Generalized multiple Dirichlet series and generalized multiple polylogarithms

by

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1 Introduction and statement of results

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the ring of rational integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{R}$ the field of real numbers, and $\mathbb{C}$ the field of complex numbers.

Let $s = \sigma + i\tau$ be a complex variable, and

$$\psi(s) = \sum_{n=0}^{\infty} \frac{a(n)}{(\beta + nw)^s} \tag{1.1}$$

be a function with complex coefficients $a(n)$, where $\beta, w \in \mathbb{R}$ with $0 < \beta \leq w$. We assume the following:

(Assumption I) *There exists a certain $q > 0$ such that $\psi(s)$ is convergent absolutely for $\sigma > q$.*

Throughout this paper we fix $\delta \in \mathbb{R}$ with $\delta > 0$ and let $u \in \mathbb{R}$ with $1 \leq u \leq 1 + \delta$. We let

$$\psi(s; u) = \sum_{n=0}^{\infty} \frac{a(n)u^{-n}}{(\beta + nw)^s}. \tag{1.2}$$

By Assumption I, we can check that if $1 < u \leq 1 + \delta$ then the right-hand side of (1.2) is convergent absolutely for any $s \in \mathbb{C}$, so $\psi(s; u)$ is holomorphic for all $s \in \mathbb{C}$. Corresponding to $\psi(s; u)$, let

$$G_1(t; \psi; u) = \sum_{n=0}^{\infty} a(n)u^{-n}e^{(\beta+nw)t}, \tag{1.3}$$

where t is a complex variable. By Assumption I, we see that the series (1.3) is convergent when $\Re t < 0$. We further assume the following:

(Assumption II) *$\psi(s)$ can be continued analytically to the whole complex plane $\mathbb{C}$, and holomorphic for all $s \in \mathbb{C}$. In any fixed strip $\sigma_1 \leq \sigma \leq \sigma_2$, $\psi(s; u)$ is uniformly convergent to $\psi(s)$ as $u \to 1 + 0$. Furthermore there exists a certain $\theta_0 = \theta_0(\sigma_1, \sigma_2) \in \mathbb{R}$ with $0 \leq \theta_0 < \pi/2$ such that $\psi(s; u) = O\left(e^{\theta_0|\tau|}\right)$ as $|\tau| \to \infty$.*

(Assumption III) *There exists a certain* $\rho = \rho(\psi) > 0$ *such that* $G_1(t;\psi;u)$ *can be continued holomorphically to*

$$\mathcal{D}(\rho) = \left\{ t \in \mathbb{C} \,\middle|\, |t| < \rho \right\} \tag{1.4}$$

for any $u \in [1, 1+\delta]$.

We will give typical examples which satisfy Assumptions I-III in Section 2 (see Example 2.2).

In the present paper, we consider generalized multiple Dirichlet series as follows. Let $(\alpha_0, \alpha_1, \ldots, \alpha_r) \in \mathbb{R}^{r+1}$, $(w_1, \ldots, w_r) \in \mathbb{R}^r$ such that $\alpha_0 = 0$ and $0 < \alpha_k - \alpha_{k-1} \le w_k$ $(1 \le k \le r)$. Let $\mathcal{P}_r = \{\psi_1, \ldots, \psi_r\}$, where

$$\psi_k(s) = \sum_{n=0}^{\infty} \frac{a_k(n)}{(\alpha_k - \alpha_{k-1} + n w_k)^s}. \tag{1.5}$$

We assume that $\psi_k(s)$ and the associated series $\psi_k(s;u)$, $G_1(t;\psi_k;u)$ (defined similarly to (1.2) and (1.3)) satisfy Assumptions I-III $(1 \le k \le r)$. By Assumptions I and III, there exist $\left\{ q_k = q(\psi_k)(> 0) \,\middle|\, 1 \le k \le r \right\}$ and $\left\{ \rho_k = \rho(\psi_k)(> 0) \,\middle|\, 1 \le k \le r \right\}$. We let

$$\eta_r = \min_{1 \le k \le r} \left\{ \frac{\rho_k}{2^{r-1}} \right\}. \tag{1.6}$$

We define the generalized multiple Dirichlet series associated with $\mathcal{P}_r$ by

$$\Psi_r(s_1, \ldots, s_r; u) = \sum_{n_1, \ldots, n_r = 0}^{\infty} \frac{a_1(n_1) \cdots a_r(n_r) u^{-\sum_{\nu=1}^{r} n_\nu}}{\prod_{j=1}^{r} (\alpha_j + \sum_{\nu=1}^{j} n_\nu w_\nu)^{s_j}} \tag{1.7}$$

for $s_1, \ldots, s_r \in \mathbb{C}$ and $u \in [1, 1+\delta]$. The special case $u = 1$ and $a_j(n) = 1$ $(1 \le j \le r)$ has been studied by the first author in [12, 13], which can be regarded as a generalization of both the Euler-Zagier multiple zeta function and the Barnes multiple zeta function. On the other hand, the special case $u = 1$, $\alpha_j = j$ and $w_j = 1$ $(1 \le j \le r)$ has also been studied before; Arakawa-Kaneko [2] when a_js are periodic functions on $\mathbb{Z}$, and Matsumoto-Tanigawa [14] for more general a_js.

First we prove the following result by using the method introduced by Matsumoto-Tanigawa (see [14]; see also [11, 12, 13]). Indeed, this can be regarded as a generalization of Theorem 2 in [14].

Theorem 1.1. *For* $s_1, \ldots, s_r \in \mathbb{C}$ *and* $u \in [1, 1+\delta]$, $\Psi_r(s_1, \ldots, s_r; u)$ *is convergent absolutely for* $s_j = \sigma_j + i\tau_j \in \mathbb{C}$ $(1 \le j \le r)$ *with each* $\sigma_j > q_j$. *Furthermore* $\Psi_r(s_1, \ldots, s_r; u)$ *can be continued analytically to the whole complex space* $\mathbb{C}^r$ *and holomorphic on* $\mathbb{C}^r$, *and satisfies that*

$$\lim_{u \to 1+0} \Psi_r(s_1, \ldots, s_r; u) = \Psi_r(s_1, \ldots, s_r; 1) \tag{1.8}$$

for any $(s_1, \ldots, s_r) \in \mathbb{C}^r$.

Remark 1.2. We can prove the meromorphic continuation of $\Psi_r(s_1, ..., s_r; u)$ even if $\psi_k(s)$ has poles. When $u > 1$, the multiple series (1.7) is convergent absolutely, hence is holomorphic, for any $(s_1, ..., s_r) \in \mathbb{C}^r$. When $u = 1$, if we assume that $\psi_k(s)$ has a pole of order at most one at $s = q_k$ and holomorphic elsewhere (and satisfies $\psi_k(s) = O(e^{\theta_0|\tau|})$) for $1 \le k \le r$, then we can show the following results, which generalize Theorem 1 in [14]:

The function $\Psi_r(s_1, ..., s_r; 1)$ can be continued meromorphically to the whole space $\mathbb{C}^r$, and its possible singularities are located only on the subsets of $\mathbb{C}^r$ each of which is defined by one of the following equations:

$$s_j + \cdots + s_r = q_j + \delta_{j+1}q_{j+1} + \cdots + \delta_r q_r - n$$

$$(1 \le j \le r,\ \delta_k = 0 \text{ or } 1\ (2 \le k \le r),\ n \in \mathbb{N}_0).$$

Moreover, (i) if $j = r \ge 2$ and $q_r \in \mathbb{N}$, then $n \le q_r - 1$, (ii) if $2 \le j \le r-1$, $q_j \in \mathbb{N}$ and $\delta_{j+1} = \cdots = \delta_r = 1$, then $n \le q_r - 1$, (iii) if $j = r = 1$ or if $j = 1$ and $\delta_2 = \cdots = \delta_r = 1$, then $n = 0$.

The proof of this fact can be given by the same method as in the proof of Theorem 1 in [14].

We further consider generalized multiple polylogarithms related to (1.5) as follows. Let $\mathbf{d}_r = (d_1, \ldots, d_r) \in \mathbb{C}^r$ with $\Re d_j > q_j$ for each j. With the above notation, and for $u \in [1, 1+\delta]$, let

$$\begin{aligned} (1.9)\qquad & F_r(t_1, \ldots, t_r; \mathbf{d}_r; \mathcal{P}_r; u) \\ &= \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{a_1(n_1)\cdots a_r(n_r) u^{-\sum_{l=1}^r n_l} \prod_{j=1}^r e^{(\alpha_j + \sum_{\mu=1}^j n_\mu w_\mu)t_j}}{\prod_{j=1}^r (\alpha_j + \sum_{\mu=1}^j n_\mu w_\mu)^{d_j}}. \end{aligned}$$

This multiple series is convergent when $\Re t_j \le 0$ $(1 \le j \le r)$. If we formally let $\psi_k(s) = \zeta(s)$ which is the Riemann zeta function and $d_k \in \mathbb{N}$ $(1 \le k \le r)$ in (1.9), then $F_r(\log x_1, \ldots, \log x_r; \mathbf{d}_r; \mathcal{P}_r; 1)$ is the multiple polylogarithm defined by Goncharov (see [6]; see also [4]). However $\zeta(s)$ does not satisfy Assumption II, so we will not consider the Goncharov multiple polylogarithms in this paper. Instead, we prove the following result.

