Mckay graph 入門.

k : alg. closed field. char $k=0$.

$G \subset GL(2,k)$: finite subgroup.

V: k-vector space. $\dim V=2$. with basis $\{x,y\}$, $G \to \{x,y\}$.

$S = k[[x,y]]$.

$G \curvearrowright S$. $\sigma f(x,y) = f(\sigma(x),\sigma(y))$

$R := S^G = \{z \in S \mid \sigma(z)=z \text{ for } \forall\sigma \in G\}$

$S*G$: skew group ring.

$\overset{\|}{\bigoplus_{\sigma\in G} S\sigma}$ $\qquad (s_1\sigma_1)(s_2\sigma_2) = s_1\sigma_1(s_2)\sigma_1\sigma_2$

$M \in \mathrm{mod}(S*G)$ is S-module with G-action.

$\begin{cases} \sigma m = (1\sigma)m \\ sm = (se)m \end{cases}$

$M, N \in \mathrm{mod}(S*G)$, $f: M \to N$: $S*G$-hom $\Leftrightarrow$ f: G-hom and S-hom

$\mathrm{Hom}_S(M,N)$ has the structure of $S*G$-module with G-action. s.t. $(\sigma f)(m) = \sigma(f(\sigma^{-1}(m)))$

$\mathrm{Hom}_{S*G}(M,N) = \mathrm{Hom}_S(M,N)^G \Rightarrow \mathrm{Ext}^i_{S*G}(M,N) = \mathrm{Ext}^i_S(M,N)^G \Rightarrow M \in \mathrm{mod}(S*G)$

$M \in \mathrm{Proj}(S*G) \Leftrightarrow M \in \mathrm{Proj}(S)$.

$F: \mathrm{mod}(kG) \longrightarrow \mathrm{mod}(S*G)$ defined by

(ob). $F(W) = S \otimes_k W$ $\quad$: $S*G$-module

(mor) $F(f) = id_S \otimes_k f$ $\quad$: $S*G$-hom.

$\begin{cases} (s\sigma)\cdot(t\otimes w) = s\sigma(t) \otimes \sigma(w) \\ (s\sigma)\cdot(id_S \otimes f) = s \otimes \sigma(f) \end{cases}$

<u>Lem 1</u> : F gives the functor $\mathrm{mod}(kG) \to \mathrm{Proj}(S*G)$.

furthermore, this functor gives

$\{\text{iso. class of } kG\text{-modules}\} \overset{1-1}{\longleftrightarrow} \{\text{iso. class of projective } S*G\text{-module}\}$

(Proof). $W \in \mathrm{mod}(kG)$, $F(W)$: free S-module $\Rightarrow F(W) \in \mathrm{Proj}(S*G)$.

F gives the fun. $\mathrm{mod}(kG) \to \mathrm{Proj}(S*G)$ which we also denote F.

$F' : \mathrm{Proj}(S*G) \longrightarrow \mathrm{mod}(kG)$

$M \longmapsto S/\mathfrak{n} \otimes M$. $\mathfrak{n} = (x,y)S \subset S$: maximal.

$F' \cdot F$ は iso. over $\mathrm{mod}(kG)$

ETS: 「$F \cdot F'$: iso over $\mathrm{Proj}(S*G)$」

$M \in \mathrm{Proj}(S*G)$, $F \cdot F'(M) = S \otimes_k (M/\mathfrak{n}M)$

$\pi : M \to M/\mathfrak{n}M$: minimal projective cover.

