathbb{C}$, $\lambda\neq 0$, $|\arg\lambda|<\pi$, $\Re s>0$, c is real with $-\Re s<c<0$, and the path of integration is the vertical line from $c-\sqrt{-1}\infty$ to $c+\sqrt{-1}\infty$. To prove Theorem 2, at first assume that $\Re s_j$ $(j=1,2,3)$ are sufficiently large. Applying (5.8) with $\lambda=m_2/m_1$, we have

$$\begin{aligned} m_1^{-s_1}m_2^{-s_2}(m_1+m_2)^{-s_3} &= m_1^{-s_1-s_3}m_2^{-s_2}\left(1+\frac{m_2}{m_1}\right)^{-s_3} \\ &= \frac{1}{2\pi\sqrt{-1}}\int_{(c)}\frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)}m_1^{-s_1-s_3-z}m_2^{-s_2+z}dz, \end{aligned} \tag{5.9}$$

where $-\Re s_3 < c < 0$, and hence

$$\begin{aligned} &\zeta_2(s_1,s_2,s_3;A_2) \\ &\quad = \frac{1}{2\pi\sqrt{-1}}\int_{(c)} \frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)}\zeta(s_1+s_3+z)\zeta(s_2-z)dz. \end{aligned} \tag{5.10}$$

The meromorphic continuation can be shown by shifting the path of integration on the right-hand side of (5.10) sufficiently to the right. This shifting is possible, because applying Stirling's formula to the gamma factors of the integrand one can see that the integrand is of rapid decay when $|\Im z| \to \infty$. Let M be a sufficiently large positive integer, ε be a small positive number. When we shift the path of integration to $\Re z = M - \varepsilon$, the relevant poles are at $z = 0, 1, \ldots, M-1$ and $z = s_2 - 1$. Therefore we have

$$\begin{aligned} &\zeta_2(s_1,s_2,s_3;A_2) \\ &= \frac{\Gamma(s_2+s_3-1)\Gamma(1-s_2)}{\Gamma(s_3)}\zeta(s_1+s_2+s_3-1) \\ &+ \sum_{k=0}^{M-1}\binom{-s_3}{k}\zeta(s_1+s_3+k)\zeta(s_2-k) \\ &+ \frac{1}{2\pi\sqrt{-1}}\int_{(M-\varepsilon)} \frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)}\zeta(s_1+s_3+z)\zeta(s_2-z)dz. \end{aligned} \tag{5.11}$$

The last integral is holomorphic in the region $\Re s_3 > -M+\varepsilon$, $\Re(s_1+s_3) > 1-M+\varepsilon$, $\Re s_2 < 1+M-\varepsilon$. Since M is arbitrary, this implies the meromorphic continuation of $\zeta_2(s_1,s_2,s_3;A_2)$ to $\mathbb{C}^3$. Moreover, we find that $s_1+s_3 = 1-l$, $s_2+s_3 = 1-l$, and $s_1+s_2+s_3 = 2$ are singularities of the residue terms on the right-hand side of (5.11). Apparently $s_2 = 1+l$ also seems singular, but this singularity is cancelled. The proof of Theorem 2 is complete.

Recently the first-named author [9] obtained an alternative proof of the meromorphic continuation of $\zeta_2(s_1,s_2,s_3;A_2)$, whose main tool is surface integration.

Next consider the B_r case. We have

$$\begin{aligned}
&\zeta_r(\mathbf{s};B_r)\\
&=\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\prod_{1\leq i\leq r}(2(m_i+\cdots+m_{r-1})+m_r)^{-s_i}\\
&\quad\times\prod_{1\leq i<j\leq r}(m_i+\cdots+m_{j-1})^{-s_{ij}^{-}}\\
&\quad\times\prod_{1\leq i<j\leq r}(m_i+\cdots+m_{j-1}+2(m_j+\cdots+m_{r-1})+m_r)^{-s_{ij}^{+}},
\end{aligned}\tag{5.12}$$

where $s_i, s_{ij}^{-}, s_{ij}^{+}$ correspond to $2e_i, e_i-e_j, e_i+e_j$, respectively. In particular,

$$\begin{aligned}
&\zeta_2(s_1,s_2,s_3,s_4;B_2)\\
&\qquad=\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}m_1^{-s_1}m_2^{-s_2}(m_1+m_2)^{-s_3}(2m_1+m_2)^{-s_4}
\end{aligned}\tag{5.13}$$

($\mathbf{s}=(s_1,s_2,s_3,s_4)$). This multi-variable zeta-function for B_2 was introduced by the second-named author in [17], which inspired the general definition (4.5) of $\zeta_r(\mathbf{s};\mathfrak{g})$, given in [18] (in the A_r case), [10], and [11]. In [17], similarly to (5.11), it has been shown that

$$\begin{aligned}
\zeta_2(s_1,s_2,s_3,s_4;B_2)&=\frac{1}{2\pi\sqrt{-1}}\int_{(c)}\frac{\Gamma(s_4+z)\Gamma(-z)}{\Gamma(s_4)}\\
&\quad\times\zeta_2(s_1,s_2-z,s_3+s_4+z;A_2)dz.
\end{aligned}\tag{5.14}$$

The analytic properties of $\zeta_2(\mathbf{s};B_2)$, which can be derived from (5.14), are discussed in [11].

In [28], the third-named author showed that $\zeta_2(r_1,r_2,r_3,r_4;B_2)$, where $r_1,r_2,r_3,r_4\in\mathbb{N}_0$ for which the series is convergent, can be expressed as a sum of special values of $\zeta_2(\mathbf{s};A_2)$ and the series (5.7). Therefore, using the results in [6] and [29], we can deduce evaluation formulas for $\zeta_2(r_1,r_2,r_3,r_4;B_2)$. A typical example is

$$\zeta_2(2,2,1,2;B_2)=-\frac{185}{16}\zeta(7)+\frac{55}{48}\pi^2\zeta(5).\tag{5.15}$$

6. Mellin-Barnes reductions

In the preceding section, we gave the Mellin-Barnes integral expressions of $\zeta_2(\mathbf{s};A_2)$ and $\zeta_2(\mathbf{s};B_2)$ ((5.10) and (5.13), respectively), from which the meromorphic continuation of those functions can be proved. In [17], this argument has been much more generalized. Let $M_{nr} = (a_{ij})_{1\le i\le n, 1\le j\le r}$ be an $n\times r$ matrix, where a_{ij} are non-negative real numbers. Assume that all rows and all columns of M_{nr} include at least one non-zero element. Define

$$\begin{aligned}\zeta_r(s_1,\ldots,s_n;M_{nr}) = \sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}(a_{11}m_1+\cdots+a_{1r}m_r)^{-s_1} \\ \times\cdots\times(a_{n1}m_1+\cdots+a_{nr}m_r)^{-s_n}.\end{aligned} \tag{6.1}$$

Then

Theorem 3 ([17])**.** *The function $\zeta_r(s_1,\ldots,s_n;M_{nr})$ can be continued meromorphically to the whole space $\mathbb{C}^n$.*

From this theorem it immediately follows that $\zeta_r(\mathbf{s};\mathfrak{g})$, defined by (4.5), can be continued meromorphically to $\mathbb{C}^n$.

Note that Essouabri [3] [4] developed a method of proving the continuation of very general form of multiple Dirichlet series, which is quite different from our Mellin-Barnes argument. The continuation of $\zeta_r(\mathbf{s};\mathfrak{g})$, and even the above Theorem 3, is actually included in Essouabri's theorem.

