| |
|---|
| $k$ : field |
| $S := k[[x_0, x_1, x_2, \cdots, x_d]]$ |
| $0 \neq f \in (x_0, x_1, x_2, \cdots, x_d)S$ |
| $R := S/(f)$ : hypersurface |

Lemma 1

$M \in CM(R) \Rightarrow {}^{\exists}(\varphi, \psi) \in MF_S(f)$ s.t. $M \cong \mathrm{Coker}(\varphi)$

(Proof) $M$ : MCM $R$-module により、$\mathrm{depth}_R M = \dim R$ 」

$S \twoheadrightarrow R \curvearrowright M$、$M \in \mathrm{mod}(S)$ & $S$ : RLR により

$$\begin{aligned} \underset{\overset{\wedge}{\infty}}{\mathrm{proj\,dim}_S} M &= \mathrm{depth} S - \mathrm{depth}_S M \\ &= \dim S - \mathrm{depth}_R M \\ &= \dim S - \dim R \\ &= 1 \end{aligned}$$

$\therefore 0 \to S^m \to S^n \to M \to 0$ ex. seq in $\mathrm{mod}\, S$

$f$ で localize (i.e. $W := \{f^n \mid n \geqq 0\}$) すると、

$$0 \to S_f^m \to S^n \to \underset{\overset{\parallel}{0}}{M_f} \to 0 \text{ ex} \qquad \therefore m = n$$

$$\begin{tikzcd} 0 \arrow[r] & S^n \arrow[r, "\varphi"] \arrow[d, "\begin{pmatrix} f & & \\ & \ddots & \\ & & f \end{pmatrix}"'] & S^n \arrow[r] \arrow[d, "\begin{pmatrix} f & & \\ & \ddots & \\ & & f \end{pmatrix}"] \arrow[dl, "\exists\psi"'] & M \arrow[r] \arrow[d, "f=0"] & 0 \\ 0 \arrow[r] & S^n \arrow[r, "\varphi"'] & S^n \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

$\varphi \circ \psi = \begin{pmatrix} f & & \\ & \ddots & \\ & & f \end{pmatrix}$ $\quad \varphi \circ \psi \circ \varphi = \varphi \circ \begin{pmatrix} f & & \\ & \ddots & \\ & & f \end{pmatrix}$

since $\varphi$ is a monomorphism

$\psi \circ \varphi = \begin{pmatrix} f & & \\ & \ddots & \\ & & f \end{pmatrix}$ ☑

snake lemma により

$$0 \to M \to R^n \xrightarrow{\bar{\varphi}} R^n \to \overset{\mathrm{Coker}(\varphi)}{\overset{\wr\wr}{M}} \to 0 \quad \text{(ex)}$$

Proposition $\quad M \in CM(R) \Rightarrow M$ は、周期 2 の自由分解をもつ!! □

## Definition

・$(\varphi, \psi)$ : matrix factorization of $f$

$\underset{\text{def}}{\iff}$ $S^n \xrightarrow{\psi} S^n \xrightarrow{\varphi} S^n$

st $\varphi \circ \psi = \begin{pmatrix} f & \\ & \ddots \\ & & f \end{pmatrix} = \psi \circ \varphi$

・$(\varphi, \psi), (\varphi', \psi')$ : mat factr of $f$

$(\alpha, \beta) : (\varphi, \psi) \longrightarrow (\varphi', \psi')$

$\underset{\text{def}}{\iff}$
$$\begin{array}{ccccc} S^n & \xrightarrow{\psi} & S^n & \xrightarrow{\varphi} & S^n \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \alpha \\ S^m & \xrightarrow[\psi']{} & S^m & \xrightarrow[\varphi']{} & S^m \end{array}$$

・$MF_S(f)$ : the category of mat factr of $f$

$\underset{\text{def}}{\iff}$ $\begin{cases} Ob(MF_S(f)) : \text{mat factr of } f \\ Hom_{MF_S(f)}((\varphi, \psi), (\varphi', \psi')) = \left\{ (\alpha, \beta) \;\middle|\; \begin{array}{ccccc} S^n & \xrightarrow{\psi} & S^n & \xrightarrow{\varphi} & S^n \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \alpha \\ S^m & \xrightarrow[\psi']{} & S^m & \xrightarrow[\varphi']{} & S^m \end{array} \right\} \end{cases}$

・$(\varphi, \psi), (\varphi', \psi') \in MF_S(f)$.