$\forall N \subsetneq M$. $\pi(N) \neq M/\mathfrak{n}M$ (by NAK)

$$\begin{array}{ccc} S \otimes_k M/\mathfrak{n}M & \longrightarrow & M/\mathfrak{n}M \to 0 \text{ : proj. cover.} \\ \exists g \uparrow \ f\downarrow & \circlearrowright & \| \\ \pi : M & \longrightarrow & M/\mathfrak{n}M \to 0 \end{array}$$

M : proj $\exists g$. s.t. $fg = \mathrm{id}_M$. $\therefore M \oplus$-summand of $S \otimes_k M/\mathfrak{n}M$ ($M \overset{\oplus}{\subset} S \otimes_k M/\mathfrak{n}M$)

$\mathrm{rank}_S M = \mathrm{rank}_S\, S \otimes_k M/\mathfrak{n}M$

$\Rightarrow M \cong S \otimes_k M/\mathfrak{n}M$ //.

Rem 2.

Lem1 gives $\hat{G} := \{$iso. class of irr. representation of $G\} = \{V_0, \dots, V_d\} \overset{1:1}{\longleftrightarrow} \{$iso. class of indec. proj. $S*G$-module$\} = \{P_0, \dots, P_d\}$.

$F(V_i) = P_i$

Def 3 (McKay graph)

- Vertex : $\{V_0, \dots, V_d\} = \hat{G}$
- Arrow : $V_i \Rrightarrow V_j$ — $\mathrm{multi}_i(V \otimes_k V_j)$ 本.

→ $\mathrm{McKay}(V, G)$

$W \in \mathrm{mod}(kG)$,

$\mathrm{multi}_i(W) = \dim_k \mathrm{Hom}_{kG}(V_i, W)$.

Ex: $G = C_n = \left\langle \begin{pmatrix} \zeta_{n+1} & \\ & \zeta_{n+1}^{-1} \end{pmatrix} \right\rangle$: order $= n+1$ の cyclic group.

ζ_m : primitive m-th root of 1 in k.

$C_n = \{g^i\}_{i=0}^{n}$, $g^{n+1} = g^0 = e$

$g^i \neq e \ (1 \leq i \leq n)$

fact: $\#\hat{G} = \#\{\text{conjuacy class}\} = n+1$

$kG = \bigoplus_{W \in \hat{G}} \mathrm{End}_{kG}(W)$, $\hat{G} = \{V_0, \cdots, V_n\}$, $n+1 = \sum_{i=0}^{n} (\dim V_i)^2$. $\therefore \dim V_i = 1$.

ρ: irr. rep. $\rho: C_n \to GL(1,k) = k^\times$, $g \mapsto \omega$

とすると、 $1 = \rho(e) = \rho(g)^{n+1} = \omega^{n+1}$

$\omega = \zeta_{n+1}^{i}$, $\{(\rho_i, V_i)\} = \hat{G}$, $\rho_i : C_n \to GL(V_i)$, $g \longmapsto (V_i \xrightarrow{\zeta_{n+1}^i} V_i)$

$a \otimes h \in V_i \otimes_k V_j$.

$g(a\otimes h) = \rho_i(g) a \otimes \rho_j(g) h = \zeta_{n+1}^{i} a \otimes \zeta_{n+1}^{j} h = \zeta_{n+1}^{i+j} a \otimes h = \rho_{i+j}(g) a \otimes h$.

$\therefore V_i \otimes V_j \simeq V_{i+j}$.

$V \cong V_1 \oplus V_{-1}$

$V \otimes_k V_j \cong V_1 \otimes V_j \oplus V_{-1} \otimes V_j \cong V_{j+1} \oplus V_{j-1}$

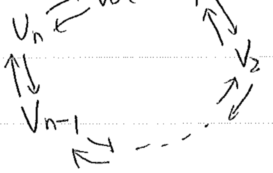
Schur's Lemma $(\pi, V), (\pi', V') \in \hat{G}$

$$\begin{cases} \text{(i)} \ \pi \simeq \pi' \Leftrightarrow \mathrm{Hom}_{kG}(V, V') \neq 0 \\ \text{(ii)}. \ \mathrm{End}_{kG}(V) \simeq k. \end{cases}$$

$$\mathrm{multi}(V \otimes V_j) = \dim_k \mathrm{Hom}(V_i, V_{j+1}) + \dim_k \mathrm{Hom}(V_i, V_{j-1}) = \begin{cases} 1 & i = j \pm 1 \\ 0 & i \neq j \pm 1 \end{cases}$$

$V_{j-1} \to V_j \leftarrow V_{j+1}$.