However, our argument has an advantage; from the Mellin-Barnes integral expression it is not difficult to deduce various explicit information, such as location of singularities. Another important point is that, by our method, we can find a recursive structure among the family of multiple zeta-functions. In fact, (5.10) is an expression of $\zeta_2(\mathbf{s};A_2)$ by a "simpler" zeta-function, that is $\zeta(s)$. Similarly, (5.14) expresses $\zeta_2(\mathbf{s};B_2)$ by $\zeta_2(\mathbf{s};A_2)$, which is "simpler" than $\zeta_2(\mathbf{s};B_2)$. We may understand that there is the recursive structure

$$A_2\to(A_1,A_1),\qquad B_2\to A_2. \tag{6.2}$$

The same structure can be found in higher-rank situation. For this purpose, now we introduce the notion of *multiple zeta-functions of root sets*. Let Δ^* be a subset of $\Delta_+=\Delta_+(\mathfrak{g})$. We call Δ^* a *root set* if for any λ_j $(1\le j\le r)$ there exists an element $\alpha\in\Delta^*$ such that $\langle\alpha^\vee,\lambda_j\rangle\neq 0$. If Δ^* is a root set, we can define

$$\zeta_r(\mathbf{s};\Delta^*) = \sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\prod_{\alpha\in\Delta^*}\langle\alpha^\vee, m_1\lambda_1+\cdots m_r\lambda_r\rangle^{-s_\alpha}, \tag{6.3}$$

where $\mathbf{s} = \mathbf{s}(\Delta^*) = (s_\alpha)_{\alpha\in\Delta^*} \in \mathbb{C}^{n^*}$ with $n^* = |\Delta^*|$. We call $\zeta_r(\mathbf{s};\Delta^*)$ the zeta-function of the root set Δ^*. When $\Delta^* = \Delta_+(\mathfrak{g})$, $\zeta_r(\mathbf{s};\Delta^*)$ coincides with $\zeta_r(\mathbf{s};\mathfrak{g})$ defined by (4.5). From this viewpoint it is suitable to call $\zeta_r(\mathbf{s};\mathfrak{g})$ the *zeta-function of the root system* Δ.

Consider the zeta-function of A_r-type, that is (5.3). Define the root set

$$\Delta_h^*(A_r) = \{\varepsilon_1 - \varepsilon_j \mid 2 \le j \le h\} \cup \{\varepsilon_i - \varepsilon_j \mid 2 \le i < j \le r+1\} \tag{6.4}$$

for $2 \le h \le r+1$ and

$$\Delta^*(A_r) = \{\varepsilon_i - \varepsilon_j \mid 2 \le i < j \le r+1\}. \tag{6.5}$$

The term $(m_1 + \cdots + m_r)^{-s_{1,r+1}}$ corresponds to the coroot $e_1 - e_{r+1}$, or the root $\varepsilon_1 - \varepsilon_{r+1}$. Applying the Mellin-Barnes formula (5.8) we have

$$\begin{aligned}(m_1 + \cdots + m_r)^{-s_{1,r+1}} &= (m_1 + \cdots + m_{r-1})^{-s_{1,r+1}} \\ &\times \frac{1}{2\pi\sqrt{-1}} \int_{(c)} \frac{\Gamma(s_{1,r+1}+z)\Gamma(-z)}{\Gamma(s_{1,r+1})} \left(\frac{m_r}{m_1 + \cdots + m_{r-1}}\right)^z dz,\end{aligned} \tag{6.6}$$

and hence

$$\begin{aligned}\zeta_r(\mathbf{s};A_r) = \frac{1}{2\pi\sqrt{-1}} &\int_{(c)} \frac{\Gamma(s_{1,r+1}+z)\Gamma(-z)}{\Gamma(s_{1,r+1})} \\ &\times \zeta_r(\mathbf{s}^*(A_r,z);\Delta_r^*(A_r))dz,\end{aligned} \tag{6.7}$$

where

$$\mathbf{s}^*(A_r,z) = (s_{12},\ldots,s_{1,r-1},s_{1r}+s_{1,r+1}+z,s_{23},\ldots,s_{r,r+1}-z). \tag{6.8}$$

The formula (6.7) gives the recursive relation $\zeta_r(\cdot\,;A_r) \to \zeta_r(\cdot\,;\Delta_r^*(A_r))$, which corresponds to removing one root $\varepsilon_1 - \varepsilon_{r+1}$ from $\Delta_+(A_r)$. Similarly, $\zeta_r(\cdot\,;\Delta_r^*(A_r))$ can be expressed as the Mellin-Barnes integral involving $\zeta_r(\cdot\,;\Delta_{r-1}^*(A_r))$. Repeating this procedure, we finally arrive at $\zeta_r(\cdot\,;\Delta_2^*(A_r))$. In the definition of $\zeta_r(\cdot\,;\Delta_2^*(A_r))$, the only term including m_1 is $m_1^{-s_{12}}$, so the sum with respect to m_1 can be completely separated. Hence $\zeta_r(\cdot\,;\Delta_2^*(A_r))$ can be written as a product of $\zeta_{r-1}(\cdot\,;\Delta^*(A_r))$ and $\zeta(\cdot)$. However the root set $\Delta^*(A_r)$ is actually equivalent to $\Delta_+(A_{r-1})$. Therefore, corresponding to the reduction of root sets

$$\begin{aligned}\Delta_+(A_r) = \Delta_{r+1}^*(A_r) \supset \Delta_r^*(A_r) \supset \Delta_{r-1}^*(A_r) \supset \cdots \\ \cdots \supset \Delta_2^*(A_r) \supset \Delta^*(A_r) = \Delta_+(A_{r-1}),\end{aligned} \tag{6.9}$$

the recursive structure among zeta-functions

$$\begin{aligned}\zeta_r(\cdot\,;A_r) \to \zeta_r(\cdot\,;\Delta_r^*(A_r)) \to \cdots \to \zeta_r(\cdot\,;\Delta_3^*(A_r)) \\ \to \zeta_r(\cdot\,;\Delta_2^*(A_r)) = \zeta_{r-1}(\cdot\,;A_{r-1})\zeta(\cdot)\end{aligned} \tag{6.10}$$

exists, which can be described by Mellin-Barnes integrals. The conclusion of (6.10) can be summarized as

$$A_r \to (A_{r-1}, A_1), \tag{6.11}$$

a generalization of the first relation of (6.2).

The same type of recursive structures can be found for zeta-functions of the other root systems. For the details, see [11].

7. Dynkin diagrams and Dynkin reductions

In this section we introduce the notion of Dynkin diagrams, and explain the recursive structure given in the preceding section in terms of Dynkin diagrams.

Let $\mathfrak{g}$ be a semisimple Lie algebra, and $\Psi = \Psi(\mathfrak{g}) = \{\alpha_1, \ldots, \alpha_r\}$ be a fundamental system. We define the Dynkin diagram $\Gamma = \Gamma(\mathfrak{g})$ associated with $\mathfrak{g}$ as follows. First, to each α_i, we associate a vertex, with the weight $\langle \alpha_i, \alpha_i \rangle$. Any two different vertices α_i and α_j are connected by $a_{ij} \cdot a_{ji}$ edges, where $a_{ij} = a(\alpha_j, \alpha_i)$ is the Cartan integer defined by (2.3). In particular, if $\langle \alpha_i, \alpha_j \rangle = 0$, then there is no edge which connects α_i and α_j. The number of edges connecting two vertices are 1, 2, or 3. In the case when number of edges are 2 or 3, we add an arrow, pointing from the vertex of higher weight to that of lower weight.

Since simple Lie algebras are corresponding to irreducible root systems, it is easy to see that $\Gamma(\mathfrak{g})$ is connected if and only if $\mathfrak{g}$ is simple. Therefore the problem of the classification of simple Lie algebras can be reduced to that of the classification of connected Dynkin diagrams.

In the case of A_r type, the fundamental system consists of $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \le i \le r)$. For $i \ne j$,

$$\alpha_i + t\alpha_j = \varepsilon_i - \varepsilon_{i+1} + t(\varepsilon_j - \varepsilon_{j+1})$$

is a root if and only if $t = 0$ or $t = 1, i + 1 = j$ or $t = 1, j + 1 = i$. Therefore if $|i - j| = 1$, then the α_i-string through α_j consists of just two elements, so by (2.4) we have $a_{ji} = -1$. Hence the vertices α_i and α_j are connected by just one edge. If $|i - j| \ge 2$, then $a(\alpha_j, \alpha_i) = 0$, or in other words α_i and α_j are orthogonal to each other, and there is no edge between the corresponding vertices. Therefore the Dynkin diagram $\Gamma(A_r)$ is as follows:

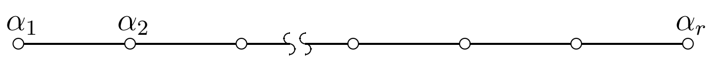
(A_r) 

Figure 1

The Mellin-Barnes reduction process for A_r, described in the preceding section, is actually the process of removing all terms corresponding to the roots $\varepsilon_1 - \varepsilon_h$ $(2 \leq h \leq r+1)$. These roots are exactly the roots which include the term α_1 when one writes them as sums of fundamental roots. Therefore we can summarize that the process is to separate α_1 from the other fundamental roots, that is, to cut off the leftmost edge of the above Dynkin diagram. Then the resulting diagram has two connected components, which are diagrams of A_1 and A_{r-1}.

