Introduction to Auslander-Reiten duality for Cohen-Macaulay modules

<u>reference</u> [Auslander 76] Functors and morphism… } ⊃[Yoshino] §2~4
[———— 86] Isolated singularities and… }
[AR 74?] Representation theory of artin algebra III, IV
→ AR theory for fin.dim.alg was established
→ AR theory for "order" was established.
$\cup$
CM ring

<u>Throughout</u> (R, m): regular local ring
Λ: $\boxed{\text{R-algebra. module-finite}}$ R-order
d := dim R $\Lambda \in \mathrm{mod}\, R$
Λ: ring. Mod Λ: cat of right Λ-mod
mod Λ: —— fin.gen. right Λ-mod
proj Λ: cat. of fin.gen. proj Λ-mod
f.l Λ: —— finite length Λ-mod

§1

commutative	Λ	fin.dim.alg
regular	non-singular R-order	semisimple alg
$\cap$	$\cap$	$\cap$
Gorenstein	Gorenstein R-order	selfinj alg
$\cap$	$\cap$	$\cap$
Cohen-Macaulay	R-order	all

Def Λ : R-order

$\overset{\text{def}}{\Longleftrightarrow}$ $\Lambda \in \operatorname{proj} R$

Ex ① Λ : comm complete local CM ring containing a field

$\exists$(nonunique) $R \subset \Lambda$ $\begin{cases} R \subset \mathrm{Cen}(\Lambda) \\ R \text{ is complete reg local} \\ \Lambda \in \operatorname{proj} R \end{cases}$

In particular Λ is an R-order

② $(d=0)$ R : field

R-order = fin. dim. R-algebra

③ $(d=1)$ R : discrete valuation ring

R-order = "classical" R-order

[Curtis-Reiner] Methods of rep. th.

[Roggenkamp] Almost split seq ~ $\rightsquigarrow$ $\begin{pmatrix} d=1 \text{ order} \\ \text{AR theory} \end{pmatrix}$

Def $X \in \operatorname{mod} \Lambda$ is CM $\overset{\text{def}}{\Longleftrightarrow}$ $X \in \operatorname{proj} R$

$\mathrm{CM}(\Lambda)$: cat of CM Λ-modules

properties 1.1 (0) Λ is left and right noetherian $\Lambda \in \mathrm{CM}(\Lambda)$

(1) $0 \to X \to Y \to Z \to 0$; exact seq of Λ-mod

- $X, Z \in \mathrm{CM}(\Lambda) \Rightarrow Y \in \mathrm{CM}(\Lambda)$
- Y, Z ———— X ————

(2) $0 \to X_d \to \underbrace{X_{d-1} \to \cdots \to X_1 \to X_0}$: exact

$\overset{\cap}{\mathrm{CM}}(\Lambda) \Longleftarrow \mathrm{CM}(\Lambda)$

(3) $(\ )^\vee := \mathrm{Hom}_R(-, R) : \mathrm{CM}(\Lambda) \overset{\sim}{\longleftrightarrow} \mathrm{CM}(\Lambda^{op})$

right Λ^{op}-mod = left Λ-mod $(\ )^\vee (\ )^\vee \simeq \mathrm{id}$

canonical duality $\omega_\Lambda := \Lambda^\vee$: canonical mod

Def Λ : R-order Λ : non-singular $\overset{\text{def}}{\Longleftrightarrow}$ $\mathrm{gl.dim}\,\Lambda = d$

Ex $k[[x_1, \cdots, x_d]] * G$ $\left(G \underset{\text{finite}}{\subset} GL_d(k)\ \ \mathrm{char}\,k \nmid |G|\right)$
is a non-singular order

Thm 1.2 Λ : R-order $\circlearrowright$
(1) Λ : non-singular (2) $\mathrm{gl.dim}\,\Lambda \le d$
(3) $\mathrm{CM}(\Lambda) = \mathrm{proj}\,\Lambda$

Sketch **Lem 1.3** $X \in \mathrm{mod}\,\Lambda$ pd = proj dim
(1) $r \in \mathfrak{m}$: non-zerodivisor $\Rightarrow \mathrm{pd}_\Lambda X + 1 = \mathrm{pd}_\Lambda (X/rX)$
(2) $\mathrm{pd}_\Lambda X < \infty \Rightarrow \mathrm{pd}_\Lambda X + \mathrm{depth}_R X \le \mathrm{gl.dim}\,\Lambda$

Sketch $0 \to X \xrightarrow{r} X \to X/rX \to 0$ $(*)$
(1): long exact seq assoc with $\mathrm{Hom}_\Lambda((*), -)$
Use Nakayama's Lemma for R-modules
(2) Use (1) repeatedly □

pf. of Thm 1.2 (2) ⇒ (1)
$d = \mathrm{pd}_\Lambda \Lambda + \mathrm{depth}_R \Lambda \le \mathrm{gl.dim}\,\Lambda$
(1) ⇒ (3)
$X \in \mathrm{CM}(\Lambda) \Rightarrow X \in \mathrm{proj}\,\Lambda$
$\left(\mathrm{pd}_\Lambda X + \boxed{\mathrm{depth}_R X}_{d} \le d \Rightarrow \mathrm{pd}_\Lambda X = 0\right)$
(3) ⇒ (2)

$$X \in \mathrm{mod}\,\Lambda \quad 0 \to \underset{\substack{\mathrm{proj}\,R \\ \mathrm{CM}(\Lambda) \\ \shortparallel \\ \mathrm{proj}\,\Lambda}}{K} \to \underbrace{P_{d-1} \to \cdots \to P_0}_{\substack{\mathrm{proj}\,\Lambda \\ \mathrm{proj}\,R}} \quad \mathrm{pd}_\Lambda X \le d.$$

□

$\cdot\ X, Y \in \mathrm{Mod}\,\Lambda.\ \mathrm{mod}\,\Lambda$

$\left.\begin{array}{l}\cdot\ \mathrm{Hom}_\Lambda(X,Y)\\ \cdot\ \mathrm{Ext}^i_\Lambda(X,Y)\end{array}\right\}\begin{array}{l}R\text{-mod}\\ \mathrm{mod}\,R\end{array}$

Tor

$\mathfrak{p}$: prime ideal of R $\Lambda_\mathfrak{p}$: $R_\mathfrak{p}$-order

$$\begin{tikzcd} (\ )_\mathfrak{p} := - \otimes_R R_\mathfrak{p} : \mathrm{mod}\,\Lambda \arrow[r] \arrow[d, phantom, "\cup"] & \mathrm{mod}\,\Lambda_\mathfrak{p} \arrow[d, phantom, "\cup"] \\ \mathrm{CM}(\Lambda) \arrow[r] & \mathrm{CM}(\Lambda_\mathfrak{p}) \end{tikzcd}$$