McKay(V, C_n):



Lem 4. $\nu_i(P) :=$ the number of copies of P_i appearing in decomposition of P.

$P \in \mathrm{Proj}(S * G)$, $\mathrm{multi}(V \otimes_k V_j) = \nu_i(F(V \otimes V_j))$ $0 \le i \le d$.

(pf) $V \otimes_k V_j \simeq \sum V_i^{\mu_i}$ $\mu_i := \mathrm{multi}(V \otimes V_j)$

$\therefore F(V \otimes V_j) \simeq \sum P_i^{(\mu_i)}$, $\nu_i(F(V \otimes V_j)) = \mu_i$ //.

Fact: A: normal local domain. $M \in \mathrm{mod}(A)$. TFAE:

(i). M : second syzygy

(ii). M : reflexive i.e. $M \cong \mathrm{Hom}_A(\mathrm{Hom}_A(M, A), A)$

(iii). M satisfies (S_2).

(S_2) : $\forall \mathfrak{p} \in \mathrm{Spec} A$, $\mathrm{depth}\, M_\mathfrak{p} \geqq \min(2, \mathrm{ht}\,\mathfrak{p})$

R : normal local domain $\Rightarrow$ $M \in \mathrm{CM}(R) \Leftrightarrow M$: reflexive f.g. R-module.

Prop5. $\mathrm{add}_R(S) := \{$direct R-summand of free S-module$\}$.

$\subset \mathrm{Mod}(R)$: full subcategory

$= \mathrm{CM}(R)$.

furthermore, $\{M \in \mathrm{CM}(R) : \text{indec}\} = \{$indec. direct R-summand of $S\}$

hence R : rep-fin.

(pf). S : reflexive R-module $\Rightarrow \mathrm{add}_R(S) \subset \mathrm{CM}(R)$.

$M \in \mathrm{CM}(R)$,

$\psi : R \hookrightarrow S$: split mono. $\left(\because \varphi : S \to R,\ \varphi(s) = \frac{1}{|G|} \sum_{\sigma \in G} \sigma(s)\right)$

$\mathrm{Hom}_R(-, R) = (\ )^*$

$\mathrm{Hom}_R(M^*, \psi)$: $\underset{\underset{M}{\cong}}{\mathrm{Hom}_R(M^*, R)} \longrightarrow \underset{\underset{\mathrm{CM}(S)}{\in}}{\underline{\mathrm{Hom}_R(M^*, S)}}$: split mono

S : RLR $\Rightarrow \mathrm{Hom}_R(M^*, S)$: free S-module

Fact : A : RLR, $M \in \mathrm{mod}(A)$

$M \in \mathrm{CM}(A) \Leftrightarrow M$: free A-module.

$(\because)$ proj. dim $M \leqq$ gl.dim $A < \infty$.

Auslander-Buchsbaum formula

depth M + proj.dim M = depth A.

$\therefore M \simeq \mathrm{Hom}_R(M^*, R)$: direct summand of free S-module,

$M \in \mathrm{add}_R(S)$ //

Def 6: $\sigma \in GL(2,k)$: pseudo-reflection $\overset{def}{\Longleftrightarrow}$ $\mathrm{rank}(\sigma - e) \leq 1$

$A \subset B$: ring extension.

$A \subset B$: unramified in codim 1 $\overset{def}{\Longleftrightarrow}$ $\forall \mathfrak{p} \in \mathrm{Spec} B$, $ht\,\mathfrak{p} \leq 1 \Rightarrow (\mathfrak{p} \cap A) B_{\mathfrak{p}} = \mathfrak{p} B_{\mathfrak{p}}$

Prop 7. (i). R : regular $\Leftrightarrow$ G : generated by pseudo-ref.