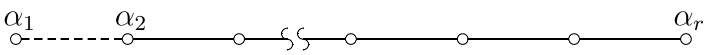
(A_r) 

FIGURE 2

Therefore the above cutting process expresses the recursive relation (6.11).

The cutting of some other edge gives a different recursive relation. If one cuts the edge between α_{l-1} and α_l, one finds that the two connected components of the resulting diagram are the diagrams of A_{l-1} and A_{r-l+1}. This implies that $\zeta_r(\mathbf{s}; A_r)$ can be expressed as an integral of the Mellin-Barnes type whose integrand includes $\zeta_{l-1}(\cdot\ ; A_{l-1})$ and $\zeta_{r-l+1}(\cdot\ ; A_{r-l+1})$. We write this structure as $A_r \to (A_{l-1}, A_{r-l+1})$. In general, we can show the following theorem.

Theorem 4 ([11]). *By cutting off any edge of a Dynkin diagram, we find that the zeta-function of the corresponding root system can be written as a (multiple) integral, whose integrand includes zeta-functions of every connected components of the resulting Dynkin diagram.*

Examine the B_r case. In this case, for $1 \leq i, j \leq r-1$, $i \neq j$, we have $a_{ij} = 1$ if $|i-j| = 1$ and $a_{ij} = 0$ otherwise. Also we have $a_{r-1,r} = 1$. However $a_{r,r-1} = 2$, because

$$\alpha_{r-1} + t\alpha_r = \varepsilon_{r-1} - \varepsilon_r + t\varepsilon_r$$

is a root for $t = 0, 1, 2$. Moreover, since $\langle \alpha_i, \alpha_i \rangle = 1/(2r-1)$ $(1 \leq i \leq r-1)$ and $\langle \alpha_r, \alpha_r \rangle = 1/2(2r-1)$ (see (4.4.49) of [30]), the direction of the arrow is from α_{r-1} to α_r. Therefore the Dynkin diagram for B_r is as follows.

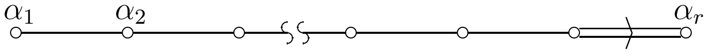
(B_r) 

FIGURE 3

If one cuts the leftmost edge, one obtains the recursive structure $B_r \to (B_{r-1}, A_1)$. This is an analogue of (6.11). On the other hand, if one cuts the rightmost two edges, one obtains a different recursive structure, that is $B_r \to (A_{r-1}, A_1)$. Another way is to cut only one of the rightmost edges. In this case the resulting diagram is still connected, that is the diagram of A_r. Therefore the corresponding recursive relation is $B_r \to A_r$.

We do not mention the recursive structures for the C_r and D_r cases, but those are also discussed in [11]. Those structures give a way of expressing zeta-functions of higher-rank root systems as a (multiple) integral involving zeta-functions of lower-rank root systems. Therefore, analytic properties of zeta-functions of root systems can be obtained inductively, by going upstream the arrows in the recursive structures. Following this way, we have determine all the singularity sets of $\zeta_3(\mathbf{s}; A_3)$ in [18]. The list of possible singularity sets of $\zeta_3(\mathbf{s}; B_3)$ and $\zeta_3(\mathbf{s}; C_3)$ are given in [11].

Among the five exceptional algebras, the most accessible one is G_2. It is known that the Dynkin diagram of G_2 is as follows.

(G_2)

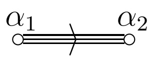


FIGURE 4

Therefore, by cutting one edge we have $G_2 \to B_2$. By using this structure we can study the properties of the zeta-function of G_2, which will be given in [13].

8. The Weyl group symmetry

It is known that in Lie theory, the Weyl groups play essential roles. For example, Weyl's dimension formula is derived by using this symmetry. Therefore it is natural to investigate the Weyl group symmetry of zeta-functions of root systems. Our starting point is functional relations among Lerch zeta-functions $\varphi(s, y)$ and Bernoulli polynomials $B_k(y)$, namely,

$$\varphi(k, y) + (-1)^k \varphi(k, -y) = -B_k(\{y\}) \frac{(2\pi\sqrt{-1})^k}{k!}, \tag{8.1}$$

where $k \in \mathbb{Z}_{\geq 2}$, $y \in \mathbb{R}$, $\varphi(s, y)$ is the Lerch zeta-function defined by

$$\varphi(s, y) = \sum_{m=1}^{\infty} \frac{e^{2\pi\sqrt{-1}my}}{m^s} \tag{8.2}$$

and

$$\frac{te^{t\{y\}}}{e^t-1}=\sum_{k=0}^{\infty}B_k(\{y\})\frac{t^k}{k!}, \tag{8.3}$$

with $\{y\}=y-[y]$ (i.e. fractional part).

Motivated by this observation, we introduce Lerch-type generalizations of (4.5) as

$$\begin{aligned}\zeta_r(\mathbf{s},\mathbf{y};\mathfrak{g})=&\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}e^{2\pi\sqrt{-1}\langle\mathbf{y},m_1\lambda_1+\cdots+m_r\lambda_r\rangle}\\&\times\prod_{\alpha\in\Delta_+}\langle\alpha^\vee,m_1\lambda_1+\cdots+m_r\lambda_r\rangle^{-s_\alpha},\end{aligned} \tag{8.4}$$

where $\mathbf{y}\in\mathfrak{h}_0$. To define an action of the Weyl group, we identify $\mathbf{s}=(s_\alpha)_{\alpha\in\Delta_+}$ with $(s_\alpha)_{\alpha\in\Delta}$ by $s_\alpha=s_{-\alpha}$. Since $w(-\alpha)=-w(\alpha)$ for $\alpha\in\Delta$ and $w\in W$, an action of the Weyl group is naturally induced on any function f in $\mathbf{s}$ and $\mathbf{y}$ as follows: For $w\in W$,

$$(wf)(\mathbf{s},\mathbf{y})=f(w^{-1}\mathbf{s},w^{-1}\mathbf{y}), \tag{8.5}$$

where for $\beta\in\Delta$,

$$(\sigma_\beta\mathbf{s})_\alpha=s_{\sigma_\beta\alpha}, \tag{8.6}$$

$$\sigma_\beta\mathbf{y}=\mathbf{y}-\langle\mathbf{y},\beta\rangle\beta^\vee. \tag{8.7}$$

We define a main object of the following sections as follows:

$$S(\mathbf{s},\mathbf{y};\mathfrak{g})=\sum_{w\in W}\Big(\prod_{\alpha\in\Delta_+\cap w\Delta_-}(-1)^{-s_\alpha}\Big)(w\zeta_r)(\mathbf{s},\mathbf{y};\mathfrak{g}). \tag{8.8}$$

Here we give two examples, from which we will observe that the function $S(\mathbf{s},\mathbf{y};\mathfrak{g})$ plays an role of periodic Bernoulli functions in the classical theory.

Example 1. In the A_1 case, we have $\Delta_+=\{\alpha=\alpha_1\}$ and $W=\{\mathrm{id},\sigma_\alpha\}$. By putting $\mathbf{y}=y\alpha^\vee$ and $\mathbf{s}=(k)$ with $k\in\mathbb{Z}_{\geq 2}$, we obtain

$$\zeta_1(\mathbf{s},\mathbf{y};A_1)=\sum_{m=1}^{\infty}e^{2\pi\sqrt{-1}my}m^{-k}=\varphi(k,y) \tag{8.9}$$

and

$$S(\mathbf{s},\mathbf{y};A_1)=\varphi(k,y)+(-1)^{-k}\varphi(k,-y), \tag{8.10}$$

which is reduced to the left-hand side of (8.1).

