可換環論からの準備

k : field ← coefficient field.
$R\ (\supset k)$: Noetherian complete local ring.
$\mathrm{rad}R$: the unique maximal ideal of R.
$d := \dim R$: Krull dimension of R.

Theorem.

(1). Cohen's structure theorem of complete local ring.

$R \cong k[[x_1, x_2, \cdots, x_n]]/I$. (i.e. $R/\mathrm{rad}R \cong k$).

(2). Noether's normalization theorem

$\exists t_1, t_2, \cdots, t_d \in \mathrm{rad}R$: system of parameters

($\overset{\mathrm{def}}{\Longleftrightarrow}$ $R/(t_1 t_2 \cdots t_d)R$: finite length) s.t. $T := k[[t_1, t_2, \cdots, t_d]] \subset R$

& R is a finitely generated T-module

$\mathrm{mod}(R)$: the category of finitely generated R-modules.

(3). Hensel's lemma.

- $M \in \mathrm{mod}(R)$: indecomposable $\Longleftrightarrow$ $\mathrm{End}_R(M)$: local ring
- For $M \in \mathrm{mod}(R)$, M has the unique decomposition

← Krull-Schmidt.

Definitions - Remarks

(1). For $M \in \mathrm{mod}(R)$, $\mathrm{depth}_R M := \inf\{n \mid \mathrm{Ext}_R^n(k, M) \neq 0\}$.

$\dim_R M := \dim(R/\mathrm{Ann}_R(M))$ $\quad (\mathrm{Ann}_R(M) := (x \in R \mid xM = 0)R)$

注 $\mathrm{depth}_R M \leq \dim_R M \leq \dim R$

記号 $\mathrm{depth}\, R := \mathrm{depth}_R R$

(2). R : regular local ring (RLR) $\overset{\mathrm{def}}{\Longleftrightarrow}$ $\dim R = \mu(\mathrm{rad}R)$ (:= $\mathrm{rad}R$ の極小生成元の個数) $\Longleftrightarrow$ $\mathrm{gl.dim}R = \mathrm{depth}\,R$ $\Longleftrightarrow$ $R \cong k[[x_1, x_2, \cdots, x_d]]$.

(3). R : Cohen-Macaulay ring (CM) $\overset{\mathrm{def}}{\Longleftrightarrow}$ $\dim R = \mathrm{depth}\,R$.

$\Longleftrightarrow$ $R \cong T^e$ as T-module

(4). $M \in \mathrm{mod}(R)$: maximal CM R-module (MCM)
$\overset{\text{def}}{\Longleftrightarrow}$ $\dim R = \mathrm{depth}_R M$ $\Longleftrightarrow$ $M \cong T^n$ as T-module

Examples

① if $\dim R = 1$, then M:MCM $\Longleftrightarrow$ $\mathrm{Hom}_R(k, M) = 0$.

②. $\dim R = 1$, R:domain $\Rightarrow$ R: CM.

③. $k[[x,y]]/(xy)$: CM, $k[[x,y]]/(x^2,xy)$: CM

$k[[x+y]] \hookrightarrow$ $\qquad$ $k[[y]] \hookrightarrow$

以下では, R: CM を仮定する.

$\mathrm{mod}(R) \underset{\substack{\text{full}\\ \text{subcat.}}}{\supseteq} \mathrm{CM}(R) := \{\text{MCM } R\text{-modules}\}$.