<u>Def</u> Λ: R-order Λ: <u>isolated singularity</u> $\overset{\text{def}}{\Longleftrightarrow}$ $\forall \mathfrak{p} \neq \mathfrak{m}$, $\Lambda_\mathfrak{p}$ is a non-singular $R_\mathfrak{p}$-order

iso. sing

$\boxed{\text{Assume } R\text{: complete}}$

<u>ThmA</u> Λ: R-order $\boxed{\text{iso sing}}$

(1) $\exists$ equivalence $\tau: \underline{\mathrm{CM}}(\Lambda) \xrightarrow{\sim} \overline{\mathrm{CM}}(\Lambda)$ <u>AR translation</u>

$\boxed{\S 2\sim4}$ (2) $\exists$ functorial iso $\mathrm{Hom}_\Lambda(X,Y) \xrightarrow{\sim} D\,\mathrm{Ext}^1_\Lambda(Y, \tau X)$ $(X, Y \in \mathrm{CM}(\Lambda))$

<u>AR duality</u>

(3) $\mathrm{CM}(\Lambda)$ has almost split sequences

<u>ThmB</u> Λ: R-order $\circlearrowright$

(1) Λ: iso. sing

$\boxed{\S 4}$ (2) $\mathrm{CM}(\Lambda)$ has almost split seq

(3) $\underline{\mathrm{CM}}(\Lambda)$ is Hom-finite

$\boxed{\S 5}$ <u>Cor C</u> Λ: R-order representation-finite $\Rightarrow$ isolated sing

<u>Rem</u> Λ: R-order

$\mathrm{CM}_0(\Lambda) := \{X \in \mathrm{CM}(\Lambda) \mid \forall \mathfrak{p} \neq \mathfrak{m}, X_\mathfrak{p} \in \mathrm{proj}\,\Lambda_\mathfrak{p}\}$

$\mathrm{CM}^0(\Lambda) := \{ \text{———————} X_\mathfrak{p} \in \mathrm{add}\,W_{\Lambda_\mathfrak{p}}\}$

$X_\mathfrak{p}$ is a summand of $(W_{\Lambda_\mathfrak{p}})^{\oplus \ell}$

<u>Rem</u> Λ: isolated sing $\Rightarrow \mathrm{CM}_0(\Lambda) = \mathrm{CM}(\Lambda) = \mathrm{CM}^0(\Lambda)$

Thm A' (1) $\exists$ equivalence $\tau : \underline{CM}_0(\Lambda) \xrightarrow{\sim} \overline{CM}^0(\Lambda)$

(2) $\exists$ funct. iso $\underline{Hom}_\Lambda(X,Y) \simeq D\,Ext^1_\Lambda(Y, \tau X)$

$(X \in CM_0(\Lambda),\ Y \in CM(\Lambda))$

(3) $\forall X \in CM_0(\Lambda)$, $\exists$ almost split seq

indec non-proj $\quad 0 \to \underset{\overset{\cap}{CM^0}}{\tau X} \to M \to \underset{\overset{\cap}{CM_0}}{X} \to 0$ in $CM(\Lambda)$

§2 AR translation

$\tau = (\underset{\downarrow}{R\text{-dual}})\underbrace{(\Lambda\text{-dual})}_{\Omega^d \mathrm{Tr}}$ $\quad ()^* = \mathrm{Hom}_\Lambda(-,\Lambda) : \underset{\cup}{\mathrm{mod}}\,\Lambda \to \underset{\cup}{\mathrm{mod}}\,\Lambda^{op}$ far from duality!

$\mathrm{Hom}_R(-,R) = ()^\vee$ $\qquad \boxed{\mathrm{proj}\,\Lambda \xrightarrow{\sim} \mathrm{proj}\,\Lambda^{op}}$ duality

- For $X \in \mathrm{mod}\,\Lambda$, $\quad \underbrace{P_1 \xrightarrow{f} P_0}_{\mathrm{proj}\,\Lambda} \to X \to 0$: exact $\quad P_0^* \xrightarrow{f^*} P_1^* \to \mathrm{Tr}X \to 0$: exact.

  Define $\mathrm{Tr}X \in \mathrm{mod}\,\Lambda^{op}$

<u>caution</u>
- $\mathrm{Tr}X$ depends on proj resol.
- $\mathrm{Tr} : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda^{op}$ is not a functor

<u>Def</u> <u>$\underline{\mathrm{mod}}\,\Lambda$: stable category</u>

obj : same with $\mathrm{mod}\,\Lambda$.

morphism : $\underline{\mathrm{Hom}}_\Lambda(X,Y) := \dfrac{\mathrm{Hom}_\Lambda(X,Y)}{P(X,Y)}$ $\quad P(X,Y) := \left\{ \begin{tikzcd} X \arrow[rr] \arrow[dr, dashed] & \circlearrowright & Y \\ & \exists P \in \mathrm{proj}\,\Lambda \arrow[ur, dashed] & \end{tikzcd} \right\}$

<u>Exercise</u> $\underline{\mathrm{mod}}\,\Lambda$ has a structure of additive category

<u>Notation</u> : For a full subcat $\mathcal{C}$ of $\mathrm{mod}\,\Lambda$,

we denote by $\underline{\mathcal{C}}$ the corresponding full subcat of $\underline{\mathrm{mod}}\,\Lambda$

<u>Thm 2.1</u> [Auslander-Bridger]

- Tr gives a functor $\underline{\mathrm{mod}}\,\Lambda \to \underline{\mathrm{mod}}\,\Lambda^{op}$ which does not depend on choice of proj. resolutions
- Tr gives a duality $\underline{\mathrm{mod}}\,\Lambda \xleftrightarrow{\sim} \underline{\mathrm{mod}}\,\Lambda^{op}$ $\quad (\mathrm{Tr})^2 \simeq \mathrm{id}$

<u>Sketch</u> ① For $X \in \mathrm{mod}\,\Lambda$, $\mathrm{Tr}X \in \underline{\mathrm{mod}}\,\Lambda$ does not depend on choice of a proj resol.