Example 2. In the A_2 case with $\mathbf{y}=\mathbf{0}$, we have $\Delta_+=\{\alpha_1,\alpha_2,\alpha_1+\alpha_2\}$ and $W=\{\mathrm{id},\sigma_1,\sigma_2,\sigma_1\sigma_2,\sigma_2\sigma_1,\sigma_1\sigma_2\sigma_1=\sigma_2\sigma_1\sigma_2\}$, where $\sigma_1=\sigma_{\alpha_1}$ and $\sigma_2=\sigma_{\alpha_2}$. For simplicity we set $s_{\alpha_1}=k_1$, $s_{\alpha_2}=k_2$, $s_{\alpha_1+\alpha_2}=k_3$ with $k_1,k_2,k_3\in\mathbb{Z}_{\geq 2}$ and we abbreviate $\zeta_2(\mathbf{s},\mathbf{0};A_2)=\zeta_2(s_1,s_2,s_3;A_2)$. We obtain

$$
\begin{aligned}
\Delta_+\cap \mathrm{id}\,\Delta_- &= \emptyset,\\
\Delta_+\cap \sigma_1\Delta_- &= \{\alpha_1\},\\
\Delta_+\cap \sigma_2\Delta_- &= \{\alpha_2\},\\
\Delta_+\cap \sigma_1\sigma_2\Delta_- &= \{\alpha_1,\alpha_1+\alpha_2\},\\
\Delta_+\cap \sigma_2\sigma_1\Delta_- &= \{\alpha_2,\alpha_1+\alpha_2\},\\
\Delta_+\cap \sigma_1\sigma_2\sigma_1\Delta_- &= \{\alpha_1,\alpha_2,\alpha_1+\alpha_2\},
\end{aligned}
\tag{8.11}
$$

which implies

$$
\begin{aligned}
&S(\mathbf{s},\mathbf{y};A_2)\\
&=\zeta_2(k_1,k_2,k_3;A_2)+(-1)^{-k_1}\zeta_2(k_1,k_3,k_2;A_2)\\
&\quad+(-1)^{-k_2}\zeta_2(k_3,k_2,k_1;A_2)+(-1)^{-k_1-k_3}\zeta_2(k_2,k_3,k_1;A_2)\\
&\quad+(-1)^{-k_2-k_3}\zeta_2(k_3,k_1,k_2;A_2)\\
&\qquad+(-1)^{-k_1-k_2-k_3}\zeta_2(k_2,k_1,k_3;A_2)\\
&=(1+(-1)^{k_1+k_2+k_3})\\
&\quad\times(\zeta_2(k_1,k_2,k_3;A_2)+(-1)^{k_2}\zeta_2(k_2,k_3,k_1;A_2)\\
&\qquad+(-1)^{k_1}\zeta_2(k_3,k_1,k_2;A_2))
\end{aligned}
\tag{8.12}
$$

by use of $\zeta_2(k_1,k_2,k_3;A_2)=\zeta_2(k_2,k_1,k_3;A_2)$. When $k_1+k_2+k_3$ is even, then (8.12) coincides with the linear combination in Theorem 1 up to a constant factor. Hence they can be expressed in terms of Bernoulli numbers.

From these examples, we can expect that $S(\mathbf{k},\mathbf{y};\mathfrak{g})$ has nice properties when all k_α are positive integers. In fact, in the next section we will construct multiple generalizations of periodic Bernoulli functions $P(\mathbf{k},\mathbf{y};\mathfrak{g})$, so that $S(\mathbf{k},\mathbf{y};\mathfrak{g})$ is expressed in terms of them. More precisely we have

Theorem 5 ([12])**.**

$$
S(\mathbf{k},\mathbf{y};\mathfrak{g})=(-1)^n\Bigl(\prod_{\alpha\in\Delta_+}\frac{(2\pi\sqrt{-1})^{k_\alpha}}{k_\alpha!}\Bigr)P(\mathbf{k},\mathbf{y};\mathfrak{g}).
\tag{8.13}
$$

That is, the value at $\mathbf{s}=\mathbf{k}$ of the Weyl group symmetric linear combination of $\zeta_r(\mathbf{s},\mathbf{y};\mathfrak{g})$ can be expressed in terms of generalized periodic

Bernoulli functions $P(\mathbf{s},\mathbf{y};\mathfrak{g})$. Since $P(\mathbf{s},\mathbf{y};\mathfrak{g})$ can be explicitly calculated (see the next section), Theorem 5 gives relations among special values of $\zeta_r(\mathbf{s},\mathbf{y};\mathfrak{g})$.

We omit the proof of Theorem 5 and admit this statement because it is quite lengthy. For the details, see [12].

It should be noted that in [22, 23], Szenes studied generalizations of Bernoulli polynomials from the viewpoint of the theory of arrangement of hyperplanes which include $P(\mathbf{k},\mathbf{y};\mathfrak{g})$ appearing above, and that he also gave an algorithm for calculating them by use of iterated residues of meromorphic functions at the points of indeterminacy.

Corollary 1. *Assume that* $\mathbf{k}=(k_\alpha)_{\alpha\in\Delta_+}\in 2\mathbb{N}^n$ *satisfies* $\mathbf{k}=w\mathbf{k}$ *for all* $w\in W$. *Then*

$$\zeta_r(\mathbf{k},\mathbf{0};\mathfrak{g})=\frac{1}{|W|}(-1)^n\left(\prod_{\alpha\in\Delta_+}\frac{(2\pi\sqrt{-1})^{k_\alpha}}{k_\alpha!}\right)P(\mathbf{k},\mathbf{0};\mathfrak{g}). \tag{8.14}$$

If all $k_\alpha=k$, then $\mathbf{k}=w\mathbf{k}$ for all $w\in W$ and hence Corollary 1 implies $\zeta_r((k,k,...,k),\mathbf{0};\mathfrak{g})\in\mathbb{Q}\pi^{|\mathbf{k}|}$, where $|\mathbf{k}|=\sum_{\alpha\in\Delta_+}k_\alpha$. This is called Witten's volume formula [32, 33]. Our (8.14) gives the explicit value of the rational coefficient, which was not determined in the original formula of Witten.

Example 3. We have

$$\zeta_2((2,2,2),\mathbf{0};A_2)=\frac{1}{6}(-1)^3\frac{(2\pi\sqrt{-1})^6}{(2!)^3}\frac{1}{3780}=\frac{\pi^6}{2835}, \tag{8.15}$$

where the rational number $1/3780$ is $P((2,2,2),\mathbf{0};A_2)$ and calculated by using the explicit form of the generating function given in Example 8 in the next section. This recovers Mordell's result (5.5).

In an irreducible root system, $\mathbf{k}=w\mathbf{k}$ for all $w\in W$ is equivalent to $k_\alpha=k_\beta$ if $\langle\alpha,\alpha\rangle=\langle\beta,\beta\rangle$. Hence if the root system is non-simply laced, that is, in the cases of B_r, C_r, F_4, G_2, then by Corollary 1, we can also obtain generalizations of Witten's volume formula. The following is a typical example.

Example 4. In the root system of type B_2, we have $\Delta_+=\{\alpha_1,\alpha_2,\alpha_1+\alpha_2,\alpha_1+2\alpha_2\}$ and $(\alpha_1+\alpha_2)^\vee=2\alpha_1^\vee+\alpha_2^\vee$, $(\alpha_1+2\alpha_2)^\vee=\alpha_1^\vee+\alpha_2^\vee$. Hence by setting $k_{\alpha_1} = k_{\alpha_1+2\alpha_2} = 4$, $k_{\alpha_2} = k_{\alpha_1+\alpha_2} = 2$, we obtain

$$
\begin{aligned}
&\zeta_2(\mathbf{k}, \mathbf{0}; B_2) \\
&\quad = \sum_{m_1,m_2=1}^{\infty} \frac{1}{m_1^4 m_2^2 (m_1+m_2)^4 (2m_1+m_2)^2} \\
&\quad = \frac{(-1)^4}{2^2 2!} \left(\frac{(2\pi\sqrt{-1})^2}{2!} \right)^2 \left(\frac{(2\pi\sqrt{-1})^4}{4!} \right)^2 \frac{53}{1513512000} \\
&\quad = \frac{53\pi^{12}}{6810804000}.
\end{aligned} \tag{8.16}
$$

Before proceeding to the construction of generating functions of generalized periodic Bernoulli functions, we discuss the Weyl symmetry of the function $S(\mathbf{k}, \mathbf{y}; \mathfrak{g})$ and its consequence. Denote by $\mathbb{Z}_{\geq 2}$ the set of integers ≥ 2.