(i) $(\varphi, \psi) \oplus (\varphi', \psi') := \left( \begin{pmatrix} \varphi & 0 \\ 0 & \varphi' \end{pmatrix}, \begin{pmatrix} \psi & 0 \\ 0 & \psi' \end{pmatrix} \right)$

(ii) $(\varphi, \psi) \sim (\varphi', \psi')$ : equivalent

$\underset{\text{def}}{\iff}$ $\exists (\alpha, \beta) \in Hom_{MF_S(f)}((\varphi, \psi), (\varphi', \psi'))$, $\alpha, \beta$ : iso.

(iii) $(\varphi, \psi)$ : reduced

$\underset{\text{def}}{\iff}$ 行列$\varphi, \psi$の成分は全てnon-unit

・$(1, f) : S \xrightarrow{f} S \xrightarrow{1} S$ (i.e. $Coker(1) = 0$)

$(f, 1) : S \xrightarrow{1} S \xrightarrow{f} S$ (i.e. $Coker(f) = R$)

・$\underline{MF}_S(f) := MF_S(f) / \{(1, f)\}$

i.e. $\begin{cases} Ob(\underline{MF}_S(f)) := Ob(MF_S(f)) \\ Hom_{\underline{MF}_S(f)}((\varphi, \psi), (\varphi', \psi')) := Hom_{MF_S(f)}((\varphi, \psi), (\varphi', \psi')) \Big/ \left\{ \begin{array}{ccc} (\varphi, \psi) & \longrightarrow & (\varphi', \psi') \\ & \searrow \circlearrowright \nearrow & \\ & (1, f)^{\oplus} & \end{array} \right\} \end{cases}$

$\underline{RMF}_S(f) := MF_S(f) / \{(1, f), (f, 1)\}$

## Lemma 2

$(\varphi, \psi) \in MF_S(f) \Rightarrow \varphi, \psi$ : monomorphism

(Proof) $S^n \xrightarrow{\psi} S^n \xrightarrow{\varphi} S^n \xrightarrow{\psi} S^n$

$$\begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} \longmapsto 0 \longmapsto 0$$

$$\begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} \longmapsto \begin{pmatrix} ft_1 \\ \vdots \\ ft_n \end{pmatrix}$$

$ft_1 = \cdots = ft_n = 0 \Rightarrow t_1 = \cdots = t_n = 0$

$\varphi, \psi$ : monom // $\blacksquare$

## Lemma 3

$(\varphi, \psi) \in MF_S(f)$ に対して、

chain complex $\cdots \xrightarrow{\overline{\Psi}} R^n \xrightarrow{\overline{\varphi}} R^n \xrightarrow{\overline{\Psi}} R^n \xrightarrow{\overline{\varphi}} R^n \xrightarrow{\overline{\Psi}} \cdots$

$0 \to M \to 0$, $\quad 0 \to \Omega M \to 0$, $\quad M \to 0$

is exact !!

