Auslander-Reiten quiver 入門

1-1 Motivation

1-2. quiver and path algs

1-3. quiver representations

K : alg closed field.

char $K = 0$.

$\boxed{1\text{-}1.}$ From a quiver, we can construct an alg. $Q \rightsquigarrow KQ$ (path alg)

- From an alg, we can construct a quiver $A \rightsquigarrow Q_A$ and $\exists$ ideal $I \subset KQ_A$ s.t. $A \simeq KQ_A/I$

$\Rightarrow$ we can present an alg by quiver (with relations)

$\Rightarrow$ visualize algs by quivers and modules by quiver rep.

$\Rightarrow$ Powerful tools and it makes all sorts calculations simple.

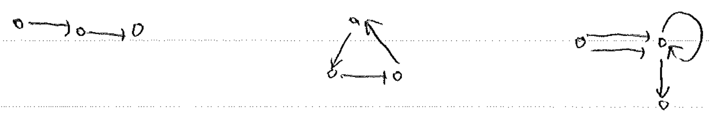
$\boxed{1\text{-}2.}$

Def: Quiver is a quadruple $Q = (Q_0, Q_1, s, e)$

- Q_0 : vertices
- Q_1 : arrows
- $s, e : Q_1 \to Q_0$.

$\alpha \in Q_1$, $s(\alpha) \xrightarrow{\alpha} e(\alpha)$.

Ex.



Def.
- Q is finite $\overset{\text{def}}{\Leftrightarrow}$ $\# Q_0, \# Q_1 < \infty$.
- Q is connected $\Leftrightarrow$ Q is connected graph.
- Path of length l $\Leftrightarrow$ sequence $\alpha_1 \alpha_2 \cdots \alpha_l$, $\alpha_i \in Q_1$ s.t. $e(\alpha_i) = s(\alpha_{i+1})$ $(1 \le i \le l-1)$

$\circ \xrightarrow{\alpha_1} \circ \xrightarrow{\alpha_2} \circ \cdots \xrightarrow{\alpha_l} \circ$

- path of cycle $\Leftrightarrow$ its start and end coincide.

- Q is acyclic $\iff$ Q has no cycle.

<u>Def</u>: • Path alg KQ is defined as follows

KQ has as its basis the set of all paths of length $\ell \geq 0$.

$KQ = \bigoplus_{\ell \geq 0} KQ_\ell$.

- For basis $\alpha, \beta \in KQ$, define the product $\alpha \cdot \beta = \begin{cases} \alpha\beta & (e(\alpha) = s(\beta)) \\ 0 & (e(\alpha) \neq s(\beta)) \end{cases}$

EX: (1) $\overset{1}{\circ} \xrightarrow{\alpha} \overset{2}{\circ} \xrightarrow{\beta} \overset{3}{\circ}$

basis $= \{e_1, e_2, e_3, \alpha, \beta, \alpha\beta\}$

$KQ = Ke_1 \oplus Ke_2 \oplus Ke_3 \oplus K\alpha \oplus K\beta \oplus K\alpha\beta$

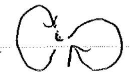
$$KQ = \begin{pmatrix} K & K & K \\ 0 & K & K \\ 0 & 0 & K \end{pmatrix}$$

(2).  α.

basis $= \{e, \alpha, \alpha^2, \cdots\}$

$\alpha^n \cdot \alpha^m = \alpha^m \cdot \alpha^n = \alpha^{n+m}$

$KQ \cong K[x]$.

(3). α  β

basis $= \{e, \alpha, \beta, \alpha^2, \alpha\beta, \beta\alpha, \beta^2, \cdots\}$

$KQ \cong K\langle x, y\rangle$

$I = \langle \alpha\beta = \beta\alpha \rangle$

$KQ/I \cong K[x, y]$

<u>Remark</u>.

- $\dim_K KQ < \infty \iff Q$: finite and acyclic.
- e: path of length 0., $\{e_i \mid i \in Q_0\}$ is a complete set of primitive orthogonal idempotent (CSPOI)

- idempotent $e \overset{\Leftrightarrow}{} e^2=e$.
- primitive idempotent $e \Leftrightarrow \begin{cases} e^2=e \\ e \text{ can't write as } e=e_1+e_2 \quad (e_1 \neq 0 \neq e_2) \end{cases}$
- orthogonal idempotents $e_1, e_2 \Leftrightarrow e_1e_2=e_2e_1=0$.
- $kQ \ni 1 = \sum_{i \in Q_0} e_i$.