Theorem 6 ([12]). *For $\mathbf{k} \in \mathbb{Z}_{\geq 2}^n$ and $\mathbf{y} \in \mathfrak{h}_0$, and for $w \in W$, we have*

$$
(wS)(\mathbf{k}, \mathbf{y}; \mathfrak{g}) = \Big(\prod_{\alpha \in \Delta_+ \cap w\Delta_-} (-1)^{-k_\alpha} \Big) S(\mathbf{k}, \mathbf{y}; \mathfrak{g}). \tag{8.17}
$$

This theorem can be easily derived from definition (8.8). Furthermore direct calculations yield the following corollary.

Corollary 2. *For $\mathbf{k} \in \mathbb{Z}_{\geq 2}^n$ and $\mathbf{y} \in \mathfrak{h}_0$, we have $S(\mathbf{k}, \mathbf{y}; \mathfrak{g}) = 0$ if there exists an element $w \in W_{\mathbf{k}} \cap W_{\mathbf{y}}$ such that*

$$
\sum_{\alpha \in \Delta_+ \cap w\Delta_-} k_\alpha \notin 2\mathbb{Z}, \tag{8.18}
$$

where $W_{\mathbf{k}}$ and $W_{\mathbf{y}}$ are the stabilizers of $\mathbf{k}$ and $\mathbf{y}$ respectively by regarding $\mathbf{y} \in \mathfrak{h}_0 \pmod{\bigoplus_{j=1}^r \mathbb{Z}\alpha_j^\vee}$.

Example 5. In the A_1 case, choosing an odd $k(>1)$ and $w = \sigma_\alpha$ in Example 1, we see that

$$
(wS)(\mathbf{k}, \mathbf{y}; A_1) = -S(\mathbf{k}, \mathbf{y}; A_1), \tag{8.19}
$$

by Theorem 6. Let $\mathbf{y} = j\alpha^\vee/2 \in (\alpha^\vee/2)\mathbb{Z}$. Then $w\mathbf{y} = -\mathbf{y} = -j\alpha^\vee/2 = j\alpha^\vee/2 \pmod{Q^\vee}$, and so

$$
(wS)(\mathbf{k}, \mathbf{y}; A_1) = S(\mathbf{k}, -j\alpha^\vee/2; A_1) = S(\mathbf{k}, j\alpha^\vee/2; A_1).
$$

This and (8.19) imply $S(\mathbf{k}, \mathbf{y}; A_1) = 0$. This is the simplest case of Corollary 2, and is nothing but the classical result $B_k(0) = B_k(1/2) = 0$ when $k(>1)$ is odd.

Example 6. In the root system of type A_2, we set $\mathbf{y} = y_1\alpha_1^\vee + y_2\alpha_2^\vee$ and consider $S((3,2,2),(y_1,y_2);A_2)$. We see that $\Delta_+ \cap \sigma_1\Delta_- = \{\alpha_1\}$, $\sigma_1(s_1,s_2,s_3) = (s_1,s_3,s_2)$ and $\sigma_1\mathbf{y} = \mathbf{y} - \langle \mathbf{y},\alpha_1\rangle\alpha_1^\vee$. Hence if $\langle \mathbf{y},\alpha_1\rangle \in \mathbb{Z}$, then $\sigma_1\mathbf{y} \equiv \mathbf{y} \pmod{Q^\vee}$, which is equivalent to

$$y_1 = \frac{2n+y}{3}, \qquad y_2 = \frac{n+2y}{3}, \tag{8.20}$$

where $n \in \mathbb{Z}$ and $y \in \mathbb{R}$. Therefore we see that

$$S\Big((3,2,2),\Big(\frac{2n+y}{3},\frac{n+2y}{3}\Big);A_2\Big) = 0 \tag{8.21}$$

for all $n \in \mathbb{Z}$ and $y \in \mathbb{R}$ by Corollary 2.

In fact, (8.21) is directly checked by use of the explicit form of $S((3,2,2),(y_1,y_2);A_2)$. We have

$$\begin{aligned}
&(8.22)\quad S((3,2,2),(y_1,y_2);A_2) = -128\sqrt{-1}\pi^7\times \\
&\Big(\frac{1}{840}\{y_1\}^7 + \frac{1}{240}\{y_2-y_1\}\{y_1\}^6 - \frac{1}{160}\{y_1\}^6 + \frac{1}{240}\{y_2-y_1\}^2\{y_1\}^5 \\
&- \frac{1}{60}\{y_2-y_1\}\{y_1\}^5 + \frac{1}{90}\{y_1\}^5 - \frac{1}{96}\{y_2-y_1\}^2\{y_1\}^4 + \frac{1}{48}\{y_2-y_1\}\{y_1\}^4 \\
&- \frac{1}{144}\{y_1\}^4 + \frac{1}{144}\{y_2-y_1\}^2\{y_1\}^3 - \frac{1}{144}\{y_2-y_1\}\{y_1\}^3 - \frac{\{y_1\}^3}{4320} \\
&- \frac{1}{480}\{y_2-y_1\}\{y_1\}^2 + \frac{1}{960}\{y_1\}^2 - \frac{\{y_2-y_1\}^2\{y_1\}}{1440} + \frac{\{y_2-y_1\}\{y_1\}}{1440} \\
&+ \frac{\{y_1\}}{12096} + \frac{\{y_2\}^7}{1260} + \frac{1}{240}\{y_1-y_2\}\{y_2\}^6 - \frac{7\{y_2\}^6}{1440} \\
&+ \frac{1}{120}\{y_1-y_2\}^2\{y_2\}^5 - \frac{1}{48}\{y_1-y_2\}\{y_2\}^5 + \frac{1}{96}\{y_2\}^5 + \frac{1}{144}\{y_1-y_2\}^3\{y_2\}^4 \\
&- \frac{1}{32}\{y_1-y_2\}^2\{y_2\}^4 + \frac{5}{144}\{y_1-y_2\}\{y_2\}^4 - \frac{5}{576}\{y_2\}^4 - \frac{\{y_1-y_2\}^3}{4320} \\
&- \frac{1}{72}\{y_1-y_2\}^3\{y_2\}^3 + \frac{5}{144}\{y_1-y_2\}^2\{y_2\}^3 - \frac{1}{48}\{y_1-y_2\}\{y_2\}^3 + \frac{1}{720}\{y_2\}^3 \\
&+ \frac{\{y_1-y_2\}^2}{2880} + \frac{1}{144}\{y_1-y_2\}^3\{y_2\}^2 - \frac{1}{96}\{y_1-y_2\}^2\{y_2\}^2 + \frac{1}{720}\{y_1-y_2\}\{y_2\}^2 \\
&+ \frac{1}{960}\{y_2\}^2 - \frac{\{y_1-y_2\}}{60480} - \frac{1}{720}\{y_1-y_2\}^2\{y_2\} + \frac{1}{720}\{y_1-y_2\}\{y_2\} \\
&- \frac{\{y_2\}}{10080} + \frac{\{y_2-y_1\}}{10080} - \frac{1}{10080}\Big),
\end{aligned}$$

which can be calculated by use of the generating function. (See the next section.)

9. Generating functions and Bernoulli polynomials of root systems

As mentioned in the previous section, we construct the generating functions of multiple periodic Bernoulli functions. To this end, we prepare some definitions.

Let $\mathscr{V}$ be the set of all bases $\mathbf{V} \subset \Delta_+$. For $\mathbf{V} \in \mathscr{V}$, let $\mathbf{V}^\vee = \{\beta^\vee\}_{\beta\in\mathbf{V}}$ and $\mathbf{V}^* = \{\mu^{\mathbf{V}}_\beta\}_{\beta\in\mathbf{V}}$, the dual basis of $\mathbf{V}^\vee$, that is, $\langle\alpha^\vee, \mu^{\mathbf{V}}_\beta\rangle = \delta_{\alpha\beta}$ for $\alpha, \beta \in \mathbf{V}$. Let $Q^\vee = \bigoplus_{i=1}^r \mathbb{Z}\alpha_i^\vee$ be the coroot lattice and $L(\mathbf{V}^\vee) = \bigoplus_{\beta\in\mathbf{V}} \mathbb{Z}\beta^\vee$. Then we see that $L(\mathbf{V}^\vee)$ is a sublattice of $Q^\vee$ with finite index.