(5). $\omega_R \in \mathrm{mod}(R)$: canonical module

$\overset{\text{def.}}{\Longleftrightarrow}$ $\begin{cases} \cdot\ \omega_R \in \mathrm{CM}(R) \\ \cdot\ \mathrm{inj.dim}_R\, \omega_R = \mathrm{depth}\, R\ (=d) \\ \cdot\ \mathrm{Ext}^d_R(k, \omega_R) \cong k. \end{cases}$

$\Longleftrightarrow$ $\omega_R \cong \mathrm{Hom}_T(R, T)$

(6). $\mathrm{Hom}_R(-, \omega_R) : \mathrm{CM}(R) \xrightarrow{\text{duality}} \mathrm{CM}(R)$ $\qquad$ $\mathrm{Hom}_R(\mathrm{Hom}_R(-, \omega_R), \omega_R) \cong (-)$

$M \longmapsto \mathrm{Hom}_R(M, \omega_R)$

㊟ $\mathrm{Hom}_R(M, \omega_R) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_T(R, T)) \cong \mathrm{Hom}_T(M \otimes_R R, T)$
$\cong \mathrm{Hom}_T(M, T)$

(7). R: Gorenstein ring $\overset{\text{def}}{\Longleftrightarrow}$ $\mathrm{inj.dim}_R R = \mathrm{depth} R\ (=d)$ $\Longleftrightarrow$ $\omega_R = R$.

㊟: $\mathrm{depth} R \leq \dim R \leq \mathrm{inj.dim}_R R \leq \mathrm{gl.dim} R$ により.

RLR $\Rightarrow$ Gorenstein $\Rightarrow$ CM.

まとめ

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$\begin{cases} \Lambda : \text{finite dimensional } k\text{-algebra} \\ \mathrm{mod}\,\Lambda : \text{the category of f.g. (left) } \Lambda\text{-modules} \end{cases}$

(1) $\Lambda \cong k^n$ as k-module

(2) $\mathrm{mod}\,\Lambda$: Krull-Schmidt category

(3) $\mathrm{mod}\,\Lambda \underset{\text{duality}}{\overset{\mathrm{Hom}_k(-,k)}{\rightleftarrows}} \mathrm{mod}\,\Lambda^{op}$

$\{\text{projectives}\} \rightleftarrows \{\text{injectives}\}$

$\begin{cases} R : \text{complete CM local ring } (\supset k) \\ \mathrm{CM}(R) \end{cases}$

(1) $R \cong T^e$ as T-module

(2) $\mathrm{CM}(R)$: Krull-Schmidt category

(3) $T \subset R$. $\mathrm{CM}(R) \underset{\text{duality}}{\overset{\mathrm{Hom}_T(-,T)}{\rightleftarrows}} \mathrm{CM}(R)$

$R \rightleftarrows \omega_R$

$\mathrm{add}\,R \rightleftarrows \mathrm{add}\,\omega_R$

Auslander–Reiten sequence 入門 (1)

R: comm. noeth. complete CM local ring
(Henselian) having the canonical module w_R

Krull-Schmidt thm

mod$R$, CM(R)

※. $M \in CM(R)$ M: ind $\Leftrightarrow$ $\mathrm{End}_R(M)$ = local

Def 1 : $M \in CM(R)$: ind

$S(M) := \{ s: 0 \to N_s \to E_s \to M \to 0 \mid$ nonsplit exact seq in $CM(R)$, N_s: ind $\}$

$N \in CM(R)$: ind.

$S'(N) := \{ 0 \to N \to G_s \to M_s \to 0 \mid$ nonsplit exact seq in $CM(R)$, M_s: ind $\}$

※. $s \in S(M)$ gives a nontrivial element of $\mathrm{Ext}_R^1(M, N_s)$
$S'(N)$ $\mathrm{Ext}_R^1(M_s, N)$

※. $(-)' = \mathrm{Hom}_R(-, w_R)$, $s: 0 \to N \to G \to M \to 0$: non-split
$s \in S(M) \Leftrightarrow s \in S'(N) \Leftrightarrow s' \in S(N')$
$N' = \mathrm{Hom}_R(N, w_R)$

Lem 2 : $M \in CM(R)$: ind , nonfree , then $S(M) \neq \phi$
N $N \not\cong w$ $S'(N)$

(Proof): M: nonfree より

$$\begin{array}{ccccccccc} 0 & \to & K & \to & F & \to & M & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & K_j & \to & F_j & \to & M & \to & 0 \end{array}$$

nonsplit ex seq in $CM(R)$ (first row); ex seq in $CM(R)$ (second row)

$K = \sum K_i$ (ind)

$F_j = F / \sum_{i \neq j} K_i$

ある j が存在して nonsplit になる. □.