Theorem 1.3. *For $\mathbf{d}_r \in \mathbb{C}^r$ with each $\Re d_j > q_j$ $(1 \le j \le r)$ and $u \in [1, 1+\delta]$, $F_r(t_1, \ldots, t_r; \mathbf{d}_r; \mathcal{P}_r; u)$ is holomorphic for all $(t_1, \ldots, t_r) \in \mathcal{D}(\eta_r)^r$, and satisfies that for $(t_1, \ldots, t_r) \in \mathcal{D}(\eta_r)^r$,*

$$\begin{aligned} (1.10)\qquad & F_r(t_1, \ldots, t_r; \mathbf{d}_r; \mathcal{P}_r; u) \\ &= \sum_{N_1,\ldots,N_r=0}^{\infty} \Psi_r(d_1 - N_1, \ldots, d_r - N_r; u) \frac{t_1^{N_1} \cdots t_r^{N_r}}{N_1! \cdots N_r!}. \end{aligned}$$

Furthermore, for any $\xi \in \mathbb{R}$ with $0 < \xi < \eta_r$, (1.10) is uniformly convergent with respect to $(t_1, \ldots, t_r, u) \in \overline{\mathcal{D}}(\xi)^r \times [1, 1+\delta]$, where $\overline{\mathcal{D}}(\xi) = \{t \in \mathbb{C} \mid |t| \leq \xi\}$.

The special case $\psi_j(s) = \sum_{n\geq 1}(-1)^n n^{-s}$ $(1 \leq j \leq r)$, $\mathbf{d}_r \in \mathbb{N}^r$ and $t_1 = \cdots = t_{r-1} = 0$ has been studied by the second author. Indeed, $F_r(0, \ldots, 0, t; \mathbf{d}_r; \mathcal{P}_r; u)$ played an important role in giving some evaluation formulas for Euler-Zagier sums (see [15]). In order to prove Theorem 1.3 and Proposition 2.1 (see below), we make use of the technique introduced in [15].

As applications, using Theorem 1.3, we prove certain estimation formulas for $\Psi_r(d_1 - N_1, \ldots, d_r - N_r; 1)$ (see Proposition 5.1 and Example 5.2). We further give certain multiple analogues of both Berndt's and Katsurada's formulas for Dirichlet L-functions proved in [3, 9] (see Example 5.3).

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2 Generalized polylogarithms

First we consider the case of $r = 1$. Let $\psi(s)$ as defined by (1.1) and $F_1(t; d; \psi; u)$ as defined by (1.9). With the notation defined in Section 1, we can prove the following.

Proposition 2.1. *For $d \in \mathbb{C}$ with $\Re d > q$ and $u \in [1, 1+\delta]$, $F_1(t; d; \psi; u)$ is holomorphic for all $t \in \mathcal{D}(\rho)$, and satisfies that for $t \in \mathcal{D}(\rho)$,*

$$F_1(t; d; \psi; u) = \sum_{N=0}^{\infty} \psi(d - N; u) \frac{t^N}{N!}. \tag{2.1}$$

Furthermore, for any $\xi \in \mathbb{R}$ with $0 < \xi < \rho$, (2.1) is uniformly convergent with respect to $(t, u) \in \overline{\mathcal{D}}(\xi) \times [1, 1+\delta]$.

Proof. By Assumption III, we can let

$$G_1(t; \psi; u) = \sum_{n=0}^{\infty} \mathfrak{B}_n(\psi; u) \frac{t^n}{n!} \tag{2.2}$$

for $|t| < \rho$. We use the method of contour integrals (see, for example, [16] Proof of Theorem 4.2). We consider the path Υ which consists of the positive real axis $[\varepsilon, \infty]$ (top side), a circle C_ε around 0 of radius ε, and the positive real axis $[\varepsilon, \infty]$ (bottom side), where $0 < \varepsilon < \rho$. Note that we interpret t^s to mean $\exp(s\log t)$, where the imaginary part of $\log t$ varies from 0 (on the top side of the real axis) to 2π (on the bottom side). Let

(2.3)

$$\begin{aligned} H_1(s;\psi;u) &= \int_{\Upsilon} G_1(-t;\psi;u)t^{s-1}dt \\ &= \left(e^{2\pi i s}-1\right)\int_{\varepsilon}^{\infty} G_1(-t;\psi;u)t^{s-1}dt + \int_{C_\varepsilon} G_1(-t;\psi;u)t^{s-1}dt, \end{aligned}$$

which is, in view of (1.3), holomorphic for all $s\in\mathbb{C}$ if $0<\varepsilon<\rho$. Putting $s=-n$ for $n\in\mathbb{N}_0$ and $\varepsilon=\xi$ with $0<\xi<\rho$ in (2.3) and using (2.2), we have

$$H_1(-n;\psi;u) = \int_{C_\xi} G_1(-t;\psi;u)t^{-n-1}dt = \frac{(2\pi i)\mathfrak{B}_n(\psi;u)(-1)^n}{n!}.$$

From Assumption III, $G_1(t;\psi;u)$ is continuous for all $(t,u)\in\mathcal{D}(\rho)\times[1,1+\delta]$. Hence there exists the value $\mathcal{M}_\xi = \max|G_1(-t;\psi;u)|$ on $\{t\in\mathbb{C}\,|\,|t|=\xi\}\times[1,1+\delta]$. By the above equation, we have

$$\frac{|\mathfrak{B}_n(\psi;u)|}{n!} \le \frac{1}{2\pi}\int_{C_\xi} |G_1(-t;\psi;u)||t|^{-n-1}|dt| \le \frac{\mathcal{M}_\xi}{\xi^n} \tag{2.4}$$

for any $n\in\mathbb{N}_0$ and $u\in[1,1+\delta]$, where ξ is an arbitrary real number with $0<\xi<\rho$.

On the other hand, let $s\in\mathbb{C}$ with $\Re s>\max(1,q)$. We see that the second term on the right-hand side of (2.3) tends to 0 as $\varepsilon\to 0$. Hence we have

$$\begin{aligned} H_1(s;\psi;u) &= \left(e^{2\pi i s}-1\right)\int_0^{\infty} G_1(-t;\psi;u)t^{s-1}dt \\ &= \left(e^{2\pi i s}-1\right)\sum_{n=0}^{\infty} a(n)u^{-n}\int_0^{\infty} t^{s-1}e^{-(\beta+nw)t}dt \\ &= \left(e^{2\pi i s}-1\right)\Gamma(s)\psi(s;u), \end{aligned} \tag{2.5}$$

where the interchange of summation and integration is valid because $\Re s>q$. Hence

$$\psi(s;u) = \frac{1}{(e^{2\pi i s}-1)\,\Gamma(s)}H_1(s;\psi;u) = \frac{\Gamma(1-s)}{2\pi i\, e^{\pi i s}}H_1(s;\psi;u), \tag{2.6}$$

because

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin\pi s} = \frac{2\pi i}{e^{\pi i s}-e^{-\pi i s}}.$$

The relation (2.6) is valid for all $s\in\mathbb{C}$ by analytic continuation.

Next, for $d \in \mathbb{C}$ with $\Re d > q$ and $N \in \mathbb{N}_0$, we put $s = d - N$ in (2.3). Then we have

$$H_1(d-N;\psi;u) = \left(e^{2\pi i d} - 1\right)\int_{\varepsilon}^{\infty} G_1(-t;\psi;u)t^{d-N-1}dt + \int_{C_\varepsilon} G_1(-t;\psi;u)t^{d-N-1}dt. \tag{2.7}$$

For simplicity, we denote by I_1 and I_2 the first and second terms on the right-hand side of (2.7), respectively. Note that if $N \geq \Re d + 1$ then

$$\left|\int_{\varepsilon}^{\infty} e^{-(\beta+nw)t} t^{d-N-1}dt\right| \leq \frac{e^{-(\beta+nw)\varepsilon}\varepsilon^{\Re d-N-1}}{\beta+nw}. \tag{2.8}$$

Hence we have

$$|I_1| \leq \varepsilon^{\Re d-N-1}|e^{2\pi i d}-1|\sum_{n=0}^{\infty}\frac{|a(n)|e^{-(\beta+nw)\varepsilon}}{\beta+nw}.$$

On the other hand, by using the fact that

$$\int_{C_\varepsilon} t^p dt = \begin{cases} 2\pi i & (p=-1) \\ \varepsilon^{p+1}\left(\dfrac{e^{2\pi i p}-1}{p+1}\right) & (p \neq -1) \end{cases} \tag{2.9}$$

for $p \in \mathbb{C}$ and by (2.2), we have

$$I_2 = \begin{cases} (2\pi i)\mathfrak{B}_{N-d}(\psi;u)\dfrac{(-1)^{N-d}}{(N-d)!} & (N-d \in \mathbb{N}_0), \\ \varepsilon^{d-N}\left(e^{2\pi i d}-1\right)\displaystyle\sum_{n=0}^{\infty}\frac{\mathfrak{B}_n(\psi;u)(-1)^n\varepsilon^n}{(n+d-N)n!} & \text{(otherwise)}. \end{cases} \tag{2.10}$$