(ii). $R \subset S$: unramified in codim 1 $\Leftrightarrow$ G has no pseudo-ref. except e.

Lem 8: $\delta : S * G \longrightarrow \mathrm{End}_R(S)$; R-alg. map.

$\delta(s\sigma)(t) = s\sigma(t)$, $\sigma \in G$, $s, t \in S$

$R \subset S$: unramified in codim 1 $\Rightarrow$ δ : isomorphism.

Prop 9. G has no ps-ref except e.

$H : \mathrm{Proj}(S * G) \to CM(R)$.

(ob). $H(M) = M^G$

(mor) $H(f) = f|_{M^G}$

gives equivalence.

(proof): $\iota : S*G \to S*G$

$\iota(s\sigma) = (\sigma^{-1}s)\sigma^{-1}$

$s \in S$, $\sigma \in G$. は同型 of rings. i.e. ι : hom of additive group

$\iota(s\sigma)\iota(t\tau) = \iota((t\tau)(s\sigma))$

$S*G$: left module M は, $m(s\sigma) = \iota(s\sigma)m$ で right module とみなせる.

逆に right module を left module となる.

M^G は 左右どちらの作用から定めても同一.

よって, right module について示せばよい

$T : CM(R) \longrightarrow \mathrm{Proj}(S*G)$, $T(M) = \mathrm{Hom}_R(S, M)$

$\varphi \in \mathrm{End}_R S$. $f \in T(M)$. $(f(\varphi))(s) = f(\varphi s)$ で $T(M)$: right $\mathrm{End}_R S$-module.

$(f(s\sigma))(t) = f(\delta(s\sigma))(t) = f(s\sigma(t))$, $T(M)$: right $S*G$-module.

$(f\sigma)(t) = f(\sigma(t))$, $T(M) \curvearrowleft G$

$p: S \to R$, $p(s) = \sum_{\sigma \in G} \sigma s$, $M \in CM(R)$.

$h: M \simeq \mathrm{Hom}_R(R, M) \xrightarrow{p^*} \mathrm{Hom}_R(S, M)^G = HT(M)$.

h: iso. を言うには $M = S$ で示せばよい.

$h(s)(t) = (p(t))(s)$. $s, t \in S$. $j: S \cong (S*G)^G$
$s \mapsto \sum_{\tau \in G} s\tau$

$(\delta_{j(s)})(t) = \sum_{\tau \in G} s\tau(t) = (p(t))s = h(s)(t)$

j, δ: iso $\Rightarrow$ h: iso. $h: M \cong HT(M)$

$N \in \mathrm{Proj}(S*G)$, $g: N \to TH(N) = \mathrm{Hom}_R(S, N^G)$, $g(n)(s) = \sum_{\sigma \in G} n s \sigma$ $\left(\begin{smallmatrix} n \in N \\ s \in S \end{smallmatrix}\right)$

g: iso を言うには. $N = S*G$ のとき示せばよい.

$j^{-1} = \pi: (S*G)^G \to S$, $\pi(\sum_\tau s_\tau \tau) = \frac{1}{|G|} \sum_\tau s_\tau$.

$S*G \xrightarrow{g} \mathrm{Hom}_R(S, (S*G)^G) \xrightarrow{\pi_*} \mathrm{Hom}_R(S, S) = \mathrm{End}_R S$.

この合成射は. $t\tau \mapsto (s \mapsto t\tau s)$ これは δ と等しい.

$\therefore g$: iso. //.

<u>Cor 10</u>: $H \circ F$ gives $\hat{G} \overset{1-1}{\longleftrightarrow} \{$iso. class of indec. CM R-module$\}$.
($\because$ Rem 2 & Prop 9).

<u>Notation 11</u>:
以下, G has no ps-ref. except e.