②

$$\begin{tikzcd} P_1 \arrow[r, "f"] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowright"] & P_0 \arrow[r] \arrow[d, "\wr"] & X \arrow[r] \arrow[d, dashed, "\wr"] & 0 & P_0^* \arrow[r, "f^*"] \arrow[dl, squiggly, "()^*"] & P_1^* \arrow[r] & \mathrm{Tr}X \arrow[r] & 0 \\ P_1^{**} \arrow[r, "f^{**}"] & P_0^{**} \arrow[r] & \mathrm{Tr}\mathrm{Tr}X \arrow[r] & 0 & & & & \end{tikzcd}$$

$P_1 \xrightarrow{f} P_0 \to X \to 0 \overset{()^*}{\rightsquigarrow} P_0^* \xrightarrow{f^*} P_1^* \to \mathrm{Tr}X \to 0$

Recall Λ : R-order $\quad \Lambda$: non-singular $\Leftrightarrow \mathrm{CM}(\Lambda) = \mathrm{proj}\,\Lambda \Leftrightarrow \underline{\mathrm{CM}}(\Lambda) = 0$

Prop 2.2 Λ: iso. sing $\Rightarrow$ · $\underline{\mathrm{CM}}(\Lambda)$ is $\underline{\text{Hom-finite}}$

$(\text{i.e. } {}^{\forall}X, Y \in \mathrm{CM}(\Lambda)\ \underline{\mathrm{Hom}}_\Lambda(X,Y) \in \mathrm{f.l}R)$

· $\boxed{\begin{array}{l}\mathrm{Ext}^i_\Lambda(X,Y) \in \mathrm{f.l}R \\ (i>0, X \in \mathrm{CM}(\Lambda), Y \in \mathrm{mod}\Lambda)\end{array}}$ → これを示す (∗)

· $\mathrm{Ext}^i_{\Lambda^{op}}(\mathrm{Tr}X, X) \in \mathrm{f.l}R$

$(i>0, X \in \mathrm{CM}(\Lambda), Z \in \mathrm{mod}\Lambda^{op})$

proof. ① $M \in \mathrm{mod}R$. $M \in \mathrm{f.l}R \Leftrightarrow X_{\mathfrak{p}} = 0\ ({}^{\forall}\mathfrak{p} \neq \mathfrak{m})$

② $(\mathrm{Ext}^i_\Lambda(X,Y))_{\mathfrak{p}} \simeq \mathrm{Ext}^i_{\Lambda_{\mathfrak{p}}}(\underset{\mathrm{CM}(\Lambda_{\mathfrak{p}}) = \mathrm{proj}\,\Lambda_{\mathfrak{p}}}{\underset{\cap}{\textcircled{X_{\mathfrak{p}}}}}, Y_{\mathfrak{p}}) = 0$

(∗)のpf. □

$(\mathrm{Tr}_\Lambda X)_{\mathfrak{p}} = \mathrm{Tr}_{\Lambda_{\mathfrak{p}}}(X_{\mathfrak{p}})$

$$\begin{tikzcd}
\mathrm{Ext}^i_\Lambda(-,Y): \mathrm{mod}\Lambda \arrow[r] \arrow[dr] & \mathrm{mod}R \\
& \underline{\mathrm{mod}}\Lambda \arrow[u, dashed]
\end{tikzcd}$$

Def $X \in \mathrm{mod}\Lambda \quad 0 \to \Omega X \to \underset{\mathrm{proj}\Lambda}{\underset{\cap}{P}} \to X \to 0$

Ω gives a functor $\underline{\mathrm{mod}}\Lambda \to \underline{\mathrm{mod}}\Lambda$ does not depend on choice of proj resol

Thm 2.3 Λ: iso sing $\Omega^d\mathrm{Tr}: \underline{\mathrm{CM}}(\Lambda) \xrightarrow{\sim} \underline{\mathrm{CM}}(\Lambda^{op})$ duality

$\mathcal{X}_d := \{X \in \mathrm{mod}\Lambda \mid \mathrm{Ext}^{1\sim d}_\Lambda(X,\Lambda) = 0\}$ $\quad F_d := \{X \in \mathrm{mod}\Lambda \mid \mathrm{Tr}X \in \mathcal{X}^{op}_d\}$

$\mathcal{X}^{op}_d \qquad \Lambda^{op} \qquad\qquad F^{op}_d \qquad \Lambda^{op} \qquad \mathcal{X}_d.$

$\boxed{\text{d-torsionfree}}$

Thm 2.4 Λ: isosing $\Rightarrow \mathrm{CM}(\Lambda) = F_d$

Lem 2.5 (1) $\mathrm{Tr}: \underline{F_d} \simeq \underline{\mathcal{X}^{op}_d}$ (immediate from $\boxed{\mathrm{Tr}: \underline{\mathrm{mod}}\Lambda \simeq \underline{\mathrm{mod}}\Lambda^{op}}$)

(2) $\Omega^d: \underline{\mathcal{X}_d} \simeq \underline{F_d}$

proof of (2)

Show $\mathrm{Tr}\Omega^d: \underline{\mathcal{X}_d} \simeq \underline{\mathcal{X}^{op}_d}$ Then (2) follows from (1) and this.

$$\begin{tikzcd}
{}^{\forall}X \in \mathcal{X}_d & P_{d+1} \arrow[r] \arrow[d, phantom, "\wr"] & P_d \arrow[r] \arrow[d, phantom, "\wr"] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, phantom, "\wr"] & X \arrow[r] \arrow[d, dashed, no head] & 0 \\
& P^{**}_{d+1} \arrow[r] & P^{**}_d \arrow[r] & \cdots \arrow[r] & P^{**}_0 \arrow[r] & (\mathrm{Tr}\Omega^d)^2X \arrow[r] & 0
\end{tikzcd}$$

proj. res $\xrightarrow{(\ )^*}$ $P^*_0 \to \cdots \to P^*_{d+1} \to \mathrm{Tr}\Omega^d X \to 0$

: exact by $X \in \mathcal{X}_d$ $\quad (\ )^*$

□

pf of 2.3 $\mathrm{CM}(\Lambda) \underset{2.4}{=} \mathcal{F}_d \xrightarrow[\mathrm{Tr}]{\sim} \mathcal{X}_d^{op} \xrightarrow[{}_\llcorner\Omega^d]{\sim} \mathcal{F}_d^{op} \underset{2.4}{=} \mathrm{CM}(\Lambda^{op})$

□

Prop2.6 (AB sequence) $\forall X \in \mathrm{mod}\,\Lambda$.

$\exists$ exact seq $0 \to \mathrm{Ext}^1_{\Lambda^{op}}(\mathrm{Tr}X, \Lambda) \to X \xrightarrow{ev} X^{**} \to \mathrm{Ext}^2_{\Lambda^{op}}(\mathrm{Tr}X, \Lambda) \to 0$

Exercise

pf of 2.4 $\underbrace{P_1 \to P_0}_{\mathrm{proj}\,\Lambda} \to X \to 0$

$X \in \mathrm{mod}\,\Lambda$

$$\begin{tikzcd}
Q_{d-1} \arrow[r] & \cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[rr] \arrow[dr] & & P_0^* \arrow[r] & P_1^* \arrow[r] & \mathrm{Tr}X \arrow[r] & 0 \\
& & & & X^* \arrow[ur] \arrow[dr] & & & & \\
& & & 0 \arrow[ur] & & 0 & & &
\end{tikzcd}$$

$\underbrace{Q_{d-1} \to \cdots \to Q_1 \to Q_0}_{\mathrm{proj}\,\Lambda^{op}}$

$E^i := \mathrm{Ext}^i(\mathrm{Tr}X, \Lambda)$ $\boxed{0 \to X \to \boxed{Q_0^* \to Q_1^* \to \cdots \to Q_{d-2}^* \to Q_{d-1}^*}}$ $(*)$

homology E^1 E^2 E^3 E^d

① $X \in \mathcal{F}_d \Rightarrow X \in \mathrm{CM}(\Lambda)$

$(*)$: exact

② $X \in \mathrm{CM}(\Lambda) \Rightarrow E^1 \sim E^d = 0$ をいう.