Note that the above infinite series in the second case is convergent because of the assumption $\varepsilon < \rho$ and (2.4). Hence we have

$$|I_2| \leq \begin{cases} 2\pi\dfrac{|\mathfrak{B}_{N-d}(\psi;u)|}{(N-d)!} & (N-d \in \mathbb{N}_0), \\ \varepsilon^{\Re d-N}\left|e^{2\pi i d}-1\right|\left|\displaystyle\sum_{n=0}^{\infty}\mathfrak{B}_n(\psi;u)\frac{(-1)^n\varepsilon^n}{(n+d-N)n!}\right| & \text{(otherwise)}. \end{cases} \tag{2.11}$$

From (2.4) with $\xi = \varepsilon$, the first case of (2.11) yields that

$$|I_2| \leq (2\pi)\mathcal{M}_\varepsilon \varepsilon^{d-N}.$$

In the second case of (2.11), we let $\gamma_d = \min|d-m|$ for all $m \in \mathbb{Z}$. Using (2.4) with ξ such that $0 < \varepsilon < \xi < \rho$, we see that the second case of (2.11) yields that

$$|I_2| \leq \varepsilon^{\Re d-N}\left|e^{2\pi i d}-1\right|\frac{\mathcal{M}_\xi}{\gamma_d(1-\varepsilon/\xi)}.$$

Hence it follows from (2.6)-(2.11) that there exists a constant $M > 0$ which depends on ε, d and ψ but is independent of N and u such that

$$\left|\frac{\psi(d-N;u)}{\Gamma(1+N-d)}\right| = \frac{1}{2\pi|e^{\pi id}|}\left|H_1(d-N;\psi;u)\right| \leq M\varepsilon^{-N} \tag{2.12}$$

for $N \in \mathbb{N}_0$ with $N \geq \Re d + 1$. Note that we can take an arbitrary ε such that $0 < \varepsilon < \rho$. Since $|s| \leq |\Re s| + |\Im s|$ for $s \in \mathbb{C}$, we have

$$\begin{aligned}|\Gamma(1+N-d)| &= |(N-d)(N-d-1)\cdots([\Re d]+1-d)\Gamma([\Re d]+1-d)| \\ &\leq (N-[\Re d]+[|\Im d|]+1)!\ |\Gamma([\Re d]+1-d)|\end{aligned}$$

for $N \in \mathbb{N}_0$ with $N \geq \Re d + 1$. Hence we have

$$\begin{aligned}\frac{|\psi(d-N;u)|}{N!} &\leq \frac{(N-[\Re d]+[|\Im d|]+1)!\ |\Gamma([\Re d]+1-d)|}{N!}\left|\frac{\psi(d-N;u)}{\Gamma(1+N-d)}\right| \\ &\leq \frac{(N-[\Re d]+[|\Im d|]+1)!\ |\Gamma([\Re d]+1-d)|}{N!}M\varepsilon^{-N}.\end{aligned} \tag{2.13}$$

Suppose $u \in (1, 1+\delta]$ and $t = i\theta$ with $\theta \in (-\rho, \rho) \subset \mathbb{R}$. Then there exists an $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon < \rho$ and $|\theta| < \varepsilon$. From the definition (1.9) we have

$$\begin{aligned}F_1(i\theta;d;\psi;u) &= \sum_{n=0}^{\infty}\frac{a(n)u^{-n}}{(\beta+nw)^d}\sum_{N=0}^{\infty}\frac{(\beta+nw)^N(i\theta)^N}{N!} \\ &= \sum_{N=0}^{\infty}\psi(d-N;u)\frac{(i\theta)^N}{N!}.\end{aligned} \tag{2.14}$$

From (2.13) we can see that each side of (2.14) is uniformly convergent with respect to $u \in [1, 1+\delta]$ because $|\theta| < \varepsilon$. Hence we can let $u \to 1$ in each side of (2.14), namely (2.14) holds for $u = 1$ when $\theta \in (-\rho, \rho)$. We can define

$$F_1(t;d;\psi;u) = \sum_{N=0}^{\infty}\psi(d-N;u)\frac{t^N}{N!}$$

for any $u \in [1, 1+\delta]$ and $t \in \mathbb{C}$ with $|t| < \rho$. From (2.13), this is uniformly convergent with respect to $(t, u) \in \overline{\mathcal{D}}(\xi) \times [1, 1+\delta]$ when $0 < \xi < \rho$. Thus we have the assertion. □

Example 2.2. Let $f : (\mathbb{Z}/m\mathbb{Z}) \to \mathbb{C}$ such that $\sum_{a=1}^{m} f(a) = 0$. It can be regarded as a periodic function defined on $\mathbb{Z}$. For example, any non-trivial primitive Dirichlet character and any non-trivial additive character defined mod m satisfy this condition. We define

$$L(s;f) = \sum_{n=1}^{\infty}\frac{f(n)}{n^s} \tag{2.15}$$

and

$$G_1(t;L;u) = \sum_{n=1}^{\infty} f(n)u^{-n}e^{nt} = \sum_{a=1}^{m} \frac{f(a)u^{-a}e^{at}}{1-u^{-m}e^{mt}}$$

for $u \in [1, 1+\delta]$. Then $L(s;f)$ and $G_1(t;L;u)$ satisfy Assumptions I-III. Note that $\rho = 2\pi/m$ and $q = 1$ in this case. For $d \in \mathbb{C}$ with $\Re d > 1$, let

$$F_1(t;d;L) = \sum_{n=1}^{\infty} \frac{f(n)e^{nt}}{n^d}.$$

It follows from Proposition 2.1 that $F_1(t;d;L)$ is holomorphic on $\mathcal{D}(2\pi/m)$ and satisfies that

$$F_1(t;d;L) = \sum_{N=0}^{\infty} L(d-N;f)\frac{t^N}{N!}. \tag{2.16}$$

In particular when f is a primitive Dirichlet character χ of conductor m, we know that $L(-2j-1,\chi) = 0$ if $\chi(-1) = -1$ and $L(-2j,\chi) = 0$ if $\chi(-1) = 1$ for $j \in \mathbb{N}_0$ (see, for example, [16] Chap. 4). Hence, applying (2.16) with $d = 2k$ and $d = 2k+1$ for $k \in \mathbb{N}$ and using $\cos x = \left(e^{ix} + e^{-ix}\right)/2$, we obtain,

$$\sum_{n=1}^{\infty} \frac{\chi(n)\cos(n\theta)}{n^{2k}} = \sum_{j=0}^{k-1} L(2k-2j,\chi)\frac{(i\theta)^{2j}}{(2j)!} \quad (\chi(-1) = 1),$$
$$\sum_{n=1}^{\infty} \frac{\chi(n)\cos(n\theta)}{n^{2k+1}} = \sum_{j=0}^{k} L(2k+1-2j,\chi)\frac{(i\theta)^{2j}}{(2j)!} \quad (\chi(-1) = -1)$$

for $\theta \in (-2\pi/m, 2\pi/m)$. These are typical examples of Berndt's result (see [3] Theorem 4.2; see also [5] (1.2.12)). Similarly, it follows from (2.16) that

$$\sum_{n=1}^{\infty} \frac{\chi(n)\cos(n\theta)}{n^{2k+1}} = \sum_{j=0}^{\infty} L(2k+1-2j,\chi)\frac{(i\theta)^{2j}}{(2j)!} \quad (\chi(-1) = 1),$$
$$\sum_{n=1}^{\infty} \frac{\chi(n)\cos(n\theta)}{n^{2k}} = \sum_{j=0}^{\infty} L(2k-2j,\chi)\frac{(i\theta)^{2j}}{(2j)!} \quad (\chi(-1) = -1)$$

for $k \in \mathbb{N}$ and $\theta \in (-2\pi/m, 2\pi/m)$. Using the functional equations for $L(s,\chi)$, we can confirm that these equations coincide with Katsurada's formulas for $L(s,\chi)$ (see [9] Theorem 3).

3 Proof of Theorem 1.1

Using the method introduced in [14] Section 2 (see also [11, 12, 13]), we give the proof of Theorem 1.1 by the induction on r. The case of $r = 1$

can be directly obtained from Assumptions I and II. Hence we assume that Theorem 1.1 holds for $r-1$, and aim to prove the case of $r(\geq 2)$.

As mentioned in Section 1, let

$$\Psi_r(s_1,\ldots,s_r;u)=\Psi_r(s_1,\ldots,s_r;\psi_1,\ldots,\psi_r;u)$$

be the function defined by (1.7). Since each $\psi_k(s)$ defined by (1.5) converges absolutely for $\Re s > q_k$ $(1 \leq k \leq r)$, we can easily check that $\Psi_r(s_1,\ldots,s_r;u)$ converges absolutely if $\sigma_k = \Re s_k > q_k$ $(1 \leq k \leq r)$.