(Proof) $R^n \xrightarrow{\overline{\Psi}} R^n \xrightarrow{\overline{\varphi}} R^n \xrightarrow{\overline{\Psi}} R^n$

$$\begin{pmatrix} \bar t_1 \\ \vdots \\ \bar t_n \end{pmatrix} \longmapsto 0$$

$$\varphi\left(\begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}\right) = \begin{pmatrix} fs_1 \\ \vdots \\ fs_n \end{pmatrix} = \begin{pmatrix} f & & \\ & \ddots & \\ & & f \end{pmatrix}\begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} = \varphi \circ \psi\left(\begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}\right)$$

$\varphi$ : monom により

$$\begin{pmatrix} \bar t_1 \\ \vdots \\ \bar t_n \end{pmatrix} = \overline{\varphi}\left(\begin{pmatrix} \bar s_1 \\ \vdots \\ \bar s_n \end{pmatrix}\right)$$

$\therefore$ exact // $\blacksquare$

## Lemma 4 $(\varphi, \psi) \in MF_S(f) \Rightarrow \mathrm{Coker}(\varphi) \in CM(R)$

(Proof)

$M := \mathrm{Coker}(\varphi)$

$$\begin{tikzcd} 0 \arrow[r] & S^n \arrow[r, "\varphi"] \arrow[dr, phantom, "\curvearrowright"] & S^n \arrow[r] \arrow[dr, phantom, "\curvearrowright"] & M \arrow[r] & 0 \quad (\mathrm{ex}) \\ 0 \arrow[r] & S^n \arrow[r, "{\begin{pmatrix} f & \\ & \ddots f \end{pmatrix}}"'] \arrow[u, "\psi"] & S^n \arrow[r] \arrow[u, equal] & R^n \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

$\Rightarrow M \in$ ~~mod(R)~~ $CM(R)$

$\dim R - \mathrm{depth}_R M$

$= \dim S - 1 - \mathrm{depth}_S M$

$= \mathrm{depth}\, S - \mathrm{depth}_S M - 1$

$= \mathrm{pd}_S M - 1 = 0$

<u>Remark</u>

$(\varphi, \psi) \xrightarrow{(\alpha,\beta)} (\varphi', \psi')$ in $\underline{MF}_S(f)$

$\rightsquigarrow$

induces

$\mathrm{Coker}(\varphi) \xrightarrow{\mathrm{Coker}\,\alpha} \mathrm{Coker}(\varphi')$ in $CM(R)$

(Proof) Theorem.

(1) $(1, f) \longmapsto 0$

Lemma 4 & Remark

$\mathrm{Coker}: \underline{MF}_S(f) \longrightarrow CM(R)$

obj $(\varphi, \psi) \longmapsto \mathrm{Coker}(\varphi)$

morph $(\alpha, \beta) \longmapsto \mathrm{Coker}(\alpha)$

$0 \neq {}^\forall M \in CM(R)$

$0 \to S^n \xrightarrow{\varphi} S^n \to M \to 0$ : minimal $S$-free resol

$\rightsquigarrow$

ここから、得られる行列分解を、$(\varphi, \psi)$と表わす。

Lemma 1により得られるmatrix factr のうちの1つを

$(\varphi', \psi')$ とおく。

$$\begin{tikzcd}
0 \arrow[r] & S^m \arrow[r, "\begin{pmatrix}\varphi_1 & & \\ & \ddots & \\ & & 1\end{pmatrix}"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\circlearrowright"] & S^m \arrow[r] \arrow[d, "\alpha"'] \arrow[dd, bend left] \arrow[dr, phantom, "\circlearrowright"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & S^m \arrow[r, "\varphi'"] \arrow[d, "\beta'"'] \arrow[dr, phantom, "\circlearrowright"] & S^m \arrow[r, "1_{S^m}"] \arrow[d, "\alpha'"'] \arrow[dr, phantom, "\circlearrowright"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & S^m \arrow[r, "\begin{pmatrix}\varphi_1 & & \\ & \ddots & \\ & & 1\end{pmatrix}"'] & S^m \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