$\Rightarrow kQ$: associate k-alg. with 1.

- Construct a quiver from an alg.

Def : A : basic $\overset{\text{def}}{\Leftrightarrow} A/\mathrm{rad}A \simeq k \times \cdots \times k$

$\Leftrightarrow e_iA \not\cong e_jA$ as A-modules if $i \neq j$.

$\{e_i\}$: CSPOI.

Prop : A : f.d. k-alg. $\Rightarrow \exists$ basic A^b s.t. $\mathrm{mod}A \simeq \mathrm{mod}A^b$

Def : A : basic indec. f.d. k-alg. $\{e_1, \cdots, e_n\}$: CSPOI. Define a quiver Q_A as follows :

- $(Q_A)_0 = \{1, \cdots, n\}$
- Draw $\dim_k(e_i(\mathrm{rad}A/\mathrm{rad}^2A)e_j)$ arrows from i to j.

Ex : $A = \begin{pmatrix} k & 0 & 0 \\ k & k & 0 \\ k & 0 & k \end{pmatrix}$, $e_1 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$

$\mathrm{rad}A = \begin{pmatrix} 0 & 0 & 0 \\ k & 0 & 0 \\ k & 0 & 0 \end{pmatrix}$.

$\mathrm{rad}^2A = 0$.

$e_2(\mathrm{rad}A)e_1 = \begin{pmatrix} 0 & 0 & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \overset{2}{\circ} \longrightarrow \overset{1}{\circ}$

$\rightsquigarrow Q_A = \left(\underset{2}{\circ} \longrightarrow \underset{1}{\circ} \longleftarrow \underset{3}{\circ} \right)$.

Thm. (1). A : basic f.d. k-alg.

$\Rightarrow \exists$ ideal $I \subset kQ_A$ s.t. $A \simeq kQ_A/I$.

Moreover, A is hereditary (gl. dim $A \leq 1$)

$A \cong kQ_A$

(2). Q : fin. connected acyclic $\Rightarrow kQ$ is hereditary.

2-3 Quiver representations

①. quiver rep. $(X_i, X_\alpha)_{i \in Q_0, \alpha \in Q_1}$ is defined as follows

(i) k-vector X_i $i \in Q_0$.

(ii) linear map $X_\alpha : X_{s(\alpha)} \to X_{e(\alpha)}$, $\alpha \in Q_1$

②. A morphism. $f : (X_i, X_\alpha) \to (Y_i, Y_\alpha)$. is a family $(f_i)_{i \in Q_0}$ of k-linear map $f_i : X_i \to Y_i$ s.t.

$$\begin{tikzcd} X_{s(\alpha)} \arrow[r, "f_{s(\alpha)}"] \arrow[d, "X_\alpha"'] \arrow[dr, phantom, "\circlearrowright"] & Y_{s(\alpha)} \arrow[d, "Y_\alpha"] \\ X_{e(\alpha)} \arrow[r, "f_{e(\alpha)}"] & Y_{e(\alpha)} \end{tikzcd}$$

commutes.

③. we can define the compositions.

①②③ define the cat $\mathrm{Rep}_k(Q)$. ($\mathrm{rep}_k(Q)$: $\dim X_i < \infty$, $\forall i \in Q_0$).

Ex :

$$\circ \leftarrow \circ \qquad \circ \leftleftarrows \circ \qquad \begin{tikzcd}[column sep=small, row sep=small] & & \circ \arrow[dl] \\ \circ & \circ \arrow[l] & \\ & & \circ \arrow[ul] \end{tikzcd}$$

$$\begin{tikzcd} k \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & k \arrow[l] \arrow[d] \\ k & k \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} k^2 & k \arrow[l, shift right, "\binom{1}{0}"'] \arrow[l, shift left, "\binom{0}{1}"] \end{tikzcd} \qquad \begin{tikzcd}[column sep=small, row sep=small] & & k \arrow[dl, "1"'] \\ k^2 & k \arrow[l, "\binom{1}{1}"'] & \\ & & k \arrow[ul, "1"] \end{tikzcd}$$

Thm $\exists$ k-linear equivalence. $\mathrm{Rep}_k(Q) \simeq \mathrm{Mod}\, kQ$

and for $A \simeq kQ/I$, $\mathrm{Rep}_k(Q, I) \simeq \mathrm{Mod}\, A$. $\Rightarrow$ we can present A-modules as k-vector spaces and linear maps.