First we assume each $\sigma_k > q_k$ $(1 \leq k \leq r)$. Recall the Mellin-Barnes formula

$$\Gamma(s)(1+\lambda)^{-s}=\frac{1}{2\pi i}\int_{(c)}\Gamma(s+z)\Gamma(-z)\lambda^z dz, \tag{3.1}$$

where $\Re s>0$, $|\arg\lambda|<\pi$, $\lambda\neq 0$, $-\Re s<c<0$, and the path of integration is the vertical line $\Re z = c$. By the above assumption, we may assume $-\sigma_r < c < -q_r$. Put $s = s_r$ and

$$\lambda=\frac{\alpha_r-\alpha_{r-1}+n_r w_r}{\alpha_{r-1}+n_1w_1+\cdots+n_{r-1}w_{r-1}}$$

in (3.1). Then multiply the both sides by

$$\frac{a_1(n_1)\cdots a_r(n_r)u^{-\sum_{\nu=1}^r n_\nu}}{\prod_{j=1}^{r-2}(\alpha_j+\sum_{\nu=1}^j n_\nu w_\nu)^{s_j}(\alpha_{r-1}+\sum_{\nu=1}^{r-1} n_\nu w_\nu)^{s_{r-1}+s_r}}$$

and sum up with respect to $n_1,\ldots,n_r$. Then we have

$$\begin{aligned}\Psi_r(s_1,\ldots,s_r;u)=\frac{1}{2\pi i}\int_{(c)}&\frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)}\\&\times\Psi_{r-1}(s_1,\ldots,s_{r-2},s_{r-1}+s_r+z;u)\psi_r(-z;u)dz.\end{aligned}\tag{3.2}$$

Let $M\in\mathbb{N}$ and $\varepsilon\in\mathbb{R}$ a small positive number. We shall shift the path to $\Re z = M-\varepsilon$. We see that

$$\Psi_{r-1}(s_1,\ldots,s_{r-2},s_{r-1}+s_r+z;u)=O(1)$$

in the region $c\leq \Re z\leq M-\varepsilon$ because $\sigma_k>q_k$ $(1\leq k\leq r-2)$, $-\sigma_r<c$ and

$$\sigma_{r-1}+\sigma_r+\Re z\geq\sigma_{r-1}+\sigma_r+c>\sigma_{r-1}.$$

From the well-known Stirling formula for $\Gamma(s)$, we have

$$|\Gamma(s)|=e^{-\frac{\pi}{2}|\tau|}(|\tau|+1)^{\sigma-\frac{1}{2}}\left(1+O\left(\frac{1}{|\tau|+1}\right)\right)\tag{3.3}$$

as $|\tau| \to \infty$, where $s = \sigma + i\tau$. Hence, by Assumption II, we see that the integrand on the right-hand side of (3.2) tends to zero as $|\Im z| \to \infty$, namely this shifting is possible. By the assumption of induction, we see that Ψ_{r-1} is holomorphic on $\mathbb{C}^{r-1}$ and ψ_r holomorphic on $\mathbb{C}$. Therefore we only have to count the residues of the poles of $\Gamma(-z)$ at $z = 0, 1, \ldots, M-1$. Since the residue of the pole of $\Gamma(s_r + z)\Gamma(-z)/\Gamma(s_r)$ at $z = k$ equals to $-\binom{-s_r}{k}$, we obtain

$$
\begin{aligned}
(3.4) \qquad & \Psi_r(s_1, \ldots, s_r; u) \\
&= \sum_{k=0}^{M-1} \binom{-s_r}{k} \Psi_{r-1}(s_1, \ldots, s_{r-2}, s_{r-1} + s_r + k; u)\psi_r(-k; u) \\
&+ \frac{1}{2\pi i} \int_{(M-\varepsilon)} \frac{\Gamma(s_r + z)\Gamma(-z)}{\Gamma(s_r)} \\
&\times \Psi_{r-1}(s_1, \ldots, s_{r-2}, s_{r-1} + s_r + z; u)\psi_r(-z; u) dz.
\end{aligned}
$$

The first term on the right-hand side is holomorphic on the whole $\mathbb{C}^r$ space by the assumption of induction. On the other hand, $\Gamma(s_r + z)$ has no pole on the path $(M - \varepsilon)$, when $\Re(-s_r) = -\sigma_r < M - \varepsilon$, namely $\sigma_r > -M + \varepsilon$. Using (3.3) and Assumption II, we see that the second term on the right-hand side of (3.4) is convergent absolutely, so is holomorphic on the region

$$
\left\{ (s_1, \ldots, s_r) \in \mathbb{C}^r \,\middle|\, \sigma_1 > q_1, \ldots, \sigma_{r-1} > q_{r-1}, \sigma_r > -M + \varepsilon \right\},
$$

where M is arbitrary.

Next we fix $s_r \in \mathbb{C}$ with $\sigma_r > -M + \varepsilon$, and consider the continuation with respect to s_k for $1 \le k \le r-1$. Since Ψ_{r-1} is holomorphic on $\mathbb{C}^{r-1}$, the integrand on the right-hand side of (3.4) is holomorphic for all $(s_1, \ldots, s_{r-1}) \in \mathbb{C}^{r-1}$. So, if we prove that the second term on the right-hand side of (3.4) converges absolutely for any $(s_1, \ldots, s_{r-1}) \in \mathbb{C}^{r-1}$ and $s_r \in \mathbb{C}$ with $\sigma_r > -M + \varepsilon$, then $\Psi_r(s_1, \ldots, s_r; u)$ is holomorphic on the whole $\mathbb{C}^r$ space because M is arbitrary. In order to prove this result, we need the following lemma.

Lemma 3.1. *For $r \in \mathbb{N}$ with $r \ge 2$, there exists a polynomial $P_r(X) \in \mathbb{R}[X]$ such that*

$$
(3.5) \qquad \Psi_r(s_1, \cdots, s_r; u) = O\left(P_r(|\tau_r|) e^{\theta_0 |\tau_r|}\right) \quad (|\tau_r| \to \infty)
$$

for any $(s_1, \cdots, s_{r-1}) \in \mathbb{C}^{r-1}$ and $u \in [1, 1+d]$, where the constant implied by the O-symbol depends on $\tau_1, \ldots, \tau_{r-1}$.

Proof. We denote (3.5) by

$$
\Psi_r(s_1, \cdots, s_r; u) \ll P_r(|\tau_r|) e^{\theta_0 |\tau_r|}.
$$

We prove this lemma by the induction on $r(\geq 2)$. First we consider the case of $r = 2$. It follows from Assumption II and (3.4) that

(3.6)
$$
\begin{aligned}
|\Psi_2(s_1, s_2; u)| \leq & \sum_{k=0}^{M-1} \left| \binom{-s_2}{k} \right| |\Psi_1(s_1 + s_2 + k; u)\psi_2(-k; u)| \\
& + \frac{1}{2\pi} \left| \int_{(M-\varepsilon)} \frac{\Gamma(s_r + z)\Gamma(-z)}{\Gamma(s_r)} \Psi_1(s_1 + s_2 + z; u)\psi_2(-z; u) dz \right| \\
\ll & \sum_{k=0}^{M-1} \left| \binom{-s_2}{k} \right| e^{\theta_0|\tau_2|} \\
& + \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\Gamma(s_r + z)\Gamma(-z)}{\Gamma(s_r)} \right| e^{\theta_0|\tau_2+y|} e^{\theta_0|y|} dy,
\end{aligned}
$$

where $z = x + iy$. For simplicity, we denote the last term on the right-hand side of (3.6) by I. Using (3.3), we have

$$
\begin{aligned}
(3.7)\quad I \ll & e^{\frac{\pi}{2}|\tau_2|}(|\tau_2| + 1)^{-\sigma_2+\frac{1}{2}} \\
& \times \int_{-\infty}^{\infty} e^{(\theta_0-\frac{\pi}{2})|\tau_2+y|} e^{(\theta_0-\frac{\pi}{2})|y|} (|\tau_2 + y| + 1)^{\sigma_2+x-\frac{1}{2}} (|y| + 1)^{-x-\frac{1}{2}} dy.
\end{aligned}
$$

Now we apply Lemma 4 in [12] with $A = B = \theta_0 - \frac{1}{2}\pi$, $p = \sigma_2 + x - \frac{1}{2}$ and $q = -x - \frac{1}{2}$. Then it follows from (3.7) that

$$
\begin{aligned}
(3.8)\qquad I \ll & e^{\frac{\pi}{2}|\tau_2|}(|\tau_2| + 1)^{-\sigma_2+\frac{1}{2}} \\
& \times \left[\left\{ 1 + (|\tau_2| + 1)^{\sigma_2+x-\frac{1}{2}} \right\} (|\tau_2| + 1)^{-x+\frac{1}{2}} e^{(\theta_0-\frac{\pi}{2})|\tau_2|} \right. \\
& \left. + \left\{ 1 + (|\tau_2| + 1)^{\sigma_2+x-\frac{1}{2}} \right\} e^{(\theta_0-\frac{\pi}{2})|\tau_2|} \right].
\end{aligned}
$$

Combining (3.6) and (3.8), we see that there exists $P_2(X) \in \mathbb{R}[X]$ such that

$$
\Psi_2(s_1, s_2; u) \ll P_2(|\tau_2|)e^{\theta_0|\tau_2|} \quad (|\tau_2| \to \infty).
$$

Thus we have the assertion for $r = 2$.