Show: $\boxed{n \leq d \quad E^1 \sim E^{n-1} = 0 \Rightarrow E^n = 0}$

$\boxed{\text{Assume } d \geq 2}$ $\Downarrow$

$0 \to \underbrace{X \to Q_0^* \to \cdots \to Q_{n-3}^*}_{\substack{\mathrm{CM}(\Lambda) \\ \text{depth } d}} \to \underbrace{({}_\llcorner\Omega^n \mathrm{Tr}X)^*}_{\text{depth} \geq 2} \to E^n \to 0$: exact

E^n: depth ≥ 1

$E^n \in \text{f.}\ell R$ $\Rightarrow E^n = 0$ □

${}_\llcorner\Omega^d \mathrm{Tr}: \underline{\mathrm{CM}}(\Lambda) \xleftrightarrow{\sim} \underline{\mathrm{CM}}(\Lambda^{op})$

$(\ )^\vee : \mathrm{CM}(\Lambda) \xleftrightarrow{\sim} \mathrm{CM}(\Lambda^{op})$

$\omega_\Lambda \longleftrightarrow \Lambda$

Def $\overline{\mathrm{CM}}(\Lambda)$ costable category

obj: same with $\mathrm{CM}(\Lambda)$

morph: $\overline{\mathrm{Hom}}_\Lambda(X, Y) := \dfrac{\mathrm{Hom}_\Lambda(X,Y)}{I(X,Y)}$ $\quad I(X,Y) := \left\{ \begin{tikzcd} X \arrow[rr] \arrow[dr, dashed] & \circlearrowleft & Y \\ & \omega_\Lambda^{\oplus} \arrow[ur, dashed] & \end{tikzcd} \right\}$

Def-Thm $\tau := \left(\underline{\mathrm{CM}}(\Lambda) \xrightarrow[\sim]{\Omega^d \mathrm{Tr}} \underline{\mathrm{CM}}(\Lambda^{op}) \xrightarrow[\sim]{(\,)^\vee} \overline{\mathrm{CM}}(\Lambda)\right)_{\text{equiv.}}$

$\underline{\text{AR translation}}$

In the rest. we define $\left.\begin{array}{l}\mathrm{Tr}\,X\\ \Omega X\end{array}\right\}$ mod Λ by minimal proj resolutions

Cor2.8 R: complete Λ: iso sing

τ gives a bijection $\{\text{indec non-proj CM } \Lambda\text{-mod}\}$

$\xleftrightarrow{1:1} \{\text{indec } \text{non-inj} \text{ CM } \Lambda\text{-mod}\}$ summand of $W_\Lambda^{\oplus}$

§3 AR duality

$$\mathrm{Ext}^i_R(\underset{\substack{\parallel \\ R/m}}{k}, R) = \begin{cases} 0 & (i<d) \\ k & (i=d) \end{cases}$$

$D := \mathrm{Ext}^d_R(-,R)$ f.l.R $\xrightarrow{\simeq}$ f.l.R Matlis duality

Thm3.1 Λ: iso sing

$\exists$ functorial iso $\underline{\mathrm{Hom}}_\Lambda(X,Y) \simeq D\mathrm{Ext}^1_\Lambda(Y, \tau X)$

$(\forall X, Y \in \mathrm{CM}(\Lambda))$ AR duality

$\frac{\uparrow ②}{\downarrow ③}$

preliminaries ① $\mathrm{Hom}_R(\mathrm{Tor}^\Lambda_i(Y,Z), I) \simeq \mathrm{Ext}^i_{\Lambda^{op}}(Z, \mathrm{Hom}_R(Y,I))$ functorial iso.

$Y \in \mathrm{Mod}\,\Lambda$. $Z \in \mathrm{Mod}\,\Lambda^{op}$. $I \in \mathrm{Inj}\,R$ $i \geq 0$ exercise

② $0 \to R \to \boxed{I^0 \to \cdots} \to I^d \to 0$: min. inj. resol. (**)

$\mathrm{Hom}_R(\text{f.l.}R, I^i) = \begin{cases} 0 & i \neq d \\ D & i = d \end{cases}$ exercise Use (*)

③ $\mathrm{Tor}^\Lambda_1(Y, \mathrm{Tr}\,X) \simeq \underline{\mathrm{Hom}}_\Lambda(X,Y)$

$X, Y \in \mathrm{mod}\,\Lambda$

Sketch. $Y \otimes X^* \xrightarrow{a_{X,Y}} \mathrm{Hom}_\Lambda(X,Y) \to \underline{\mathrm{Hom}}_\Lambda(X,Y) \to 0$: exact.

$y \otimes f \longmapsto (x \mapsto y f(x))$

X: proj $\Rightarrow$ $a_{X,Y}$: iso.

$$\begin{tikzcd}
P_1 \arrow[r] \arrow[ddddr, "\mathrm{Hom}_\Lambda(-,Y)"'] & P_0 \arrow[r] & X \arrow[r] & 0 & & & \\
& 0 \arrow[r] & X^* \arrow[r] & P_0^* \arrow[r] & P_1^* \arrow[r] \arrow[ddl, "Y\otimes_\Lambda"] & \mathrm{Tr}\,X \arrow[r] & 0 \\
& & \text{homology}\ \boxed{\mathrm{Tor}^\Lambda_1(Y,\mathrm{Tr}X)} \arrow[ddd, bend right=60, "\text{iso}"'] & & & & \\
& & Y \otimes_\Lambda X^* \arrow[r] \arrow[d, "a_{X,Y}"'] & Y \otimes_\Lambda P_0^* \arrow[r] \arrow[d, "a_{P_0,Y}"'] & Y \otimes_\Lambda P_1^* \arrow[d, "a_{P_1,Y}"'] & & \\
& 0 \arrow[r] & \mathrm{Hom}_\Lambda(X,Y) \arrow[r] \arrow[d] & (P_0,Y) \arrow[r] & (P_1,Y) & :\text{exact} & \\
& & \boxed{\underline{\mathrm{Hom}}_\Lambda(X,Y)} \arrow[r] & 0 & :\text{exact} & &
\end{tikzcd}$$

<u>pf. of 3.1</u> $\mathrm{Ext}^1_\Lambda(Y, \underset{\underset{(\Omega^d \mathrm{Tr} X)^\vee}{\parallel}}{\tau X}) \simeq \mathrm{Ext}^1_{\Lambda^{op}}(\Omega X, Y^\vee) \simeq \mathrm{Ext}^{d+1}_{\Lambda^{op}}(\mathrm{Tr} X, \underset{\downarrow}{\boxed{Y^\vee}})$

inj resol をとる

$(\ )^\vee : \mathrm{CM}(\Lambda) \xrightarrow{\sim} \mathrm{CM}(\Lambda^{op})$ exact duality.