Thm : A : basic f.d. k-alg. $D := \mathrm{Hom}_k(-, k)$, $J := \mathrm{rad} A$, $\{e_1, \cdots, e_n\}$: CSPOI

(i). $\{e_1 A, \cdots, e_n A\}$ is a complete set of indec. proj. A-modules

(ii). $\{D(Ae_1), \cdots, D(Ae_n)\}$ is a complete set of injective A-modules.
(iii) $\{e_1A/e_1J, \cdots, e_nA/e_nJ\}$ simple

· Q: finite, connected, acyclic

[indec. proj. kQ-modules]

$P(i) := (P(i)_j, P_\alpha)$ · $P(i)_j := \sum_{\mu \in kQ, s(\mu)=i, e(\mu)=j} k\mu$

· $\alpha: x \to y \in Q_1$. $P(i)_\alpha : P(i)_x \to P(i)_y$ (right multiplication)

$\mu \longmapsto \mu\alpha$.

[simple kQ-modules]

$S(i) := (S(i)_j, S_\alpha)$ · $S(i)_j = \begin{cases} 0 & i \neq j \\ k & i = j \end{cases}$.

· $S_\alpha = 0$. $\forall \alpha \in Q_1$.

Ex: $\overset{1}{\circ} \longleftarrow \overset{2}{\circ} \leftleftarrows \overset{3}{\circ}$

$S(1) = (k\ 0\ 0)$ $P_1 = (k\ 0\ 0)$ $I_1 = (k, k, k^2)$

$S(2) = (0\ k\ 0)$ $P_2 = (k\ k\ 0)$ $I_2 = (0\ k\ k^2)$

$S(3) = (0\ 0\ k)$ $P_3 = (k^2\ k^2\ k)$ $I_3 = (0\ 0\ k)$

2-1. Def and Properties of AR quiver.
2-2. Applications
2-3 Caculations.

· From an alg, we can construct a quiver. $A \rightsquigarrow Q_A$
· From a category, we can construct a quiver. $\mathrm{mod}A \rightsquigarrow Q_{\mathrm{mod}A}$

Describe the structure of category. ⇒ AR quiver.
(A の AR quiver) = ($\mathrm{mod}A$ の quiver)

A: rep-fin (∃ only finitely many indecomp. A-modules).
⊕X_i: all direct sums of indec. A-modules

(A の AR-quiver) = ($\mathrm{End}_A(\oplus X_i)$ の quiver)

<u>Def</u>: AR quiver $\Gamma(\mathrm{mod}\,A) := (\Gamma_0, \Gamma_1, \tau)$ of A. identified as follows:

- $\Gamma_0 = \{$iso. classes of indecomp. A-modules$\}$.
- $X, Y \in \Gamma_0$, draw $\dim_K \mathrm{Irr}(X, Y)$ arrows from X to Y.
- Draw dotted lines from X to τX. ($0 \to \tau X \to Y \to X \to 0$) AR-seq.

<u>Remark</u>. Γ_0 is finite $\Leftrightarrow$ A: rep-fin.