Assume that the assertion for $r - 1$ holds. Substituting the asserted bounds into (3.4) and using Assumption II, we have

$$
\begin{aligned}
\Psi_r(s_1, \ldots, s_r; u) \ll & \sum_{k=0}^{M-1} \left| \binom{-\sigma_r + i\tau_r}{k} \right| P_{r-1}(|\tau_{r-1} + \tau_r|) e^{\theta_0|\tau_{r-1}+\tau_r|} \\
& + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left| \frac{\Gamma(s_r + z)\Gamma(-z)}{\Gamma(s_r)} \right| \\
& \times P_{r-1}(|\tau_{r-1} + \tau_r + y|) e^{\theta_0|\tau_{r-1}+\tau_r+y|} e^{\theta_0|y|} dy.
\end{aligned}
$$

By the same method as mentioned above, we can see that there exists $P_r(X) \in \mathbb{R}[X]$ such that

$$\Psi_r(s_1, \cdots, s_r; u) \ll P_r(|\tau_r|)e^{\theta_0|\tau_r|}.$$

By induction, we obtain the proof of Lemma 3.1. □

Now we can complete the proof of Theorem 1.1 as follows. If we fix any $(s_1, \ldots, s_r) \in \mathbb{C}^r$, then it follows from Lemma 3.1 that

$$\Psi_{r-1}(s_1, \ldots, s_{r-2}, s_{r-1} + s_r + z; u) \ll P_{r-1}(|\tau_{r-1} + \tau_r + y|)e^{\theta_0|\tau_{r-1}+\tau_r+y|}$$

as $|y| \to \infty$, where $z = x + iy$. Since s_{r-1} is fixed, this can be written as

$$\Psi_{r-1}(s_1, \ldots, s_{r-2}, s_{r-1}+s_r+z; u) \ll \widetilde{P}_{r-1}(|\tau_r+y|)e^{\theta_0|\tau_r+y|} \quad (|y| \to \infty), \tag{3.9}$$

where $\widetilde{P}_{r-1}(X) \in \mathbb{R}[X]$. We denote the second term on the right-hand side of (3.4) by $\widetilde{I}$. Then, by using (3.3), and by (3.8) and Assumption II, we have

$$\begin{aligned}\widetilde{I} &\ll \int_{-\infty}^{\infty} \widetilde{\widetilde{P}}_{r-1}(y)e^{-\frac{\pi}{2}|y|-\frac{\pi}{2}|\tau_r+y|}e^{\theta_0|\tau_r+y|}e^{\theta_0|y|}dy \\ &= \int_{-\infty}^{\infty} \widetilde{\widetilde{P}}_{r-1}(y)e^{(\theta_0-\frac{\pi}{2})(|\tau_r+y|+|y|)}dy\end{aligned}$$

for some $\widetilde{\widetilde{P}}_{r-1}(X) \in \mathbb{R}[X]$. Since $0 \le \theta_0 < \pi/2$, $\widetilde{I}$ converges absolutely for any $(s_1, \ldots, s_r) \in \mathbb{C}^r$. By (3.4), we see that $\Psi_r(s_1, \ldots, s_r; u)$ is holomorphic on $\mathbb{C}^r$.

Lastly we prove (1.8). More strictly we prove that (1.8) is uniformly convergent with respect to s_j $(1 \le j \le r)$ in any fixed strip $\sigma_{1j} \le \Re s_j \le \sigma_{2j}$ as $u \to 1+0$. The case of $r = 1$ follows from Assumption II. Hence we assume that the case of $r-1$ holds and prove the case of r (≥ 2). Let $u \to 1+0$ in (3.4). From the assumption of induction, the integrand in the second term on the right-hand side of (3.4) is uniformly convergent with respect to z in any fixed strip $\sigma_1 \le \Re z$ $(= M-\varepsilon)$ $\le \sigma_2$ as $u \to 1+0$. Exchanging $\lim_{u\to 1+0}$ and the integral, and using the assumption of induction, we see that the right-hand side of (3.4) tends to

$$\begin{aligned}&\sum_{k=0}^{M-1}\binom{-s_r}{k}\Psi_{r-1}(s_1, \ldots, s_{r-2}, s_{r-1} + s_r + k; 1)\psi_r(-k; 1) \\ &\quad + \frac{1}{2\pi i}\int_{(M-\varepsilon)} \frac{\Gamma(s_r + z)\Gamma(-z)}{\Gamma(s_r)} \\ &\quad \times \Psi_{r-1}(s_1, \ldots, s_{r-2}, s_{r-1} + s_r + z; 1)\psi_r(-z; 1)dz\end{aligned} \tag{3.10}$$

as $u \to 1+0$. It is clear that this convergence is uniformly with respect to s_j in any fixed strip $\sigma_{1j} \le \Re s_j \le \sigma_{2j}$ $(1 \le j \le r)$. From (3.4), we see that (3.10) coincides with $\Psi_r(s_1,\ldots,s_r;1)$. Hence the assertion in case of r holds. By induction, we have the assertion that (1.8) holds. Thus we obtain the proof of Theorem 1.1.

Remark 3.2. For any $N \in \mathbb{N}_0$, let $M = N+1$ and $s_r \to -N$ in (3.4). Then the second term on the right-hand side of (3.4) tends to 0 because $\Gamma(s_r)$ has a pole at $s_r = -N$. Hence we obtain

$$\begin{aligned}(3.11)\qquad &\Psi_r(s_1,\ldots,s_{r-1},-N;u)\\ &= \sum_{\nu=0}^{N}\binom{N}{\nu}\Psi_{r-1}(s_1,\ldots,s_{r-2},s_{r-1}+\nu-N;u)\psi_r(-\nu;u)\end{aligned}$$

for $u \in [1,1+\delta]$ and $(s_1,\ldots,s_{r-1}) \in \mathbb{C}^{r-1}$. In particular, let $\psi_j(s) = L(s;f_j)$ $(1 \le j \le r)$ and $u=1$, where each f_j is defined mod m_j and satisfies a certain condition (see Example 2.2). Then we can check that Assumptions I-III hold. In this case, $\Psi_r(s_1,\ldots,s_r;1)$ coincides with the multiple L-function

$$L_r(s_1,\ldots,s_r;f_1,\ldots,f_r) = \sum_{n_1,\ldots,n_r=1}^{\infty}\frac{f_1(n_1)\cdots f_r(n_r)}{n_1^{s_1}(n_1+n_2)^{s_2}\cdots(n_1+\cdots+n_r)^{s_r}},$$

which has been studied in [2]. Hence (3.11) gives that

(3.12)

$$\begin{aligned}&L_r(s_1,\ldots,s_{r-1},-N;f_1,\ldots,f_r)\\ &= \sum_{\nu=0}^{N}\binom{N}{\nu}L_{r-1}(s_1,\ldots,s_{r-2},s_{r-1}+\nu-N;f_1,\ldots,f_{r-1})L_1(-\nu;f_r)\end{aligned}$$

for $(s_1,\ldots,s_{r-1}) \in \mathbb{C}^{r-1}$. This result was proved by Kamano (see [8]) by using the method introduced in [1]. This case can be also derived directly from the relation (2.3) in [14].

4 Proof of Theorem 1.3

In this section, we give the proof of Theorem 1.3 by the induction on r.

The case of $r=1$ is just what we argued in Proposition 2.1. Hence we assume that the assertion of Theorem 1.3 holds for $r-1$ and prove the case of $r(\ge 2)$.