$\hat{G} = \{V_0, \ldots, V_d\}$

$\{$iso class of indec. proj. $S*G$-module$\} = \{P_0, \cdots, P_d\}$

$F(V_i) = P_i$

$\tau(V_i) = (\overset{2}{\wedge} V) \otimes V_i$ $(0 \le i \le d)$

$\tau(P_i) = F(\tau(V_i))$

$\tau(V_i) \simeq \tau(V_j) \Leftrightarrow \tau(P_i) \simeq \tau(P_j) \Leftrightarrow i = j$.

$\{L_0, \ldots, L_d\}$ iso class of indec. CM R-module$\}$.

$HF(V_i) = H(P_i) = L_i$

$\tau(L_i) = H(\tau(P_i)) = HF(\tau(V_i))$

Koszul complex over S

$C: 0 \to S \otimes_k \bigwedge^2 V \xrightarrow{a} S \otimes_k V \xrightarrow{b} S \to k \to 0$ exact seq as $S*G$-module.

exact. $a(g \otimes (x \wedge y)) = gx \otimes y - gy \otimes x \quad (g \in S)$

$b(f_1 \otimes x + f_2 \otimes y) = f_1 x + f_2 y \quad (f_i \in S)$.

$C \otimes_k V_i : 0 \to S \otimes_k (\bigwedge^2 V \otimes_k V_i) \to S \otimes_k (V \otimes_k V_i) \to S \otimes_k V_i) \to V_i \to 0$. exact.

$0 \to \tau(P_i) \to F(V \otimes_k V_i) \to P_i \to V_i \to 0$.

$0 \to \tau(L_i) \to H \cdot F(V \otimes_k V_i) \to L_i \to V_i^G \to 0$.

$V_0 = k$: trivial representation

$V_0^G = k$, $V_i^G = 0 \quad (i \neq 0)$, $E_i := HF(V \otimes_k V_i)$

$$\begin{cases} 0 \to \tau(L_0) \to E_0 \xrightarrow{p_0} L_0 \to k \to 0 \\ 0 \to \tau(L_i) \to E_i \xrightarrow{p_i} L_i \to 0. \quad (i \neq 0) \end{cases}$$

Prop 13. $L \in CM(R)$. For any i $(0 \le i \le d)$, seq $(**)$ satisfies the following condition.

$\forall f : L \to L_i$: $\underset{\wedge}{\overset{\text{nonsplit. epi}}{}} CM(R)$-hom. $\exists g : L \to E_i$: R-hom. s.t. $f = p_i \circ g$.

$$\begin{tikzcd} L \arrow[d, dashed, "\exists g"'] \arrow[dr, "\forall f"] & \\ E_i \arrow[r, "p_i"] & L_i \end{tikzcd}$$

hence, if $i \neq 0$, $(**)$ is AR seq.

(pf) $H : \mathrm{Proj}(S*G) \simeq CM(R)$.

$\mathrm{Proj}(S*G)$

$$\begin{tikzcd} & H^{-1}(L) \arrow[dl, dashed, "\exists g'"'] \arrow[dr, "H^{-1}(f)"] & \\ F(V \otimes V_i) \arrow[rr, "H^{-1}(p_i)"] & & P_i \end{tikzcd}$$

f: non split epi. $\Rightarrow \mathrm{Im} f \subseteq \mathrm{Im}\, p_i \Rightarrow \mathrm{Im} H^{-1}(f) \subseteq \mathrm{Im} H^{-1}(p_i)$

$H^{-1}(L) \in \mathrm{Proj}(S*G)$. $\Rightarrow \exists g' : H^{-1}(L) \to F(V \otimes_k V_i)$ s.t.

$H^{-1}(p_i) \cdot g' = H^{-1}(f)$.

$g = H(g')$ $\quad f = p_i \circ g$ //.

Yoshino (5.5)

M, N: indec. CM R-modules. $0 \to \tau(M) \xrightarrow{f} E \to M \to 0$: AR-seq. ending in M.