<u>Ex</u>: (1) $Q = (3 \to 2 \to 1)$

$\Gamma(\mathrm{mod}\,KQ)$.

$$\begin{tikzcd}
 & & P_3=(K\ K\ K) \arrow[dr] & & \\
 & P_2=(0\ K\ K) \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & I_2=(K\ K\ 0) \arrow[dr] & \\
P_1=(0\ 0\ K) \arrow[ur] \arrow[rr, dashed, no head] & & S_2=(0\ K\ 0) \arrow[ur] \arrow[rr, dashed, no head] & & I_3=(K\ 0\ 0)
\end{tikzcd}$$

(2). $R = K[x]/(x^n)$, $R_i := K[x]/(x^i)$ $(1 \leq i \leq n)$.

$$R_1 \rightleftarrows R_2 \rightleftarrows \cdots \rightleftarrows R_{n-1} \rightleftarrows R_n$$

(3). $Q = \left(\begin{smallmatrix} 2 & & \\ & \searrow & \\ & & 1 \leftarrow 4 \\ & \nearrow & \\ 3 & & \end{smallmatrix} \right)$

$\Gamma(\mathrm{mod}\,KQ)$.

$$\begin{tikzcd}
 & P_2 \arrow[dr] & & \begin{smallmatrix}0&&\\&K&K\\K&&\end{smallmatrix} \arrow[dr] & & I_2 \\
P_1 \arrow[ur] \arrow[dr] \arrow[ddr] & & \begin{smallmatrix}K&&\\&K^2&K\\K&&\end{smallmatrix} \arrow[ur] \arrow[dr] \arrow[ddr] & & I_1 \arrow[ur] \arrow[dr] \arrow[ddr] & \\
 & P_3 \arrow[ur] & & \begin{smallmatrix}K&&\\&K&K\\0&&\end{smallmatrix} \arrow[ur] & & I_4 \\
 & P_4 \arrow[uur] & & \begin{smallmatrix}K&&\\&K&0\\K&&\end{smallmatrix} \arrow[uur] & & I_3
\end{tikzcd}$$

(4). $\overset{1}{\circ} \leftleftarrows \overset{2}{\circ}$

$$\begin{tikzcd}
 & P_2=(K^2\ K) \arrow[dr, shift left] \arrow[dr, shift right] \arrow[rr, dashed, no head] & & (K^4\ K^3) \arrow[dr, shift left] \arrow[dr, shift right] & \\
P_1=(K\ 0) \arrow[ur, shift left] \arrow[ur, shift right] \arrow[rr, dashed, no head] & & (K^3\ K^2) \arrow[ur, shift left] \arrow[ur, shift right] & & \cdots
\end{tikzcd}$$

· Structure of AR quivers.

Recall : $X, Y \in \mathrm{mod} A$.

$\mathrm{Irr}(X, Y) := \mathrm{rad}(X, Y) / \mathrm{rad}^2(X, Y)$

$\mathrm{rad}(X, Y) := \{\text{non-iso in } \mathrm{Hom}(X, Y)\}$

f : irreducible $\overset{\text{Def}}{\Leftrightarrow}$ (i) non split mono and non split epi.

(ii) $X \xrightarrow{f} Y$, $X \xrightarrow{g} Z \xrightarrow{h} Y$, $f = hg \Rightarrow g$: split mono or h : split epi.

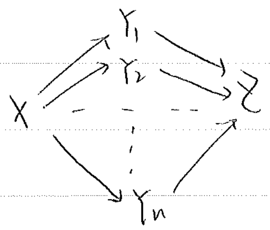
$\overset{\text{Prop}}{\Leftrightarrow}$ $f \in \mathrm{rad}(X, Y) / \mathrm{rad}^2(X, Y)$

Prop: $0 \to X \to \bigoplus_{i=1} Y_i^{n_i} \to Z \to 0$ AR seq.

Y_i : indec and pairwise nonisomphic.

$\Rightarrow n_i = \dim_k \mathrm{Irr}(X, Y_i) = \dim_k \mathrm{Irr}(Y_i, Z)$ for each i.

$\Rightarrow$ AR quiver



"mesh".

Def: Translation quiver $(\Gamma_0, \Gamma_1, \tau)$.

· (Γ_0, Γ_1) : quiver.

· $\forall x \in \Gamma_0 - \Gamma_P$, $\forall y \in \Gamma_0$, $\#\{\tau x \to y\} = \#\{y \to x\}$. $(\Gamma_P \subset \Gamma_0)$

· τ : bijection $(\Gamma_0 - \Gamma_P) \leftrightarrow (\Gamma_0 - \Gamma_I)$ $(\Gamma_I \subset \Gamma_0)$.

Thm: AR-quiver is a translation quiver. τ is defined for all non-proj modules.
$(\Gamma_P := \mathrm{proj} A,\ \Gamma_I := \mathrm{inj} A)$.