Let $\mathcal{P}_r = \{\psi_1, \ldots, \psi_r\}$ which satisfy Assumptions I-III. Then we can take $\{q_k\}_{1\leq k\leq r}$ and $\{\rho_k\}_{1\leq k\leq r}$, and define η_{r-1} and η_r by (1.6). Let

$$
\begin{aligned}
(4.1) \qquad & G_r(t_1, \ldots, t_r; \mathbf{d}_{r-1}; \mathcal{P}_r; u) \\
&= F_{r-1}(t_1, \ldots, t_{r-2}, t_{r-1} + t_r; \mathbf{d}_{r-1}; \mathcal{P}_{r-1}; u) G_1(t_r; \psi_r; u) \\
&= \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{a_1(n_1)\cdots a_r(n_r) u^{-\sum_{l=1}^{r} n_l} \prod_{j=1}^{r} e^{(\alpha_j + \sum_{\mu=1}^{j} n_\mu w_\mu) t_j}}{\prod_{j=1}^{r-1} (\alpha_j + \sum_{\mu=1}^{j} n_\mu w_\mu)^{d_j}},
\end{aligned}
$$

which is convergent when $\Re t_j < 0$ $(1 \leq j \leq r)$. By the assumption of induction, we see that $F_{r-1}(t_1, \ldots, t_{r-1} + t_r; \mathbf{d}_{r-1}; \mathcal{P}_{r-1}; u)$ is holomorphic for $(t_1, \ldots, t_r) \in \mathcal{D}(\eta_{r-1})^{r-2} \times \mathcal{D}(\eta_{r-1}/2)^2$, and $G_1(t_r; \psi_r; u)$ is holomorphic for $t_r \in \mathcal{D}(\rho_r)$. Since we have $\eta_r \leq \min(\eta_{r-1}/2, \rho_r)$, we can see that $G_r(t_1, \ldots, t_r; \mathbf{d}_{r-1}; \mathcal{P}_r; u)$ is holomorphic for $(t_1, \ldots, t_r) \in \mathcal{D}(\eta_r)^r$. Therefore, if we fix $t_r \in \mathcal{D}(\eta_r)$ then the function of real $(r-1)$ variables $G_r(i\theta_1, \ldots, i\theta_{r-1}, t_r; \mathbf{d}_{r-1}; \mathcal{P}_r; u)$ is real-analytic for $(\theta_1, \ldots, \theta_{r-1}) \in (-\eta_r, \eta_r)^{r-1} \subset \mathbb{R}^{r-1}$ (see, for example, [10] Corollary 2.3.7). Similarly, if we fix $(\theta_1, \ldots, \theta_{r-1}) \in (-\eta_r, \eta_r)^{r-1}$, then we see that $G_r(\{i\theta_k\}, t_r; \mathbf{d}_{r-1}; \mathcal{P}_r; u)$ is holomorphic for $t_r \in \mathcal{D}(\eta_r)$. Hence we define $\{\mathfrak{B}_n(\{i\theta_k\}; \mathbf{d}_{r-1}; \mathcal{P}_r; u)\}_{n\geq 0}$ by

$$
(4.2) \qquad G_r(i\theta_1, \ldots, i\theta_{r-1}, t_r; \mathbf{d}_{r-1}; \mathcal{P}_r; u) = \sum_{n=0}^{\infty} \mathfrak{B}_n(\{i\theta_k\}; \mathbf{d}_{r-1}; \mathcal{P}_r; u) \frac{t_r^n}{n!}.
$$

As well as in the proof of Proposition 2.1, we let

$$
\begin{aligned}
(4.3) \qquad & H_r(s; i\theta_1, \ldots, i\theta_{r-1}; \mathbf{d}_{r-1}; \mathcal{P}_r; u) \\
&= \int_{\Upsilon} G_r(\{i\theta_k\}, -t; \mathbf{d}_{r-1}; \mathcal{P}_r; u) t^{s-1} dt \\
&= \left(e^{2\pi i s} - 1\right) \int_{\varepsilon}^{\infty} G_r(\{i\theta_k\}, -t; \mathbf{d}_{r-1}; \mathcal{P}_r; u) t^{s-1} dt \\
&\quad + \int_{C_\varepsilon} G_r(\{i\theta_k\}, -t; \mathbf{d}_{r-1}; \mathcal{P}_r; u) t^{s-1} dt,
\end{aligned}
$$

which is holomorphic for all $s \in \mathbb{C}$ if we fix $(\theta_1, \ldots, \theta_{r-1}) \in (-\eta_r, \eta_r)^{r-1}$ and $0 < \varepsilon < \eta_r$.

Putting $s = -n$ for $n \in \mathbb{N}_0$ and $\varepsilon = \xi$ with $0 < \xi < \eta_r$ in (4.3), and using (4.2), we have

$$
\begin{aligned}
H_r(-n; i\theta_1, \ldots, i\theta_{r-1}; \mathbf{d}_{r-1}; \mathcal{P}_r; u) &= \int_{C_\xi} G_r(\{i\theta_k\}, -t; \mathbf{d}_{r-1}; \mathcal{P}_r; u) t^{-n-1} dt \\
&= \frac{(2\pi i) \mathfrak{B}_n(\{i\theta_k\}; \mathbf{d}_{r-1}; \mathcal{P}_r; u) (-1)^n}{n!}.
\end{aligned}
$$

By the assumption of induction and (4.1), we see that the Taylor expansion series of $G_r(\{i\theta_k\}, -t; \mathbf{d}_{r-1}; \mathcal{P}_r; u)$ around $t = 0$ is uniformly convergent with respect to $(\theta_1,\dots,\theta_{r-1},t,u)\in[-\xi,\xi]^{r-1}\times\overline{\mathcal{D}}(\xi)\times[1,1+\delta]$ when $\xi\in\mathbb{R}$ with $0<\xi<\eta_r$. In particular, $G_r(\{i\theta_k\},-t;\mathbf{d}_{r-1};\mathcal{P}_r;u)$ is continuous for $(\theta_1,\dots,\theta_{r-1},t,u)\in[-\xi,\xi]^{r-1}\times\overline{\mathcal{D}}(\xi)\times[1,1+\delta]$. Hence there exists the value

$$\widetilde{\mathcal{M}}_\xi=\max|G_r(\{i\theta_k\},-t;\mathbf{d}_{r-1};\mathcal{P}_r;u)|\quad\text{on}\quad[-\xi,\xi]^{r-1}\times\{|t|=\xi\}\times[1,1+\delta]$$

when $\xi\in\mathbb{R}$ with $0<\xi<\eta_r$. By the above equation, we have

$$\frac{|\mathfrak{B}_n(\{i\theta_k\};\mathbf{d}_{r-1};\mathcal{P}_r;u)|}{n!}\le\frac{\widetilde{\mathcal{M}}_\xi}{\xi^n}\tag{4.4}$$

for any $n\in\mathbb{N}_0$, $(\theta_1,\dots,\theta_{r-1})\in[-\xi,\xi]^{r-1}$ and $u\in[1,1+\delta]$.

Define

$$\begin{aligned}\mathcal{Z}_r(\mathbf{d}_{r-1},s;i\theta_1,\dots,i\theta_{r-1};\mathcal{P}_r;u)&=F_r(i\theta_1,\dots,i\theta_{r-1},0;\mathbf{d}_{r-1},s;\mathcal{P}_r;u)\\&=\sum_{n_1,\dots,n_r=0}^{\infty}\frac{a_1(n_1)\cdots a_r(n_r)u^{-\sum_{\nu=1}^r n_\nu}\prod_{j=1}^{r-1}e^{(\alpha_j+\sum_{\mu=1}^j n_\mu w_\mu)i\theta_j}}{\prod_{j=1}^{r-1}(\alpha_j+\sum_{\nu=1}^j n_\nu w_\nu)^{d_j}(\alpha_r+\sum_{\nu=1}^r n_\nu w_\nu)^s}\end{aligned}\tag{4.5}$$

for $(\theta_1,\dots,\theta_{r-1})\in(-\eta_r,\eta_r)^{r-1}$, $s\in\mathbb{C}$ with $\Re s>q_r$ and $u\in[1,1+\delta]$. Assuming $\Re s>\max(1,q_r)$ and using the same method as in the proof of Proposition 2.1, we have

$$\begin{aligned}\mathcal{Z}_r(\mathbf{d}_{r-1},s;\{i\theta_k\};\mathcal{P}_r;u)&=\frac{1}{(e^{2\pi is}-1)\,\Gamma(s)}H_r(s;\{i\theta_k\};\mathbf{d}_{r-1};\mathcal{P}_r;u)\\&=\frac{\Gamma(1-s)}{2\pi i\,e^{\pi is}}H_r(s;\{i\theta_k\};\mathbf{d}_{r-1};\mathcal{P}_r;u).\end{aligned}\tag{4.6}$$

Note that $H_r(s;\{i\theta_k\};\mathbf{d}_{r-1};\mathcal{P}_r;u)$ is holomorphic for all $s\in\mathbb{C}$ if we fix $\{\theta_k\}\in(-\eta_r,\eta_r)^{r-1}$ (as mentioned above), and that poles of $\Gamma(1-s)$ coincide with $\mathbb{N}=\{1,2,\dots\}$. Since $\mathcal{Z}_r(\mathbf{d}_{r-1},s;\{i\theta_k\};\mathcal{P}_r;u)$ is convergent absolutely for $s\in\mathbb{C}$ with $\Re s>q_r$, it follows from (4.6) that $\mathcal{Z}_r(\mathbf{d}_{r-1},s;\{i\theta_k\};\mathcal{P}_r;u)$ is defined and holomorphic for all $s\in\mathbb{C}\setminus\{1,2,\dots,[q_r]\}$ if we fix $\{\theta_k\}\in(-\eta_r,\eta_r)^{r-1}$.