$n :=$ the number of copies of N appearing in indec. decomposition of E.

$\mathrm{irr}(N, M) = n$.

$i=0$ のときも同様の主張が成立

$\mathrm{Irr}(L_j, L_i) = E_i$ の indec. decomposition に表れる L_j の個数.

Recall: $(\ , \ ) := \mathrm{Hom}_R(\ , \ )$.

$M, N \in CM(R)$. $(M, N)_n = \left\{ f \in (M,N) \;\middle|\; \begin{array}{l} \exists X_i \in CM(R),\ 0 \le i \le n,\ X_0 = M,\ X_n = N \\ \exists g_i \in (X_{i-1}, X_i) \text{ s.t. } f = g_n g_{n-1} \cdots g_2 g_1 \\ \text{各 } g_i \text{ で } \left.\begin{array}{l} X_{i-1} = \sum_j Y_j \\ X_i = \sum_\ell Z_\ell \end{array}\right\} \text{indec. decomposition} \\ g_i = (g_{ij\ell}),\ g_{ij\ell} : \text{non iso}. \end{array} \right\}$

$\mathrm{irr}(M, N) = \dim_k \mathrm{Irr}(M, N)$

$\mathrm{Irr}(M, N) = (M, N)_1 / (M, N)_2$.

$S(L_j, E_i) = (L_j, E_i)/(L_j, E_i)_1$: k-vector space.

L_j : indec. $\Rightarrow S(L_j, E_i) = \{ f : L_j \to E_i \mid \text{split mono} \}$.

X : indec. CM. R-module.

$X \not\cong L_j \Rightarrow S(L_j, X) = 0$.

$S(L_j, L_j) = \mathrm{End}_R(L_j)/\mathrm{rad}\,\mathrm{End}_R(L_j) \cong k$.

$\therefore \dim_k S(L_j, E_i)$ = the number of copies of L_j appearing in indec. decomp of E_i.

ETS: $S(L_j, E) \cong \mathrm{Irr}(L_j, L_i)$ as k-vector space.

$\varphi : S(L_j, E_i) \longrightarrow \mathrm{Irr}(L_j, L_i)$ by Prop 13, φ: surjective.

$h \longmapsto p_i \circ h$

$h \in S(L_j, E_i)$ s.t. $p_i \circ h \in (L_j, L_i)_2$.

$$\begin{tikzcd} L_j \arrow[r, "a"] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowleft"] & X \in CM(R) \arrow[d, "b"] \\ E_i \arrow[r, "p_i"] & L_i \end{tikzcd}$$

a, b : non split

by Prop 13, $\exists c : X \to E_i$ s.t. $b = p_i \circ c$

$p_i \circ (h - c \circ a) = p_i \circ h - b \circ a = 0$

$\Rightarrow \mathrm{Im}(h - c \circ a) \subset q_i(\tau(L_i)) \cong \tau(L_i)$

$$
\begin{tikzcd}
0 \arrow[r] & \tau(L_i) \arrow[r] & E_i \arrow[rr] & & L_i \\
 & & L_j \arrow[u, "h"] \arrow[rr, "a"'] & & X \arrow[u, "g"'] \arrow[ull, dashed, "\exists c"']
\end{tikzcd}
$$

$c \cdot a$ and $q_i \circ g$ are non split.

$\therefore h = q_i \circ g + c \cdot a \in (L_j, E_i)$,

$\therefore \varphi$: injective. $\lrcorner$.

$\mathrm{Proj}(S*G) \overset{H}{\cong} \mathrm{CM}(R)$.

$\mathrm{irr}(L_j, L_i) = \nu_j(F(V \otimes_k V_i)) = \mathrm{mult}_j(V \otimes_k V_i)$.

AR quiver $\Gamma \longrightarrow M_c(V, G)$: iso of graph

$L_i \longmapsto V_i$

Thm 4. G has no ps-ref. except e. AR quiver of $R \cong M_c(V, G)$.