Thm : AR quiver is locally finite (i.e. each vertex has only finite neighbours)

2-2. [Brauer-Thrall conjecture] Thm.

Any f.d. k-alg is either rep-fin. or admits indec. modules with arbitrary large dimension

<u>Thm</u>: A: basic f.d. k-alg.

If $\exists$ connected component $\Gamma^0 \subset \Gamma(\mathrm{mod}A)$ whose modules are of bounded length,
$\Rightarrow \Gamma^0$ is finite and $\Gamma^0 = \Gamma(\mathrm{mod}A)$. In particular, A is rep-fin.

<u>Cor</u>: Any alg is either rep-fin or admits indec. modules of arbitrary large length.

<u>Cor</u>: A: rep-fin, $X, Y \in \mathrm{ind}A$. $f: X \to Y$: non-iso. f = a sum of compositions of irr. maps.

② <u>Def</u>: Q: finite connected. acyclic quiver.

Define translation quiver $\mathbb{Z}Q$, vertex : $\mathbb{Z}Q_0$.

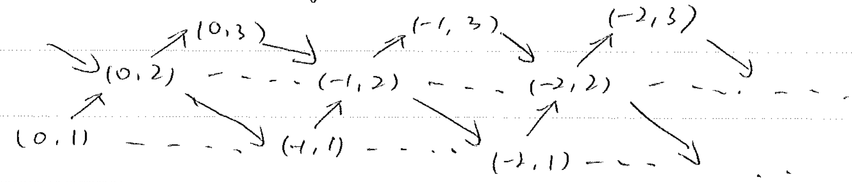
arrows: $\forall\ \alpha: i \to j \in Q_1$

$(n, \alpha): (n, i) \to (n, j)$

$(n, \alpha^*): (n+1, j) \to (n, i)$.

<u>Ex</u>:

$Q = 1 \to 2 \to 3 \quad \rightsquigarrow \quad \mathbb{Z}Q$.



<u>Thm</u>: Q: Dynkin quiver $(A, D, E_6, E_7, E_8) \Rightarrow \Gamma(D^b(\mathrm{mod}\,kQ)) \simeq \mathbb{Z}Q$.

It does not depend on the orientation of Q.

<u>2-3.</u> A: f.d. k-alg.

<u>Prop</u>: (i). P: indecomp. non-simple, proj-inj. module $\Rightarrow 0 \to \mathrm{rad}P \to \mathrm{rad}P/\mathrm{soc}P \oplus P \to P/\mathrm{soc}P \to 0$ is AR-seq.

(ii). P: indecomp. proj. Then $f: X \to P$: right minimal almost split $\Leftrightarrow f$: mono and $\mathrm{Im} f = \mathrm{rad}P$

(iii). P: simple proj and non-inj. $f: P \to X$: irreducible $\Rightarrow X$: proj.

(ii)', (iii)' dual.

$Q = (3 \to 2 \to 1)$

$\Gamma(\mathrm{mod}\,kQ)$. (1) $P_1 = (0\,0\,k)$ $\quad I_2 = (k\,k\,0)$

$P_2 = (0\,k\,k)$ $\quad I_3 = (k\,0\,0)$

$P_3 = (k\,k\,k) = I_1$

$S_2 = (0\,k\,0)$

(2). $P_3 = I_1$: proj - inj - module.

$\mathrm{rad}\,P_3 = (0, k\, k)$

$\mathrm{soc}\,P_3 = (0\;0\;k)$

By (i) $\Rightarrow$ $0 \to (0kk) \to (0k0) \oplus (kkk) \to (kk0) \to 0$. AR seq.

(3). P_1 : simple proj.

By (ii) P_1からの irreducible $P_1 \to P_2$ or ($P_1 \to P_3$ $\Rightarrow$ (ii) に反す.)

$0 \to P_1 \xrightarrow{f} P_2 \to \mathrm{coker} f \to 0$ $\quad \therefore \mathrm{coker} f \simeq S_2 = (0k0)$

left minimal almost split).

(4). Dually, we obtain $0 \to \ker f' \to I_2 \xrightarrow{f'} I_3 \to 0$