$\mathrm{Hom}_R(Y, (**))$ $0 \to Y^\vee \to \underset{\text{Show: } \boxed{\text{inj } \Lambda^{op}\text{-mod}}}{\boxed{{}_R(Y, I^0) \to \cdots \to {}_R(Y, I^{d-1})}} \to {}_R(Y, I^d) \to 0$ exact by

$$\left(\begin{array}{l} \mathrm{Ext}^1_{\Lambda^{op}}(-, {}_R(Y, I^i)) \overset{①}{\simeq} \mathrm{Hom}_R(\underbrace{\mathrm{Tor}^\Lambda_1(Y, -)}_{\text{f.l.}R}, I^i) \overset{②}{=} 0 \quad i \neq d \\ \left(\begin{array}{lll} \Lambda : \text{iso sing} & \underset{(p \neq m)}{\mathrm{Tor}^\Lambda_1(Y, -)_p} = \mathrm{Tor}^{\Lambda_p}_1(\underset{\underset{\text{proj } \Lambda_p}{\Uparrow}}{Y_p}, (-)_p) = 0 \\ Y \in \mathrm{CM}(\Lambda) & \end{array} \right) \end{array} \right)$$

$$\mathrm{Ext}^{d+1}_{\Lambda^{op}}(\mathrm{Tr} X, Y^\vee) \underset{\uparrow}{\simeq} \mathrm{Ext}^1_{\Lambda^{op}}(\mathrm{Tr} X, {}_R(Y, I^d)) \underset{①}{\simeq} \mathrm{Hom}_R(\mathrm{Tor}^\Lambda_1(Y, \mathrm{Tr} X), I^d)$$

(Use inj resolution above)

$$\begin{array}{c} \mathrm{Hom}_R(\mathrm{Tor}^\Lambda_1(Y, \mathrm{Tr} X), I^d) \\ \wr\!\wr \\ D(\mathrm{Tor}^\Lambda_1(Y, \mathrm{Tr} X)) \\ \wr\!\wr \ ③ \\ D(\underset{\underset{\text{f.l.}R}{\Uparrow}}{\boxed{\underline{\mathrm{Hom}}_\Lambda(X, Y)}}) \end{array}$$

§4 Almost split seq

$$\begin{tikzcd} 0 \arrow[r, leftrightarrow] \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & \text{split exact seq} \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] \\ \operatorname{Ext}^1_\Lambda(X,Y) \arrow[r, leftrightarrow, "1-1"] & \{\text{exact seq } 0\to Y\xrightarrow{b}M\xrightarrow{a}X\to 0\}/\simeq \end{tikzcd}$$

Recall ①

$\exists R$-bilinear map

$$\begin{tikzcd} \operatorname{Ext}^1_\Lambda(X,Y)\times\operatorname{Hom}_\Lambda(W,X) \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \operatorname{Ext}^1_\Lambda(W,Y) \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\ (\alpha, f) \arrow[r, mapsto] & \alpha f \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "b"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & M \arrow[r, "a"] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowright"] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r, "b'"] & M' \arrow[r, "a'"] & X \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} \alpha: 0 \arrow[r] & Y \arrow[r, "b"] & M \arrow[r, "a"] & X \arrow[r] & 0 \\ \alpha f: 0 \arrow[r] & Y \arrow[u, equal] \arrow[r, dotted] & M' \arrow[u, dotted, "\text{Pull back}"'] \arrow[r, dotted] & W \arrow[u, "f"'] \arrow[r] & 0 \end{tikzcd}$$

② $X, Y \in CM(\Lambda)$: indec

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "b"] & M \arrow[r, "a"] \arrow[d, phantom, "\circlearrowright"] & X \arrow[r] & 0 \\ & & & W \arrow[u, "f"'] \arrow[ul, dashed] & \end{tikzcd}$$

: alm. spl. seq

$\overset{\text{def}}{\Longleftrightarrow}$ $\begin{cases} \cdot \text{ non-split} \\ \cdot\ {}^\forall f : \underset{CM(\Lambda)}{\underset{\rotatebox{90}{$\in$}}{W}} \longrightarrow X \text{ (not split epi) factors through } \alpha \end{cases}$

Lem 4.1 $X, Y \in CM(\Lambda)$ indec $0 \neq \alpha \in \operatorname{Ext}^1_\Lambda(X,Y)$ $\circlearrowright$

(1) α is an almost split seq.

(2) $\boxed{{}^\forall f : \underset{CM(\Lambda)}{\underset{\rotatebox{90}{$\in$}}{W}} \longrightarrow X \text{ (not split epi)}, \ \alpha f = 0 \text{ in } \operatorname{Ext}^1(W,Y)}$

Thm A(3) Λ : iso sing $X \in CM(\Lambda)$ indec non-proj

$\Rightarrow {}^\exists$ alm. spl seq : $0 \to \tau X \to M \to X \to 0$.

$$\begin{tikzcd} \operatorname{End}_\Lambda(X) : \text{local} \arrow[d, phantom, "\cup"] & \underline{\operatorname{End}}_\Lambda(X) \arrow[d, two heads] & D(\underline{\operatorname{End}}_\Lambda(X)) \arrow[d, phantom, "\cup"] \arrow[r, phantom, "\underset{\sim}{\text{AR dual}}"] & \operatorname{Ext}^1_\Lambda(X, \tau X) \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\ J : \text{max ideal} \arrow[d, phantom, "\cup"] & \operatorname{End}_\Lambda(X)/J & D(\operatorname{End}_\Lambda(X)/J) \ni \beta \arrow[r, leftrightarrow] \arrow[d, phantom, "\neq"] & \alpha \arrow[d, phantom, "\neq"] \\ P(X,X) = \left\{ \begin{tikzcd}[ampersand replacement=\&] X \arrow[rr] \arrow[dr, dashed] \arrow[rr, phantom, bend right=20, "\circlearrowright"] \& \& X \\ \& \text{proj} \arrow[ur, dashed] \& \end{tikzcd} \right\} & & 0 & 0 \end{tikzcd}$$

We shall prove α is an alm. spl. seq.