Besides these definitions, we need to define a fractional part of $\mathbf{y} \in \mathfrak{h}_0$. There are two possibilities of the "fractional part" even in the one-dimensional case, namely, for $y \in \mathbb{R}$,

$$\begin{aligned} &\{y\}_{\mathrm{r}} = y - \lfloor y\rfloor, \\ &\{y\}_{\mathrm{l}} = 1 + y - \lceil y\rceil = 1 - (-y) + \lfloor -y\rfloor = 1 - \{-y\}_{\mathrm{r}}, \end{aligned} \tag{9.1}$$

where

$$\begin{aligned} \lfloor y\rfloor &= \max\{m \in \mathbb{Z} \mid m \le y\}, \\ \lceil y\rceil &= \min\{m \in \mathbb{Z} \mid m \ge y\}. \end{aligned} \tag{9.2}$$

Note that $\{y\}_{\mathrm{r}}$ is right-continuous while $\{y\}_{\mathrm{l}}$ is left-continuous and that $\{y\}_{\mathrm{r}} = \{y\}_{\mathrm{l}}$ for $y \in \mathbb{R}\setminus\mathbb{Z}$. Although $\{y\}_{\mathrm{r}}$ is usually called the fractional part $\{y\}$ of y and used extensively, we may work with $\{y\}_{\mathrm{l}}$ instead. In multiple cases, there are more possibilities and no standard choice. Hence we need to fix a direction from which the "fractional part" is continuous. To do so, we fix $\phi \in \mathfrak{h}_0$ such that $\langle\phi, \mu^{\mathbf{V}}_\beta\rangle \ne 0$ for all $\mathbf{V} \in \mathscr{V}$ and all $\beta \in \mathbf{V}$, and we define

$$\{\mathbf{y}\}_{\mathbf{V},\beta} = \begin{cases} \{\langle\mathbf{y}, \mu^{\mathbf{V}}_\beta\rangle\} & (\langle\phi, \mu^{\mathbf{V}}_\beta\rangle > 0), \\ 1 - \{-\langle\mathbf{y}, \mu^{\mathbf{V}}_\beta\rangle\} & (\langle\phi, \mu^{\mathbf{V}}_\beta\rangle < 0) \end{cases} \tag{9.3}$$

for $\mathbf{y} \in \mathfrak{h}_0$, where $\{y\} = \{y\}_{\mathrm{r}}$ denotes the fractional part of y in the usual sense. It is clear that $\{\mathbf{y}\}_{\mathbf{V},\beta}$ depends on a specific choice of ϕ. However it can be shown that the generating functions $F(\mathbf{t}, \mathbf{y}; \mathfrak{g})$ defined just below with this symbol, are independent of ϕ if the root system is not of type A_1.

Now we are in position to define the generating functions of multiple analogues of periodic Bernoulli functions. By introducing new variables

$\mathbf{t} = (t_\alpha)_{\alpha\in\Delta_+}$ and using the definitions above, we define

$$F(\mathbf{t},\mathbf{y};\mathfrak{g}) = \sum_{\mathbf{V}\in\mathscr{V}} \Big(\prod_{\gamma\in\Delta_+\setminus\mathbf{V}} \frac{t_\gamma}{t_\gamma - \sum_{\beta\in\mathbf{V}} t_\beta \langle \gamma^\vee, \mu_\beta^{\mathbf{V}} \rangle} \Big) \times \frac{1}{|Q^\vee/L(\mathbf{V}^\vee)|} \sum_{q\in Q^\vee/L(\mathbf{V}^\vee)} \Big(\prod_{\beta\in\mathbf{V}} \frac{t_\beta \exp(t_\beta\{\mathbf{y}+q\}_{\mathbf{V},\beta})}{e^{t_\beta}-1} \Big). \tag{9.4}$$

It seems that $F(\mathbf{t},\mathbf{y};\mathfrak{g})$ has a singularity at the origin with respect to $\mathbf{t}$. However, $F(\mathbf{t},\mathbf{y};\mathfrak{g})$ is indeed holomorphic in the neighborhood of the origin. (In fact, this statement follows from its construction. See [14].) Hence this function is expanded as

$$F(\mathbf{t},\mathbf{y};\mathfrak{g}) = \sum_{\mathbf{k}\in\mathbb{N}_0^n} P(\mathbf{k},\mathbf{y};\mathfrak{g}) \prod_{\alpha\in\Delta_+} \frac{t_\alpha^{k_\alpha}}{k_\alpha!}, \tag{9.5}$$

by which we define the *periodic Bernoulli function* $P(\mathbf{k},\mathbf{y};\mathfrak{g})$ *of type* $\mathfrak{g}$.

Example 7. With the same notation as in Example 1, we put $\mathbf{t} = (t)$. Then we have $\mathscr{V} = \{\mathbf{V}\}$ with

$$\mathbf{V}^\vee = \{\alpha^\vee\}, \qquad \mathbf{V}^* = \{\lambda\}. \tag{9.6}$$

We choose $\phi = \alpha^\vee$, so that $\langle \phi, \lambda\rangle = 1 > 0$ and

$$\{\mathbf{y}\}_{\mathbf{V},\alpha} = \{\langle \mathbf{y}, \lambda\rangle\} = \{y\}. \tag{9.7}$$

Therefore we obtain

$$F(\mathbf{t},\mathbf{y};A_1) = \frac{te^{t\{y\}}}{e^t-1} = \sum_{k=0}^\infty P(k,y;A_1)\frac{t^k}{k!}, \tag{9.8}$$

where $P(k,y;A_1) = B_k(\{y\})$.

Here we observe what will happen if we choose $\phi = -\alpha^\vee$. Then (9.7) is replaced by

$$\{\mathbf{y}\}_{\mathbf{V},\alpha} = 1 - \{-\langle \mathbf{y},\lambda\rangle\} = 1 - \{-y\} \tag{9.9}$$

and the resulting periodic Bernoulli functions are

$$P(k,y;A_1) = B_k(1-\{-y\}) = B_k(\{y\}), \tag{9.10}$$

due to the property $B_k(0) = B_k(1)$ if $k \neq 1$, which coincide with those of (9.8) if $k \neq 1$. Hence the replacement (9.9) only affects the definition of $P(1,y;A_1)$. In the case of the root systems other than A_1, this phenomenon does not happen. Namely, $P(\mathbf{k},\mathbf{y};X_r)$ does not depend on ϕ if $X_r \neq A_1$.

Example 8. We treat the A_2 case. As in Example 2, we put $\mathbf{t} = (t_{\alpha_1}, t_{\alpha_2}, t_{\alpha_1+\alpha_2}) = (t_1, t_2, t_3)$ and set $\mathbf{y} = y_1\alpha_1^\vee + y_2\alpha_2^\vee$. Fix $\phi = \alpha_1^\vee + \varepsilon\alpha_2^\vee$ with a sufficiently small $\varepsilon > 0$. Since $\Delta_+^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \alpha_1^\vee + \alpha_2^\vee\}$, we have $\mathscr{V} = \{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3\}$, where

$$\mathbf{V}_1^\vee = \{\alpha_1^\vee, \alpha_2^\vee\}, \qquad \mathbf{V}_1^* = \{\lambda_1, \lambda_2\}, \tag{9.11a}$$

$$\mathbf{V}_2^\vee = \{\alpha_1^\vee, \alpha_1^\vee + \alpha_2^\vee\}, \qquad \mathbf{V}_2^* = \{\lambda_1 - \lambda_2, \lambda_2\}, \tag{9.11b}$$

$$\mathbf{V}_3^\vee = \{\alpha_2^\vee, \alpha_1^\vee + \alpha_2^\vee\}, \qquad \mathbf{V}_3^* = \{\lambda_2 - \lambda_1, \lambda_1\}. \tag{9.11c}$$

Correspondingly we obtain

$$F(\mathbf{t}, \mathbf{y}; A_2) = \frac{t_3}{t_3 - t_1 - t_2} \frac{t_1 e^{t_1\{y_1\}}}{e^{t_1} - 1} \frac{t_2 e^{t_2\{y_2\}}}{e^{t_2} - 1} \tag{9.12a}$$

$$+ \frac{t_2}{t_2 + t_1 - t_3} \frac{t_1 e^{t_1\{y_1 - y_2\}}}{e^{t_1} - 1} \frac{t_3 e^{t_3\{y_2\}}}{e^{t_3} - 1} \tag{9.12b}$$

$$+ \frac{t_1}{t_1 + t_2 - t_3} \frac{t_2 e^{t_2(1 - \{y_1 - y_2\})}}{e^{t_2} - 1} \frac{t_3 e^{t_3\{y_1\}}}{e^{t_3} - 1}. \tag{9.12c}$$