$\therefore (\varphi', \psi') \sim (\varphi, \psi) \oplus (1, f)^{(m-n)}$

$\Gamma : CM(R) \longrightarrow \underline{MF_S}(f)$

$\quad\ \ M \longmapsto (\varphi, \psi)$ : ( i.e. min. $S$-free resol of $M$ から得られる mat factr )

$$\begin{tikzcd}
0 \arrow[r] & S^n \arrow[r, "\varphi"] \arrow[d, dashed, "\beta"'] \arrow[dr, phantom, "\circlearrowright"] & S^n \arrow[r] \arrow[d, dashed, "\alpha"] \arrow[dr, phantom, "\circlearrowright"] & M \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & S^m \arrow[r, "\varphi'"'] & S^m \arrow[r] & N \arrow[r] & 0
\end{tikzcd} \qquad \Gamma(g) := (\alpha, \beta)$$

右のようにして得られるもののもう1つを$(\alpha', \beta')$とおく.

$$\begin{tikzcd}
0 \arrow[r] & S^n \arrow[r, "\varphi"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & S^n \arrow[r] \arrow[d, "\alpha-\alpha'"] \arrow[dr, phantom, "\circlearrowright"] & M \arrow[r] \arrow[d, "g-g=0"] & 0 \\
0 \arrow[r] & S^m \arrow[r, "\varphi'"'] & S^m \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
 & s \arrow[d, mapsto] \arrow[dl, "\mu"'] & \\
\exists 1\, t \arrow[r, mapsto] & (\alpha-\alpha')(s) \arrow[r, mapsto] & 0
\end{tikzcd}$$

$\varphi' \circ \mu = \alpha - \alpha'$.

$\varphi' \circ \mu \circ \varphi = (\alpha - \alpha') \circ \varphi$

$\qquad\qquad\quad = \varphi' \circ (\beta - \beta')$

$\therefore\ \mu \circ \varphi = \beta - \beta'$.

$$\begin{tikzcd}
0 \arrow[r] & S^n \arrow[r, "\varphi"] \arrow[d, "\beta-\beta'"'] \arrow[dd, bend right, "\beta-\beta'"'] \arrow[dr, phantom, "\circlearrowright"] & S^n \arrow[r] \arrow[dl, "\mu"'] \arrow[d, "\mu"] \arrow[dd, bend left, "\alpha-\alpha'"] \arrow[dr, phantom, "\circlearrowright"] & M \arrow[r] \arrow[d] \arrow[dd, bend left, "0"] & 0 \\
0 \arrow[r] & S^m \arrow[r, "\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}"'] \arrow[dr, phantom, "\circlearrowright"] & S^n \arrow[r] \arrow[d, "\varphi'"] \arrow[dr, phantom, "\circlearrowright"] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S^m \arrow[r, "\varphi'"'] & S^n \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

$\therefore\ (\alpha, \beta) = (\alpha', \beta')$ in $\underline{MF_S}(f)$

$k=\overline{k}$, $\mathrm{char}(k)=0$

$\Lambda$: CM local ring

・$\mathrm{CM}(\Lambda)$ $\quad \mathrm{CM}_0(\Lambda)=\{M\in \mathrm{CM}(\Lambda) \mid M_{\mathfrak{p}}: \Lambda_{\mathfrak{p}}\text{-free } \forall \mathfrak{p}\neq \mathfrak{m}\}$

・$\mathrm{ind}(\mathrm{CM}_0(\Lambda)) := \{\text{indec MCM } \Lambda\text{-module } M\in \mathrm{CM}_0(\Lambda)\}/\cong$

・$\Lambda$: finite (countable) CM representation type

$\underset{\mathrm{def}}{\Leftrightarrow}$ $\mathrm{ind}(\mathrm{CM}(\Lambda))$: finite (countable) set.