Furthermore, we can prove that $\mathcal{Z}_r(\mathbf{d}_{r-1},s;\{i\theta_k\};\mathcal{P}_r;u)$ has no pole as follows. We fix an arbitrary $s\in\mathbb{C}$. If $1<u\le 1+\delta$ then from (1.7) and (4.5), and by substituting the Taylor expansion series for each $\exp((\alpha_j+\sum_{\nu=1}^j n_\nu w_\nu)i\theta_j)$ and changing the order of summations, we have

(4.7)

$$\begin{aligned}&\mathcal{Z}_r(\mathbf{d}_{r-1},s;\{i\theta_k\};\mathcal{P}_r;u)\\&\quad=\sum_{N_1,\dots,N_{r-1}=0}^{\infty}\Psi_r(d_1-N_1,\dots,d_{r-1}-N_{r-1},s;u)\frac{(i\theta_1)^{N_1}\cdots(i\theta_{r-1})^{N_{r-1}}}{N_1!\cdots N_{r-1}!}.\end{aligned}$$

We see that (4.3) is uniformly convergent with respect to $(\theta_1,\ldots,\theta_{r-1},u)\in[-\xi,\xi]^{r-1}\times[1,1+\delta]$, when we take any $\xi\in\mathbb{R}$ with $0<\xi<\eta_r$. Hence, for $u\in[1,1+\delta]$, $H_r(s;\{i\theta_k\};\mathbf{d}_{r-1};\mathcal{P}_r;u)$ is real-analytic for $(\theta_1,\ldots,\theta_{r-1})\in(-\eta_r,\eta_r)^{r-1}$. Put $\theta_1=\cdots=\theta_{r-1}=\theta$. Then for $u\in[1,1+\delta]$, $H_r(s;\{i\theta\};\mathbf{d}_{r-1};\mathcal{P}_r;u)$ is real-analytic for $\theta\in(-\eta_r,\eta_r)$, and its Taylor expansion series around $\theta=0$ is uniformly convergent with respect to $(\theta,u)\in[-\xi,\xi]\times[1,1+\delta]$. It follows from (4.6) that $\mathcal{Z}_r(\mathbf{d}_{r-1},s;\{i\theta\};\mathcal{P}_r;u)$ also has these properties. Hence, for any $u\in[1,1+\delta]$, we define the one variable complex function $\mathcal{Z}_r(\mathbf{d}_{r-1},s;\{t\};\mathcal{P}_r;u)$ which is holomorphic for $t\in\mathcal{D}(\eta_r)$ and its Taylor expansion series around $t=0$ is uniformly convergent with respect to $(t,u)\in\overline{\mathcal{D}}(\xi)\times[1,1+\delta]$. In particular, $\mathcal{Z}_r(\mathbf{d}_{r-1},s;\{t\};\mathcal{P}_r;u)$ is continuous for $(t,u)\in\overline{\mathcal{D}}(\xi)\times[1,1+\delta]$. Putting $\xi=\varepsilon$ with $0<\varepsilon<\eta_r$, there exists the value

$$\mathcal{M}_\varepsilon'=\max|\mathcal{Z}_r(\mathbf{d}_{r-1},s;\{t\};\mathcal{P}_r;u)|\quad\text{on } \{t\in\mathbb{C}\,|\,|t|=\varepsilon\}\times[1,1+\delta].$$

Using the same method as in the proof of (4.4) and by (4.7) and the continuity of $\Psi_r(d_1-N_1,\ldots,d_{r-1}-N_{r-1},s;u)$ on $u\in[1,1+\delta]$ (see Theorem 1.1), we see that

$$\left|\sum_{N_1+\cdots+N_{r-1}=n}\frac{\Psi_r(d_1-N_1,\ldots,d_{r-1}-N_{r-1},s;u)}{N_1!\cdots N_{r-1}!}\right|\leq\frac{\mathcal{M}_\varepsilon'}{\varepsilon^n}\tag{4.8}$$

for $u\in[1,1+\delta]$ and $n\in\mathbb{N}_0$, where ε is an arbitrary real number with $0<\varepsilon<\eta_r$. This means that the right-hand side of (4.7) is uniformly convergent with respect to $(\theta_1,\ldots,\theta_{r-1},u)\in[-\xi,\xi]^{r-1}\times[1,1+\delta]$ for any $\xi\in\mathbb{R}$ with $0<\xi<\eta_r$. Hence we can let $u\to1$ in (4.7), namely (4.7) holds for $u\in[1,1+\delta]$. Since s is an arbitrary complex number, $\mathcal{Z}_r(\mathbf{d}_{r-1},s;\{i\theta_k\};\mathcal{P}_r;u)$ has no pole, namely it is holomorphic for all $s\in\mathbb{C}$ when $u\in[1,1+\delta]$, and is real-analytic for $(\theta_1,\ldots,\theta_{r-1})\in(-\eta_r,\eta_r)^{r-1}$ when $s\in\mathbb{C}$ and $u\in[1,1+\delta]$.

For $d_r\in\mathbb{C}$ with $\Re d_r>q_r$ and $N\in\mathbb{N}_0$ with $N\geq\Re d_r+1$, we put $s=d_r-N$ in (4.3). Then we have

$$\begin{aligned}&H_r(d_r-N;\{i\theta_k\};\mathbf{d}_{r-1};\mathcal{P}_r;u)\\&\quad=\left(e^{2\pi id_r}-1\right)\int_\varepsilon^\infty G_r(\{i\theta_k\},-t;\mathbf{d}_{r-1};\mathcal{P}_r;u)t^{d_r-N-1}dt\\&\qquad+\int_{C_\varepsilon}G_r(\{i\theta_k\},-t;\mathbf{d}_{r-1};\mathcal{P}_r;u)t^{d_r-N-1}dt.\end{aligned}\tag{4.9}$$

For simplicity, we denote by I_1 and I_2 the first and second terms of the right-hand side of (4.9), respectively. Since $N\geq\Re d_r+1$, we have

$$\left|\int_\varepsilon^\infty e^{-(\alpha_r+\sum_{\mu=1}^r n_\mu w_\mu)t}t^{d_r-N-1}dt\right|\leq\frac{e^{-(\alpha_r+\sum_{\mu=1}^r n_\mu w_\mu)\varepsilon}|\varepsilon^{d_r-N-1}|}{\alpha_r+\sum_{\mu=1}^r n_\mu w_\mu}.$$

Hence we have

$$(4.10)\quad |I_1| \le \varepsilon^{\Re d_r - N-1}|e^{2\pi i d_r}-1| \times \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{|a_1(n_1)\cdots a_r(n_r)|e^{-(\alpha_r+\sum_{\mu=1}^{r} n_\mu w_\mu)\varepsilon}}{\prod_{j=1}^{r-1}(\alpha_j+\sum_{\mu=1}^{j} n_\mu w_\mu)^{\Re d_j}(\alpha_r+\sum_{\mu=1}^{r} n_\mu w_\mu)}.$$

On the other hand, by using (2.9), we have

$$I_2 = \begin{cases} (2\pi i)\mathfrak{B}_{N-d_r}(\{i\theta_k\};\mathbf{d}_{r-1};\mathcal{P}_r;u)\dfrac{(-1)^{N-d_r}}{(N-d_r)!} & (N-d_r\in\mathbb{N}_0), \\ \varepsilon^{d_r-N}\big(e^{2\pi i d_r}-1\big)\displaystyle\sum_{n=0}^{\infty}\frac{\mathfrak{B}_n(\{i\theta_k\};\mathbf{d}_{r-1};\mathcal{P}_r;u)(-1)^n\varepsilon^n}{(n+d_r-N)n!} & (\text{otherwise}). \end{cases}$$

Note that the above infinite series in the second case is uniformly convergent with respect to $(\theta_1,\ldots,\theta_{r-1},u)\in[-\varepsilon,\varepsilon]^{r-1}\times[1,1+\delta]$ because of the assumption $\varepsilon<\eta_r$ and (4.4). Hence we have either

$$(4.11)\quad |I_2| \le 2\pi\frac{|\mathfrak{B}_{N-d_r}(\{i\theta_k\};\mathbf{d}_{r-1};\mathcal{P}_r;u)|}{(N-d_r)!} \qquad (N-d_r\in\mathbb{N}_0)$$

or

$$(4.12)\quad |I_2| \le \varepsilon^{\Re d_r-N}\left|e^{2\pi i d_r}-1\right|\left|\sum_{n=0}^{\infty}\mathfrak{B}_n(\{i\theta_k\};\mathbf{d}_{r-1};\mathcal{P}_r;u)\frac{(-1)^n\varepsilon^n}{(n+d_r-N)n!}\right| \qquad (\text{otherwise}).$$

As well as (2.12), it follows from (4.4), (4.6), (4.8)-(4.12) that there exists a constant $M>0$ independent of N and $\{\theta_k\}$ such that

$$(4.13)\quad \left|\frac{\mathcal{Z}_r(\mathbf{d}_{r-1},d_r-N;\{i\theta_k\};\mathcal{P}_r;u)}{\Gamma(1+N-d_r)}\right| \le \frac{1}{2\pi|e^{\pi i d_r}|}\left|H_r(d_r-N;\{i\theta_k\};\mathbf{d}_{r-1};\mathcal{P}_r;u)\right| \le M\varepsilon^{-N}$$

for $N\in\mathbb{N}$ with $N\ge\Re d_r+1$. Note that we can take an arbitrary ε such that $0<\varepsilon<\eta_r$. As well as (2.13), we have

$$(4.14)\quad \frac{|\mathcal{Z}_r(\mathbf{d}_{r-1},d_r-N;\{i\theta_k\};\mathcal{P}_r;u)|}{N!} \le \frac{(N-[\Re d_r]+[|\Im d_r|]+1)!\ |\Gamma([\Re d_r]+1-d_r)|}{N!}M\varepsilon^{-N}$$

for $N\in\mathbb{N}$ with $N\ge\Re d_r+1$ and $u\in[1,1+\delta]$.