$^{\forall} f : \underset{\mathrm{CM}(\Lambda)}{\overset{\cap}{W}} \to X$ $\boxed{\text{not split epi}}$

$\Downarrow$

$$\begin{tikzcd}
 & I^d & \\
\underline{\mathrm{End}}_\Lambda(X)/J \arrow[ur, dashed] & & \\
 & \underline{\mathrm{End}}_\Lambda(X) \ni fg : \text{non-iso} \arrow[ul] \arrow[uu, "\beta"] \arrow[ull, phantom, "\circlearrowleft"] & \\
 & \mathrm{Hom}_\Lambda(X,W) \ni g \arrow[u, "(f\cdot)"] \arrow[uul, bend left, dash] & \\
 & X \arrow[r, shift left, "g"] & W \arrow[l, shift left, "f"]
\end{tikzcd}$$

$$\begin{tikzcd}
fg \arrow[r, phantom, "\in"] & \underline{\mathrm{End}}_\Lambda(X) \\
g \arrow[u] \arrow[r, phantom, "\in"] & \underline{\mathrm{Hom}}_\Lambda(X,W) \arrow[u, "f\cdot"']
\end{tikzcd}$$

By functoriality of AR duality.

$$\begin{tikzcd}
\beta \arrow[d, mapsto] \arrow[r, phantom, "\in"] & D(\underline{\mathrm{End}}_\Lambda(X)) \arrow[r, dash, "\sim"] \arrow[d, "D(f\cdot)"] & \mathrm{Ext}^1_\Lambda(X,\tau X) \arrow[d, "(\cdot f)"] \arrow[r, phantom, "\ni"] & \alpha \arrow[d, mapsto] \arrow[lll, bend right, leftrightarrow] \\
0 \arrow[r, phantom, "\in"] & D(\underline{\mathrm{Hom}}_\Lambda(X,W)) \arrow[r, dash, "\sim"] & \mathrm{Ext}^1_\Lambda(W,\tau X) \arrow[r, phantom, "\ni"] & \alpha f \arrow[lll, bend left, leftrightarrow]
\end{tikzcd}$$

$\alpha f = 0$

α is an alm spl. seq by Lem 4.1 □

Thm B ∴ $\circlearrowleft$ (1) Λ: iso sing. (3) $\underline{\mathrm{CM}}(\Lambda)$ is Hom-finite.

(2) $\mathrm{CM}(\Lambda)$ has alm. spl. seq

Lem 4.2 Λ: not iso sing $^{\exists}\mathfrak{p} \neq \mathfrak{m}$ $^{\exists}X \in \mathrm{CM}(\Lambda)$ $X_{\mathfrak{p}} \notin \mathrm{proj}\,\Lambda_{\mathfrak{p}}$

Rem $(\ )_{\mathfrak{p}} : \mathrm{CM}(\Lambda) \to \mathrm{CM}(\Lambda_{\mathfrak{p}})$ dense up to summands | $(\ )_{\mathfrak{p}} : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda_{\mathfrak{p}}$

Λ: R-order $\quad {}^{\exists}Y \quad {}^{\forall}X$ is a summand of $Y_{\mathfrak{p}}$ | dense.

proof of 4.2 $^{\exists}\mathfrak{p} \neq \mathfrak{m}$ $\Lambda_{\mathfrak{p}}$ singular $\mathrm{ht}\,\mathfrak{p} = d-1$ $\mathrm{gl.dim}\,\Lambda_{\mathfrak{p}} > d-1$.

$^{\exists}S$: simple $\Lambda_{\mathfrak{p}}$-module $\mathrm{pd}_{\Lambda_{\mathfrak{p}}}(S) \geq d$.

$^{\exists}X \in \mathrm{mod}\,\Lambda$ $X_{\mathfrak{p}} \simeq S$ without loss of generality,

we may assume depth $X \geq 1$.

(Replace X by X/X' ← maximal fin. length sub)

$Y := \Omega^{d-1}(X) \in \mathrm{CM}(\Lambda)$

$Y_{\mathfrak{p}} := \Omega^{d-1}_{\Lambda_{\mathfrak{p}}}(X_{\mathfrak{p}}) \simeq \Omega^{d-1}_{\Lambda_{\mathfrak{p}}}(S) \notin \mathrm{proj}\,\Lambda_{\mathfrak{p}}$ □

<u>proof of 3.1</u> $(3) \Rightarrow (1)$ Assume Λ is not iso sing

Take $\mathfrak{p} \neq \mathfrak{m}$ and $X \in \mathrm{CM}(\Lambda)$ in <u>Lem 4.2</u>

$\underline{\mathrm{End}}_\Lambda(X)_\mathfrak{p} \simeq \underline{\mathrm{End}}_{\Lambda_\mathfrak{p}}(\underset{\substack{\uparrow \\ \text{non-proj}}}{X_\mathfrak{p}}) \neq 0 \quad \underline{\mathrm{End}}_\Lambda(X) \notin \text{f.l.} R$ ⌟

$(2) \Rightarrow (1)$ In <u>Lem 4.1</u>

$(1) \Leftrightarrow (2) \Leftrightarrow (3)$ $\forall \beta_{\neq 0} \in \mathrm{Ext}^1_\Lambda(X, \underset{\substack{\Uparrow \\ \mathrm{CM}(\Lambda)}}{Z})$ $\exists g : \mathrm{Hom}_\Lambda(Z, Y)$ $\alpha = g\beta$

Assume Λ is not iso sing

Take $\mathfrak{p} \neq \mathfrak{m}$ and $X \in \mathrm{CM}(\Lambda)$ in Lem 4.2 $(\boxed{X_\mathfrak{p} \notin \mathrm{proj}\, \Lambda_\mathfrak{p}})$

Assume $\alpha \in \mathrm{Ext}^1_\Lambda(X, Y)$ is an alm. spl. seq.

$0 \to \Omega X \to P \to X \to 0$ Let $\beta \in \mathrm{Ext}^1_\Lambda(X, \Omega X)$ be proj. resol. $\mathrm{Ext}^1_{\Lambda_\mathfrak{p}}(X_\mathfrak{p}, \Omega_{\Lambda_\mathfrak{p}}(X_\mathfrak{p})) \neq 0$

$\beta \neq 0$ in $\mathrm{Ext}^1_\Lambda(X, \Omega X)_\mathfrak{p}$

Fix $r \in \mathfrak{m} \setminus \mathfrak{p}$ $\quad r^n \beta \neq 0 \; (\forall n \geq 0)$

By <u>4.1</u> $(1) \Rightarrow (3)$ $\quad \exists g_n \in \mathrm{Hom}_\Lambda(\Omega X, Y)$

$\alpha = g_n(r^n \beta) = r^n (g_n \beta) \in r^n \mathrm{Ext}^1_\Lambda(X, Y)$

$\alpha \in \bigcap_{n \geq 0} \mathfrak{m}^n \mathrm{Ext}^1_\Lambda(X, Y) = 0$

$\alpha = 0$ × □ <u>Thm B おわり</u>

§5 Representation-finite case.

Λ: R-order R: complete.

Def Λ: representation-finite $\overset{\text{def}}{\Longleftrightarrow}$ $\exists$only finitely many isomorphism classes of indecomp CM Λ-modules.