**Theorem** Knörrer's periodicity

$$\mathrm{MF}_S(f) \longrightarrow \mathrm{MF}_{S[\![y,z]\!]}(f+yz)$$

$$(A,B) \longmapsto \left( \begin{pmatrix} A & y \\ z & -B \end{pmatrix}, \begin{pmatrix} B & y \\ z & -A \end{pmatrix} \right)$$

induces $\underline{\mathrm{CM}}(R) \xrightarrow{\sim} \underline{\mathrm{CM}}(S[\![y,z]\!]/(f+yz))$

**Theorem** [Knörrer(1987)][Buchweitz-Greuel-Schreyer(1987)] [Arnold(1972)], [Kiyek-Steinke(1985)]

$R$: finite CM rep type $\Leftrightarrow$ $R$: simple singularity

$(\underset{\mathrm{def}}{\Leftrightarrow} \#\{I\subsetneq S \mid f\in I^2\}<\infty)$

$$\Leftrightarrow \quad f=\begin{cases} x_0^2+x_1^{n+1}+x_2^2+\cdots+x_d^2 & (A_n)\ (n\geqq 1) \\ x_0^2x_1+x_1^{n-1}+x_2^2+\cdots+x_d^2 & (D_n)\ (n\geqq 4) \\ x_0^3+x_1^4+x_2^2+\cdots+x_d^2 & (E_6) \\ x_0^3+x_0x_1^3+x_2^2+\cdots+x_d^2 & (E_7) \\ x_0^3+x_1^5+x_2^2+\cdots+x_d^2 & (E_8) \end{cases}$$

**Theorem** [BGS]

$R$: infinite countable CM rep. type

$$\Leftrightarrow f=\begin{cases} x_0^2+x_2^2+\cdots+x_d^2 & (A_\infty) \\ x_0^2x_1+x_2^2+\cdots+x_d^2 & (D_\infty) \end{cases}$$

**Theorem** [Schreyer (1987)] [Solberg (1989)]

- finite CM rep type のとき
  - $CM(R)$ の AR-quiver を描いた。
- infinite countable CM rep type のとき
  - $\underline{CM}_0(R)$ の AR-quiver を描いた。

**Theorem** [Araya-Takahashi-I (2010)]

- countable type のとき

(1) $\operatorname{ind} CM(R) \setminus \operatorname{ind} CM_0(R) = \{X_R, \Omega X_R\}$

(2) $X_R, \Omega X_R$ の non-free locus $= \{(x_0, x_2, \dots, x_d), (x_0, x_1, x_2, \dots, x_d)\}$

(3) $^\forall M \in \operatorname{ind}(CM_0(R))$ に対して、$^\exists n \geqq 0$ $L, N \in \{X, \Omega X\}$

$$\text{s.t. } 0 \to L \to M \oplus R^n \to N \to 0 \text{ ex in } CM(R)$$

(i.e. $L \to M \to N \to L[1]$ triangle $\underline{CM}(R)$)

**Example**

$R = k[[x, y]]/(x^2)$

[BGS] $\operatorname{ind} CM(R) = \{R, \underset{\substack{\shortparallel \\ X_R}}{R/(x)}, \underset{\substack{\shortparallel \\ M_n}}{\operatorname{Coker}\left(\begin{pmatrix} x & y^n \\ 0 & -x \end{pmatrix}\right)} \mid n \geqq 1\}$

$$\begin{array}{ccccccc}
R & \xrightarrow{x} & R & \xrightarrow{x} & R & \xrightarrow{x} & R \\
\downarrow{\scriptstyle\begin{pmatrix} x & y^n \\ 0 & -x \end{pmatrix}} & & \downarrow{\scriptstyle\begin{pmatrix} x & y^n \\ 0 & -x \end{pmatrix}} & & \downarrow{\scriptstyle\begin{pmatrix} x & y^n \\ 0 & -x \end{pmatrix}} & & \downarrow \\
R^2 & \longrightarrow & R^2 & \longrightarrow & R^2 & \longrightarrow & R^2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
R & \xrightarrow[-x]{} & R & \xrightarrow[-x]{} & R & \xrightarrow[-x]{} & R
\end{array}$$

$\xrightarrow{x} R \xrightarrow{x} R \to X.$

$0 \to X_R \to M_n \to X_R \to 0$

$0 \to X_R \to R \to X_R \to 0$