(R, m, k): complete normal local domain. $\dim R = 2$. $(\Rightarrow \mathrm{CM})$.

Q: Rの商体

$L \supset Q$: finite Galois extension field.

$G = \mathrm{Gal}(L/Q)$

S: integral closure of R in L.

(11.1) <u>Lem</u>: functor $F: \mathrm{mod}(kG) \longrightarrow \mathrm{Proj}(S*G)$

$F(W) = W \otimes_k S$

$(W \in \mathrm{mod}(kG))$

gives $\{$iso class of f.g. kG-module$\} \overset{1-1}{\longleftrightarrow} \{$iso. class of Projective $S*G$-module$\}$.

(Pf): Lem 1 //.

(11.2). $R \subset S$: unramified in codim 1. $H: \mathrm{Proj}(S*G) \cong \mathrm{add}_R(S)$: equivalence,

$H(M) = M^G$.

Pf: Prop 9 //.

(11.3). Lem [Mumford] : R satisfies the following condition:

$L \supset Q$: Galois extension. S: integral closure of R in L.

$R \subset S$: unramified in codim 1. $\Rightarrow S = R$.

then, R : regular (i.e. $R \simeq k[[x,y]]$).

(11.4). Thm: R: rep-fin. $\exists\, \Omega \supset Q$: finite Galois extension. s.t. S: integral closure of R in Ω, $S \simeq k[[x,y]]$. $G = Gal(\Omega/Q) \curvearrowright S$, $S^G \simeq R$.

(pf): fix $\bar{Q}$: alg. closure of Q.

$$\Delta = \left\{ L \;\middle|\; \begin{array}{l} L/Q : \text{finite extension field} \therefore S_L : \text{integral closure of } R \text{ in } L \\ R \subset S_L : \text{unramified in codim 1} \end{array} \right\}$$

$\Omega := \bigcup_{L \in \Delta} L$: Δの元を全て含む

minimal field.

Ω/Q : Galois ext. $\left[\because\ \forall \sigma \in Aut_Q \bar{Q},\ \Omega^\sigma = \bigcup_{L \in \Delta} L^\sigma \subset \Omega \therefore \Omega^\sigma = \Omega \right]$

$[\Omega : Q] < \infty$. を示す. if $[\Omega : Q] = \infty$, $\exists$ sequence Galois extension of Q s.t.

$Q \subset L_1 \subset L_2 \subset \cdots \subset L_n \subset L_{n+1} \subset \cdots \subset \Omega$

$G_n := Gal(L_n/Q)$

S_n: integral closure of R in L_n. $S_n <^{\oplus} S_{n+1}$ as S_n-module.

$(\because L_n <^{\oplus} L_{n+1},\ L_n \cap S_{n+1} = S_n)$

hence, $add_R(S_1) \subset add_R(S_2) \subset \cdots \subset add_R(S_n) \subset add_R(S_{n+1}) \subset \cdots \subset CM(R)$

R: rep-fin $\Rightarrow \#\{\text{indec. CM } R\text{-module}\}/\cong < \infty$

$n \gg 0$. $add_R(S_n)$ and $add_R(S_{n+1})$ has the same number of indec. object.

by (11.1), (11.2)

$\{M \in add_R(S_n) : \text{indec}\}/\cong \overset{1:1}{\longleftrightarrow} \hat{G_n}$

$\therefore G_n, G_{n+1}$ は共役類の数が等しい. (***)

$Q \subset L_n \subset L_{n+1}$, $Gal(L_{n+1}/L_n) = H$, $G_n \simeq G_{n+1}/H$, by (***), $G_n \simeq G_{n+1}$.

$\therefore L_n \simeq L_{n+1}$. $\therefore [\Omega : Q] = |G| < \infty$. by (11.3), $S \simeq k[[x,y]]$. //