For example, we have

$$\begin{aligned}
P((2,2,2),(y_1,y_2);A_2) &= \frac{1}{3780} + \frac{1}{90}(\{y_1\} - \{y_1 - y_2\} - \{y_2\}) \\
&+ \frac{1}{90}(-\{y_1\}^2 - 2\{y_1 - y_2\}\{y_1\} + \{y_1 - y_2\}^2 - \{y_2\}^2 + 2\{y_1 - y_2\}\{y_2\}) \\
&+ \frac{1}{18}(-\{y_1\}^3 + 3\{y_1 - y_2\}\{y_1\}^2 + 3\{y_2\}^3 + 3\{y_1 - y_2\}\{y_2\}^2) \\
&+ \frac{1}{18}(\{y_1\}^4 - 2\{y_1 - y_2\}\{y_1\}^3 - 3\{y_1 - y_2\}^2\{y_1\}^2 \\
&\qquad - 5\{y_2\}^4 - 10\{y_1 - y_2\}\{y_2\}^3 - 3\{y_1 - y_2\}^2\{y_2\}^2) \\
&+ \frac{1}{30}(\{y_1\}^5 - 5\{y_1 - y_2\}\{y_1\}^4 + 10\{y_1 - y_2\}^2\{y_1\}^3 \\
&\qquad + 5\{y_2\}^5 + 15\{y_1 - y_2\}\{y_2\}^4 + 10\{y_1 - y_2\}^2\{y_2\}^3) \\
&+ \frac{1}{30}(-\{y_1\}^6 + 4\{y_1 - y_2\}\{y_1\}^5 - 5\{y_1 - y_2\}^2\{y_1\}^4 \\
&\qquad - \{y_2\}^6 - 4\{y_1 - y_2\}\{y_2\}^5 - 5\{y_1 - y_2\}^2\{y_2\}^4).
\end{aligned} \tag{9.13}$$

In particular, by putting $y_1 = y_2 = 0$, we obtain

$$P((2,2,2), \mathbf{0}; A_2) = \frac{1}{3780}, \tag{9.14}$$

which implies (8.15).

From this example, we see that if we can remove the fractional parts symbolically in $P((2,2,2),(y_1,y_2);A_2)$, then it is indeed a polynomial in y_1 and y_2 and is of total degree 6. This fact holds in the case of any root system and is formulated as follows.

Theorem 7 ([12, 14])**.** *The function $P(\mathbf{k}, \mathbf{y}; \mathfrak{g})$ is analytically continued to a polynomial function on the whole space $\mathbb{C} \otimes \mathfrak{h}_0 \simeq \mathfrak{h}$ with its total degree at most $|\mathbf{k}|$.*

We call the polynomials obtained in Theorem 7 *Bernoulli polynomials of type* $\mathfrak{g}$, which are multiple generalizations of classical Bernoulli polynomials.

10. L-functions of root systems

In the previous section, we obtained the explicit form of generating functions of multiple periodic Bernoulli functions of root systems. Here we apply them to the calculation of special values of L-functions of root systems.

For comparison, first we review some results about classical L-functions. For a primitive Dirichlet character χ of conductor f, $g(\chi)$ denotes the Gauss sum defined by

$$g(\chi) = \sum_{m=0}^{f-1} \chi(m) e^{2\pi\sqrt{-1}m/f}. \tag{10.1}$$

The Dirichlet L-function associated with χ is defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}. \tag{10.2}$$

Then it is known that special values of the L-function are given in terms of Bernoulli polynomials by the following formula: For k satisfying $(-1)^{-k}\chi(-1) = 1$,

$$L(k, \chi) g(\overline{\chi}) = \frac{-1}{2} \frac{(2\pi\sqrt{-1})^k}{k!} \sum_{a=1}^{f} \overline{\chi}(a) B_k(a/f). \tag{10.3}$$

This formula can be rewritten in terms of the classical generalized Bernoulli numbers. Let $B_{k,\chi}$ be the k-th classical generalized Bernoulli number given by

$$B_{k,\chi} = f^{k-1} \sum_{a=1}^{f} \chi(a) B_k(a/f) \tag{10.4}$$

(see [31, Proposition 4.1], [8, p.10]), which is also given in terms of the generating function as

$$\sum_{a=1}^{f} \frac{\chi(a)te^{ta}}{e^{ft}-1} = \sum_{k=0}^{\infty} B_{k,\chi}\frac{t^k}{k!}. \tag{10.5}$$

Then we have the formula

$$L(k,\chi) = \frac{(-1)^{k+1}}{2}\frac{(2\pi\sqrt{-1})^k}{k!f^k}g(\chi)B_{k,\overline{\chi}} \tag{10.6}$$

(see [8, p.12]). It is also known that the following parity result holds for generalized Bernoulli numbers:

$$B_{k,\chi} = 0 \tag{10.7}$$

if $(-1)^{-k}\chi(-1) \neq 1$ and χ is non-trivial.

In the following, we will observe how these classical results are generalized to the case of L-functions of root systems.

Let χ_α be a Dirichlet character modulo $f_\alpha \in \mathbb{N}$ for $\alpha \in \Delta$ with $\chi_\alpha = \chi_{-\alpha}$. Set $\boldsymbol{\chi} = (\chi_\alpha)_{\alpha\in\Delta}$. We define an action of W on characters by

$$(w\boldsymbol{\chi})_\alpha = \chi_{w^{-1}\alpha} \tag{10.8}$$

and define the *multiple L-function* by

$$L_r(\mathbf{s},\boldsymbol{\chi};\mathfrak{g}) = \sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\prod_{\alpha\in\Delta_+}\frac{\chi_\alpha(\langle\alpha^\vee, m_1\lambda+\cdots+m_r\lambda_r\rangle)}{\langle\alpha^\vee, m_1\lambda_1+\cdots+m_r\lambda_r\rangle^{s_\alpha}}. \tag{10.9}$$

It is possible to show that $L_r(\mathbf{s},\boldsymbol{\chi};\mathfrak{g})$ can be continued meromorphically to the whole space ([14]). In the case of multiple L-functions of Euler-Zagier type (that is, series of type (1.1) with Dirichlet characters), the meromorphic continuation and the location of possible singularities were already studied by Akiyama and Ishikawa [1].

Now we state a formula for the special value $L_r(\mathbf{k},\boldsymbol{\chi};\mathfrak{g})$ in terms of multiple periodic Bernoulli functions $P(\mathbf{k},\mathbf{y};\mathfrak{g})$. Set

$$c_\alpha(m) = \frac{1}{f_\alpha}\sum_{a_\alpha=1}^{f_\alpha}\chi_\alpha(a_\alpha)e^{-2\pi\sqrt{-1}a_\alpha m/f_\alpha}. \tag{10.10}$$

Then we have

$$\chi_\alpha(m) = \sum_{a_\alpha=1}^{f_\alpha} c_\alpha(a_\alpha)e^{2\pi\sqrt{-1}a_\alpha m/f_\alpha}. \tag{10.11}$$

Note that if χ_α is a primitive character of conductor f_α, we have

$$c_\alpha(m) = \frac{\overline{\chi_\alpha}(m)}{g(\overline{\chi_\alpha})}, \tag{10.12}$$

so that

$$\chi_\alpha(m) g(\overline{\chi_\alpha}) = \sum_{a_\alpha=1}^{f_\alpha} \overline{\chi_\alpha}(a_\alpha) e^{2\pi\sqrt{-1}a_\alpha m/f_\alpha}. \tag{10.13}$$

For $\mathbf{a} = (a_\alpha)_{\alpha\in\Delta_+} \in \mathbb{Z}^n$ and $\mathbf{f} = (f_\alpha)_{\alpha\in\Delta_+} \in \mathbb{N}^n$, let

$$\mathbf{y}(\mathbf{a};\mathbf{f}) = \sum_{\alpha\in\Delta_+} a_\alpha \alpha^\vee / f_\alpha. \tag{10.14}$$

Then we have the following result, which is regarded as a generalization of (10.3).