Cor C Λ: rep-finite $\Rightarrow$ iso sing

Prop 5.1 Λ: rep-fin $\Rightarrow$ $CM(\Lambda)$ has almost split sequences

(Thm B + Prop 5.1 $\Rightarrow$ Cor C)

proof Enough to show: $\forall X \in CM(\Lambda)$: indec $\exists a: Y \to X$ (with $Y \in CM(\Lambda)$)

$$\begin{tikzcd} Y \arrow[r, "a"] & X \\ & W \arrow[ul, dashed] \arrow[u, "f"'] \end{tikzcd}$$

$\left\{ \begin{array}{l} \cdot a \text{ is not a split epi.} \\ \cdot \forall f: W \to X \text{ (not split epi)} \\ \text{factors through } a \end{array} \right.$

$M := \bigoplus_{Y \in CM(\Lambda) \text{ indec}} Y \in CM(\Lambda)$

$\mathrm{rad}(M,X) := \{ f \in \underset{\text{mod } R}{\boxed{\mathrm{Hom}_\Lambda(M,X)}} \mid f \text{ is not a split epi} \}$

X: indec Krull-Schmidt $\Rightarrow$ $\mathrm{rad}(M,X)$ is a sub R-module of $\mathrm{Hom}_\Lambda(M,X)$

($\because \mathrm{End}_\Lambda(X)$: local)

Take generators $a_1, a_2, \cdots, a_n$ of the R-mod $\mathrm{rad}(M,X)$

$a := (a_1 a_2 \cdots a_n): M^{\oplus n} \longrightarrow X$

$\cdot a$ is not a split epi ($\because$ Krull-Schmidt)

$\cdot \forall f: M \to X$ not split epi $\exists r_1 \sim r_n \in R$. $f = r_1 a_1 + \cdots + r_n a_n$

$f \in \mathrm{rad}(M,X)$

$$\begin{tikzcd} M^{\oplus n} \arrow[rr, "(a_1 \cdots a_n)"] & \circlearrowleft & X \\ & M \arrow[ul, "\begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}"] \arrow[ur, "f"'] & \end{tikzcd}$$

② is checked $\square$

Rem Λ: rep-fin. $M := \bigoplus_{X \in CM(\Lambda): \text{indec}} X$ $\Gamma := \mathrm{End}_\Lambda(M)$ Auslander algebra of Λ.

$\boxed{\text{almost split sequences in } CM(\Lambda) = \text{min proj resolution of simple } \Gamma\text{-modules}}$

$0 \to \underset{\overset{\oplus}{\wedge}\ M}{X} \to Y \to Z \to 0$: almost split seq in $CM(\Lambda)$

$$\xrightarrow[\mathrm{Hom}_\Lambda(M,-)]{\rightsquigarrow} 0 \to \underbrace{\mathrm{Hom}_\Lambda(M,X) \to \mathrm{Hom}_\Lambda(M,Y) \to \mathrm{Hom}_\Lambda(M,Z)}_{\text{proj } \Gamma} \to \underbrace{\frac{\mathrm{Hom}_\Lambda(M,Z)}{\mathrm{rad}_\Lambda(M,Z)}}_{\text{simple } \Gamma\text{-module.}} \to 0 \text{ : exact}$$

$\Gamma = \mathrm{Hom}_\Lambda(M,M)$ $\oplus$ proj Γ

$\mathrm{Hom}_\Lambda(M,Y) \to \mathrm{rad}(M,Z) \subset \mathrm{Hom}_\Lambda(M,Z)$

$(\text{AR quiver of } \Lambda) = (\underline{\overset{}{\text{quiver of } \Gamma}})$

$\frac{\mathrm{rad}(\ \ )}{\mathrm{rad}^2(\ \ )}$ $\quad$ $\downarrow$ defined in Mizuno's lecture $\quad \frac{\mathrm{rad}\,\Gamma}{\mathrm{rad}^2\Gamma}$

<u>Recall</u> Assume $k = R/m$ is algebraically closed

Γ: module-finite R-algebra $\quad J := $ Jacobson radical

quiver of Γ: $\Gamma/J \simeq \underset{\substack{\rotatebox{90}{$\in$} \\ e_{11}+\cdots+e_{1l_1}}}{M_{l_1}(k)} \times \cdots \times \underset{\substack{\rotatebox{90}{$\in$} \\ e_{n1}+\cdots+e_{nl_n}}}{M_{l_n}(k)}$ (Artin-Wedd)

$\left[\begin{array}{l} \text{vertices}: 1 \sim n \\ \text{arrows}: \quad d_{ij} := \dim_k (e_{i1}(J/J^2)e_{j1}) \quad (i,j = 1 \sim n) \\ \qquad \text{Draw } d_{ij} \text{ arrows from } i \text{ to } j \end{array}\right.$

<u>Example</u> $\underset{\text{finite sub}}{G \subset GL_2(k)}$ char $k = 0$. $\quad S := k[\![x,y]\!] \supset S^G =: \Lambda$

<u>Hoshi's lecture</u> $\cdot\ \Lambda$ is rep-fin ($\mathrm{add}\, S = CM(\Lambda)$)

$\cdot\ S * G$ is an Auslander alg ($\mathrm{End}_\Lambda(S) \simeq S * G$)

(S, n) $\qquad \cdot\ (\text{AR quiver of } \Lambda) = (\text{quiver } S * G)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

$\qquad\qquad\qquad\qquad\qquad\qquad (\text{McKay quiver of } G)$

$\Gamma = S * G = S \otimes_k kG. \quad \boxed{\Gamma/J \simeq kG}$

$$\begin{array}{rcll} \cup & & \cup & \\ J & = & \cancel{n \otimes_k kG} & kG \otimes n \\ \cup & & \cup & \quad \cup^{k} \\ J^2 & = & \cancel{n^2 \otimes_k kG} & kG \otimes_k n^2 \end{array}$$

$\Gamma/J = kG = \overset{\substack{e_{11}+\cdots+e_{1l_1} \\ \rotatebox{90}{$\ni$}}}{M_{l_1}(k)} \times \cdots \times \overset{\substack{e_{n1}+\cdots+e_{nl_n} \\ \rotatebox{90}{$\ni$}}}{M_{l_n}(k)} \qquad V_1 \qquad V_n \qquad$ irreducible rep G

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \qquad\ \ \| \qquad\qquad\qquad \|$

quiver $S*G$. vertices: $1 \sim n \longleftrightarrow e_{11}(kG) \sim e_{n1}(kG)$: simple kG-module

$J/J^2 \simeq kG \otimes \underbrace{\left(\frac{n}{n^2}\right)}_{V = \langle x, y\rangle} \qquad \underbrace{e_{i1}(kG}_{V_i}\otimes V)e_{j1} = (V_i \otimes V)e_{j1} = \left(\begin{array}{l}\text{multiplicity of } V_j \\ \text{in } V_i \otimes V\end{array}\right)$

$\square$

<u>Rem</u> For $\underset{(\text{non rep-fin})}{\text{general}}$ order, we can prove existence of almost split sequences

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ cat of functors $F: CM(\Lambda) \to Ab$

by a similar way as in <u>Prop 5.1</u> by using functor categories $\mathrm{mod}(CM(\Lambda))$

instead of $\mathrm{mod}\,\Gamma$.