Suppose $1 < u \leq 1+\delta$ and $\theta_r \in (-\eta_r, \eta_r)$. Then by (1.9), and using the Taylor expansion series for $\exp((\alpha_r + \sum_{\nu=1}^{r} n_\nu w_\nu) i\theta_r)$, we have

$$F_r(i\theta_1, \ldots, i\theta_{r-1}, i\theta_r; \mathbf{d}_r; \mathcal{P}_r; u) = \sum_{N_r=0}^{\infty} \mathcal{Z}_r(\mathbf{d}_{r-1}, d_r - N_r; \{i\theta_k\}; \mathcal{P}_r; u) \frac{(i\theta_r)^{N_r}}{N_r!}. \tag{4.15}$$

By (4.14), we see that the right-hand side of (4.15) is uniformly convergent with respect to $(\theta_r, u) \in [-\xi, \xi] \times [1, 1+\delta]$ when $(\theta_1, \ldots, \theta_{r-1}) \in (-\eta_r, \eta_r)^{r-1}$ and $0 < \xi < \eta_r$. Hence (4.15) holds for $u = 1$. As we mentioned above, (4.7) holds for any $s \in \mathbb{C}$, $(\theta_1, \ldots, \theta_{r-1}) \in (-\eta_r, \eta_r)^{r-1}$, and $u \in [1, 1+\delta]$. Hence we have

$$\mathcal{Z}_r(\mathbf{d}_{r-1}, d_r - N_r; \{i\theta_k\}; \mathcal{P}_r; u) = \sum_{N_1, \ldots, N_{r-1}=0}^{\infty} \Psi_r(d_1 - N_1, \ldots, d_r - N_r; u) \frac{(i\theta_1)^{N_1} \cdots (i\theta_{r-1})^{N_{r-1}}}{N_1! \cdots N_{r-1}!} \tag{4.16}$$

for $u \in [1, 1+\delta]$. Hence (4.15) can also be written as

$$F_r(i\theta_1, \ldots, i\theta_r; \mathbf{d}_r; \mathcal{P}_r; u) = \sum_{N_1, \ldots, N_r=0}^{\infty} \Psi_r(d_1 - N_1, \ldots, d_r - N_r; u) \frac{(i\theta_1)^{N_1} \cdots (i\theta_r)^{N_r}}{N_1! \cdots N_r!} \tag{4.17}$$

for $u \in [1, 1+\delta]$, and (4.17) is uniformly convergent with respect to $(\theta_1, \ldots, \theta_r, u) \in [-\xi, \xi]^r \times [1, 1+\delta]$ for any $\xi \in \mathbb{R}$ with $0 < \xi < \eta_r$. Therefore, for $u \in [1, 1+\delta]$, we can define

$$F_r(t_1, \ldots, t_r; \mathbf{d}_r; \mathcal{P}_r; u) = \sum_{N_1, \ldots, N_r=0}^{\infty} \Psi_r(d_1 - N_1, \ldots, d_r - N_r; u) \frac{t_1^{N_1} \cdots t_r^{N_r}}{N_1! \cdots N_r!} \tag{4.18}$$

which is uniformly convergent with respect to $(t_1, \ldots, t_r, u) \in \overline{\mathcal{D}}(\xi)^r \times [1, 1+\delta]$ and is holomorphic for $(t_1, \ldots, t_r) \in \mathcal{D}(\eta_r)^r$ (see, for example, [7] Section 2.2). Thus we obtain the case of r. By induction, we obtain the proof of Theorem 1.3.

5 Some applications

First we prove the following estimation formulas for $\Psi_r(d_1 - N_1, \ldots, d_r - N_r; u)$ by using the same method as in the proof of Proposition 2.3.10 in [10].

Proposition 5.1. *With the same notation as in Theorem 1.3,*

$$\limsup_{N_1+\cdots+N_r\to\infty}\left\{\frac{|\Psi_r(d_1-N_1,\ldots,d_r-N_r;u)|}{N_1!\cdots N_r!}\right\}^{1/(N_1+\cdots+N_r)}\leq\frac{1}{\eta_r}. \tag{5.1}$$

Proof. Assume that the left-hand side of (5.1) is greater than $1/\eta_r$. Then we take $\kappa\in\mathbb{R}$ with $\kappa>1/\eta_r$ such that there exist infinitely many $(N_1,\ldots,N_r)\in\mathbb{N}_0^r$ such that

$$\frac{|\Psi_r(d_1-N_1,\ldots,d_r-N_r;u)|}{N_1!\cdots N_r!}>\kappa^{N_1+\cdots+N_r}.$$

This means that the right-hand side of (1.10) does not converge absolutely at $(1/\kappa,\ldots,1/\kappa)\in\mathcal{D}(\eta_r)^r$. This is contradiction. □

Example 5.2. Let $\psi_j(s)=L(s;f_j)$ $(1\leq j\leq r)$ as considered in Remark 3.2. Then (5.1) gives

$$\limsup_{N_1+\cdots+N_r\to\infty}\left\{\frac{|L_r(d_1-N_1,\ldots,d_r-N_r;f_1,\ldots,f_r)|}{N_1!\cdots N_r!}\right\}^{1/(N_1+\cdots+N_r)}\leq\frac{1}{\eta_r}, \tag{5.2}$$

where each $\Re d_j>1$ $(1\leq j\leq r)$ and $\eta_r=\min_{1\leq k\leq r}\{2\pi/2^{r-1}m_k\}$.

Secondly we give certain multiple analogues of both Berndt's and Katsurada's formulas considered in Example 2.2.

Example 5.3. As well as the above example, let $\psi_j(s)=L(s;f_j)$ $(1\leq j\leq r)$ and define a certain generalization of multiple polylogarithm by

$$\begin{aligned}&\mathfrak{F}_r(t_1,\ldots,t_r;\mathbf{d}_r;f_1,\ldots,f_r)\\&\quad=\sum_{n_1,\ldots,n_r=1}^{\infty}\frac{f_1(n_1)\cdots f_r(n_r)\prod_{j=1}^{r}e^{(\sum_{\mu=1}^{j}n_\mu)t_j}}{n_1^{d_1}(n_1+n_2)^{d_2}\cdots(n_1+\cdots+n_r)^{d_r}}\end{aligned} \tag{5.3}$$

for $d_1,\ldots,d_r\in\mathbb{C}$ with $\Re d_j>1$ $(1\leq j\leq r)$. Applying Theorem 1.3 with $\psi_j(s)=L(s;f_j)$ $(1\leq j\leq r)$ and $u=1$, we see that $\mathfrak{F}_r(t_1,\ldots,t_r;\mathbf{d}_r;f_1,\ldots,f_r)$ is defined and holomorphic for $(t_1,\ldots,t_r)\in\mathcal{D}(\eta_r)^r$ such that

$$\begin{aligned}&\mathfrak{F}_r(t_1,\ldots,t_r;\mathbf{d}_r;f_1,\ldots,f_r)\\&\quad=\sum_{N_1,\ldots,N_r=0}^{\infty}L_r(d_1-N_1,\ldots,d_r-N_r;f_1,\ldots,f_r)\frac{t_1^{N_1}\cdots t_r^{N_r}}{N_1!\cdots N_r!},\end{aligned} \tag{5.4}$$

where $\eta_r=\min_{1\leq k\leq r}\{2\pi/2^{r-1}m_k\}$. Putting $t_1=\cdots=t_{r-1}=0$ and $t_r=\pm i\theta$ for $\theta\in(-\eta_r,\eta_r)$ in (5.4), we have

$$\begin{aligned}&\sum_{n_1,\ldots,n_r=1}^{\infty}\frac{f_1(n_1)\cdots f_r(n_r)\cos\left((n_1+\cdots+n_r)\,\theta\right)}{n_1^{d_1}(n_1+n_2)^{d_2}\cdots(n_1+\cdots+n_r)^{d_r}}\\&\quad=\sum_{N=0}^{\infty}L_r(d_1,\ldots,d_{r-1},d_r-2N;f_1,\ldots,f_r)\frac{(i\theta)^{2N}}{(2N)!}.\end{aligned} \tag{5.5}$$

Remark 5.4. In the case $f_j(n) = (-1)^n$ $(1 \le j \le r)$, $\mathcal{F}_r(i\theta_1, \ldots, i\theta_r; f_1, \ldots, f_r)$ has recently been used to prove what is called the parity result for Euler-Zagier sums (see [15]).

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