Theorem 8 ([14])**.** *Let $k_\alpha \in \mathbb{Z}_{\geq 2}$ for $\alpha \in \Delta_+$, and assume*

$$\begin{aligned} k_\alpha = k_\beta, \quad \chi_\alpha = \chi_\beta \qquad &\textit{if } \langle\alpha,\alpha\rangle = \langle\beta,\beta\rangle, \\ (-1)^{-k_\alpha}\chi_\alpha(-1) = 1&. \end{aligned} \tag{10.15}$$

Then we have

$$\begin{aligned} &L_r(\mathbf{k},\boldsymbol{\chi};\mathfrak{g}) \\ &\quad = \frac{(-1)^n}{|W|}\Bigl(\prod_{\alpha\in\Delta_+} \frac{(2\pi\sqrt{-1})^{k_\alpha}}{k_\alpha!}\Bigr) \sum_{\substack{a_\alpha=1\\ \alpha\in\Delta_+}}^{f_\alpha} \Bigl(\prod_{\alpha\in\Delta_+} c_\alpha(a_\alpha)\Bigr) P(\mathbf{k},\mathbf{y}(\mathbf{a};\mathbf{f});\mathfrak{g}), \end{aligned} \tag{10.16}$$

and in particular, if all χ_α's are primitive,

$$\begin{aligned} &L_r(\mathbf{k},\boldsymbol{\chi};\mathfrak{g}) \prod_{\alpha\in\Delta_+} g(\overline{\chi_\alpha}) \\ &\quad = \frac{(-1)^n}{|W|}\Bigl(\prod_{\alpha\in\Delta_+} \frac{(2\pi\sqrt{-1})^{k_\alpha}}{k_\alpha!}\Bigr) \sum_{\substack{a_\alpha=1\\ \alpha\in\Delta_+}}^{f_\alpha} \Bigl(\prod_{\alpha\in\Delta_+} \overline{\chi_\alpha}(a_\alpha)\Bigr) P(\mathbf{k},\mathbf{y}(\mathbf{a};\mathbf{f});\mathfrak{g}). \end{aligned} \tag{10.17}$$

We define the *generalized Bernoulli numbers $B_{\mathbf{k},\boldsymbol{\chi}}(\mathfrak{g})$ of type $\mathfrak{g}$*, by its generating function $G(\mathbf{t},\boldsymbol{\chi};\mathfrak{g})$ as

$$\begin{aligned} G(\mathbf{t},\boldsymbol{\chi};\mathfrak{g}) &= \sum_{\substack{a_\alpha=1\\ \alpha\in\Delta_+}}^{f_\alpha} \Bigl(\prod_{\alpha\in\Delta_+} \chi_\alpha(a_\alpha)/f_\alpha\Bigr) F(\mathbf{f}\,\mathbf{t},\mathbf{y}(\mathbf{a};\mathbf{f});\mathfrak{g}) \\ &= \sum_{\mathbf{k}\in\mathbb{N}_0^n} B_{\mathbf{k},\boldsymbol{\chi}}(\mathfrak{g}) \prod_{\alpha\in\Delta_+} \frac{t_\alpha^{k_\alpha}}{k_\alpha!}, \end{aligned} \tag{10.18}$$

where $\mathbf{f}\,\mathbf{t} = (f_\alpha t_\alpha)_{\alpha\in\Delta_+}$. Then we have a multiple generalization of (10.4).

Theorem 9 ([14]). *We have*

$$B_{\mathbf{k},\boldsymbol{\chi}}(\mathfrak{g}) = \Big(\prod_{\alpha\in\Delta_+} f_\alpha^{k_\alpha-1}\Big) \sum_{\substack{a_\alpha=1\\ \alpha\in\Delta_+}}^{f_\alpha} \Big(\prod_{\alpha\in\Delta_+} \chi_\alpha(a_\alpha)\Big) P(\mathbf{k},\mathbf{y}(\mathbf{a};\mathbf{f});\mathfrak{g}). \tag{10.19}$$

By combining Theorems 8, 9 and the formula

$$\chi(-1)g(\chi)g(\overline{\chi}) = f \tag{10.20}$$

for a primitive character χ of conductor f, we immediately obtain the following theorem corresponding to (10.6).

Theorem 10 ([14]). *Assume* (10.15) *and that all* χ_α*'s are primitive. Then*

$$L_r(\mathbf{k},\boldsymbol{\chi};\mathfrak{g}) = \frac{(-1)^{|\mathbf{k}|+n}}{|W|} \Bigg(\prod_{\alpha\in\Delta_+} \frac{(2\pi\sqrt{-1})^{k_\alpha}}{k_\alpha!\, f_\alpha^{k_\alpha}} g(\chi_\alpha)\Bigg) B_{\mathbf{k},\overline{\boldsymbol{\chi}}}(\mathfrak{g}). \tag{10.21}$$

We give a sufficient condition for $B_{\mathbf{k},\overline{\boldsymbol{\chi}}}(\mathfrak{g}) = 0$ by use of the action of the Weyl group. For $w \in W$, we define

$$(wG)(\mathbf{t},\boldsymbol{\chi};\mathfrak{g}) = G(w^{-1}\mathbf{t}, w^{-1}\boldsymbol{\chi};\mathfrak{g}). \tag{10.22}$$

Theorem 11 ([14]). *Assume that* $\mathfrak{g}$ *is simple. Moreover assume that* $f_\alpha > 1$ *if* $\mathfrak{g}$ *is of type* A_1. *Then for* $w \in W$,

$$(wG)(\mathbf{t},\boldsymbol{\chi};\mathfrak{g}) = \Big(\prod_{\alpha\in\Delta_+\cap w\Delta_-} \chi_\alpha(-1)\Big) G(\mathbf{t},\boldsymbol{\chi};\mathfrak{g}), \tag{10.23}$$

$$B_{w^{-1}\mathbf{k},w^{-1}\boldsymbol{\chi}}(\mathfrak{g}) = \Big(\prod_{\alpha\in\Delta_+\cap w\Delta_-} (-1)^{-k_\alpha}\chi_\alpha(-1)\Big) B_{\mathbf{k},\boldsymbol{\chi}}(\mathfrak{g}). \tag{10.24}$$

As a direct consequence of this theorem, we obtain a multiple analogue of the parity result (10.7).

Theorem 12 ([14]). *Under the same assumptions as in Theorem 11, we have* $B_{\mathbf{k},\boldsymbol{\chi}}(\mathfrak{g}) = 0$ *if there exists an element* $w \in W_{\mathbf{k}} \cap W_{\boldsymbol{\chi}}$ *such that*

$$\prod_{\alpha\in\Delta_+\cap w\Delta_-} (-1)^{-k_\alpha}\chi_\alpha(-1) \neq 1, \tag{10.25}$$

where $W_{\mathbf{k}}$ *and* $W_{\boldsymbol{\chi}}$ *are the stabilizers of* $\mathbf{k}$ *and* $\boldsymbol{\chi}$ *respectively.*

The above theorems show that basic properties of classical L-functions and Bernoulli numbers (polynomials) can be successfully generalized in our framework.

Example 9. For the quadratic character ρ_5 of conductor 5, namely $\rho_5(1) = \rho_5(4) = 1$, $\rho_5(2) = \rho_5(3) = -1$, we have

(10.26)
$$L_2((2,2,2),(\rho_5,\rho_5,\rho_5);A_2) = \frac{(-1)^{6+3}}{6}\left(\frac{(2\pi i)^2}{2!5^2}\sqrt{5}\right)^3\left(-\frac{28}{125}\right) = -\frac{112\sqrt{5}}{1171875}\pi^6,$$

by Theorem 10. As seen in Example 6, we have $\Delta_+ \cap \sigma_1\Delta_- = \{\alpha_1\}$. Furthermore, $\rho_5(-1) = 1$ and $(-1)^{k_{\alpha_1}} = -1$. Hence by Theorem 12, we have

$$B_{(3,2,2),(\rho_5,\rho_5,\rho_5)}(A_2) = 0. \tag{10.27}$$

This can be directly checked by (8.22), Theorems 5 and 9.

As Corollary 1 follows from Theorem 5, we obtain Theorem 8 from an L-analogue of Theorem 5, which we omitted in this article. As generalizations of Theorem 5 and its L-analogue, we can show certain functional relations which include those theorems as special cases. This is another important topic in our theory, but here we have no room for discussing this direction. For the details, see [12, 14, 15].

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