§6 Gorenstein and symm orders

fin. dim. alg

Def Λ: R-order Λ: Gorenstein $\overset{\text{def}}{\Longleftrightarrow}$ $\omega_\Lambda \in \operatorname{proj}\Lambda$ $\longleftrightarrow$ selfinj

Λ: symmetric $\overset{\text{def}}{\Longleftrightarrow}$ $\Lambda \simeq \omega_\Lambda$ as (Λ,Λ)-modules $\longleftrightarrow$ symm

Λ: Iwanaga-Gorenstein

$\overset{\text{def}}{\Longleftrightarrow}$ $\operatorname{inj}\dim_\Lambda \Lambda < \infty$ & $\operatorname{inj}\dim \Lambda_\Lambda < \infty$ $\longleftrightarrow$ Iwanaga -Gorenstein

symm $\Rightarrow$ Gorenstein $\Rightarrow$ Iwanaga-Gorenstein

$(\operatorname{inj}\dim_\Lambda \Lambda = d = \operatorname{inj}\dim \Lambda_\Lambda)$

properties Λ: Gorenstein order isolated sing

$\Omega : \underline{CM}(\Lambda) \xrightarrow{\sim} \underline{CM}(\Lambda)$

- $\underline{CM}(\Lambda) = \overline{CM}(\Lambda)$

$\tau : \underline{CM}(\Lambda) \xrightarrow{\sim} \underline{CM}(\Lambda)$

Prop Λ: symm R-order isolated sing $\Rightarrow$ (1) $\tau \simeq \Omega^{2-d}$

proof $\tau = (\ )^\vee \Omega^d \operatorname{Tr} \simeq (\ )^\vee \Omega^{d-2} (\ )^* \overset{\times}{\simeq} \Omega^{2-d} (\ )^\vee (\ )^* \simeq \Omega^{2-d}$.

$\Omega^2 \operatorname{Tr} \simeq (\ )^*$ $\quad (\ )^\vee \Omega \simeq \Omega^{-1} (\ )^\vee$ $\quad (\ )^* = \operatorname{Hom}_\Lambda(-, \underset{\omega_\Lambda}{\overset{\shortparallel}{\Lambda}}) = (\ )^\vee$ $\quad \square$

(2) $\boxed{\begin{array}{l} \exists \text{ functorial iso.} \\ \underline{\operatorname{Hom}}_\Lambda(X,Y) \simeq D\underline{\operatorname{Hom}}_\Lambda(Y, \Omega^{1-d}X) \end{array}}$ $\quad (X, Y \in CM_0(\Lambda))$

$\Omega^{1-d}X = X[d-1]$ $\qquad CM_0(\Lambda) = \{X \in CM(\Lambda) \mid \begin{array}{l}\forall \mathfrak{p} \neq \mathfrak{m}. \\ X_\mathfrak{p} \in \operatorname{proj}\Lambda_\mathfrak{p}\end{array}\}$

Rem Λ: Gorenstein $\Rightarrow$ $\underline{CM}(\Lambda)$ has a natural structure of triangulated category

$\begin{cases} \cdot\ [1] = \Omega^{-1} : \underline{CM}(\Lambda) \xrightarrow{\sim} \underline{CM}(\Lambda) \\ \cdot \text{ triangles are given by short exact seq in } CM(\Lambda) \end{cases}$

Λ: symmetric $\Rightarrow$ $\underline{CM}(\Lambda)$ is a $(d-1)$-Calabi-Yau triangulated category
isolated sing

Application AR conjecture (Tachikawa) Λ: Gorenstein R-order $M \in CM(\Lambda)$

$\operatorname{Ext}^{>0}_\Lambda(M,M) = 0 \Rightarrow M \in \operatorname{proj}\Lambda$.

$$\boxed{\begin{array}{l}\underline{\text{Thm[Araya]}}\ \Lambda : \text{symmetric } R\text{-order} \ \ \mathrm{gl.dim}\,\Lambda_\mathfrak{p} = 1 \ (\forall \mathfrak{p} \text{ ht } 1. \text{ prime ideal}) \\ \Rightarrow \text{ARC is true}\end{array}}$$

proof. $\underset{\text{non-proj}}{M} \in CM(\Lambda)$ $\mathrm{Ext}^{>0}_\Lambda(M,M)=0.$

Take minimal $\mathfrak{p}$ s.t. $\underline{M_\mathfrak{p} \notin \mathrm{proj}\,\Lambda_\mathfrak{p}}$ $\mathrm{ht}\,\mathfrak{p} = n > 1$

$M'_\mathfrak{p} \in CM_0(\Lambda_\mathfrak{p})$

By AR duality for $CM_0(\Lambda_\mathfrak{p})$

$$\begin{aligned} 0 \neq \underline{\mathrm{Hom}}_{\Lambda_\mathfrak{p}}(M_\mathfrak{p}, M_\mathfrak{p}) &\simeq D\,\underline{\mathrm{Hom}}_{\Lambda_\mathfrak{p}}(M_\mathfrak{p}, \Omega^{1-n}_{\Lambda_\mathfrak{p}} M_\mathfrak{p}) \\ &= D\,\mathrm{Ext}^{n-1}_{\Lambda_\mathfrak{p}}(M_\mathfrak{p}, M_\mathfrak{p}). \\ &= D\,\mathrm{Ext}^{n-1}_{\Lambda}(M,M)_\mathfrak{p} = 0 \quad \times \quad \square \end{aligned}$$

$\underline{d=1}$ $\begin{pmatrix} R & m^i & m^j \\ m^n & R & m^k \\ m^l & m^n & R \end{pmatrix}$ $\underline{\text{tiled. } R\text{-order.}}$

$\underset{\substack{\cup \\ \text{simple sing type } A_n}}{\underline{\text{Bass order}}}$ $:= (\forall$ overorder is Gorenstein order) type (I) ~ (V)

(I) $M_2(k)$ $M_2(k[[x]]) \supset \Lambda_n := \left\{ \begin{array}{c|c} A_0 + A_1 x + A_2 x^2 + \cdots & A_0 \sim A_{n-1} \in K \\ A_i \in M_2(k) & \end{array} \right\}$

$\cup$

K

$\cup$ quad ext

k $\quad (k[[x]]\ k[[x]]) \underset{(2\ 1)}{\overset{(1\ 2)}{\rightleftarrows}} \Lambda_1 \rightleftarrows \Lambda_2 \rightleftarrows \cdots \rightleftarrows \Lambda_n$