Def 3. $s, t \in S(M)$. $\quad s, t \in S'(N)$

(i). $s > t \overset{\text{Def}}{\iff} \exists f \in \mathrm{Hom}_R(N_s, N_t)$ s.t. $\mathrm{Ext}^1_R(M, F)(s) = (t)$

$$\iff \begin{tikzcd} s: 0 \arrow[r] & N_s \arrow[r] \arrow[d, dashed, "\exists f"'] \arrow[dr, phantom, "\circlearrowright"] & E_s \arrow[r] \arrow[d, dashed] \arrow[dr, phantom, "\circlearrowright"] & M \arrow[r] \arrow[d, equal] & 0 \\ t: 0 \arrow[r] & N_t \arrow[r] & E_t \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

(i)' $s >' t \overset{\text{Def}}{\iff} \exists f \in \mathrm{Hom}_R(M_t, M_s)$ s.t. $\mathrm{Ext}^1_R(F, N)(s) = (t)$

$$\iff \begin{tikzcd} s: 0 \arrow[r] & N \arrow[r] \arrow[dr, phantom, "\circlearrowright"] & G_s \arrow[r] \arrow[dr, phantom, "\circlearrowright"] & M_s \arrow[r] & 0 \\ t: 0 \arrow[r] & N \arrow[r] \arrow[u, equal] & G_t \arrow[r] \arrow[u, dashed] & M_t \arrow[r] \arrow[u, dashed, "f"'] & 0 \end{tikzcd}$$

(ii). $s \sim t \overset{\text{Def}}{\iff} f$: iso in (i)

(ii)' $s \sim' t \overset{\text{Def}}{\iff} f$: iso in (i)'

Lem 4. $s, t \in S(M)$ Then $s > t$, $t > s$, $\Rightarrow s \sim t$ $\quad$ ($S(M)$: partially ordered set)

$\quad S'(N) \qquad >' \quad >' \qquad \sim'$ $\quad$ ($S'(N)$: partially ordered set)

Lem 6. $s, t \in S(M)$. Then $\exists u \in S(M)$ s.t. $s > u$ and $t > u$.

$\quad S'(N) \qquad S'(N) \qquad >' \qquad >'$

(proof)

$$\begin{tikzcd} s: 0 \arrow[r] & N_s \arrow[r] \arrow[d, dashed] \arrow[dr, phantom, "\circlearrowright"] & E_s \arrow[r, "p"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K \arrow[r] \arrow[d] & E \arrow[r, "(p,q)"] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\ u: 0 \arrow[r] & K_j \arrow[r] & E_j \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

$t: 0 \to N_t \to E_t \xrightarrow{q} M \to 0$

$E = E_s \oplus E_t$, $\quad K = \ker(p, q)$

$K = \Sigma K_i$ (K_i: ind)

$E_j = E / \sum_{i \neq j} K_i$

あるjで nonsplit ex seq. in CM(R) □

Cor 7: s: minimal in $S(M) \Rightarrow s$: minimum in $S(M)$.

$\qquad S'(N) \qquad\qquad S'(N)$

Def 8 $M \in CM(R)$: ind.

N

$s: 0 \to N_s \to E_s \to M \to 0$: AR-seq ending in M $\overset{\text{Def}}{\iff}$ s: minimum in $S(M)$

$s: 0 \to N \to G_s \to M_s \to 0$: AR-seq starting from N $\overset{\text{Def}}{\iff}$ s: minimum in $S'(N)$

存在すれば一意的。

N_s, E_s は同型を除いて一意的。
M_s, G_s.

S: AR-seq ending in M のとき, N_s を $\tau(M)$ とかく, $\tau(M)$ をAR-translation という。
starting from N $\quad M_s \quad \tau^{-1}(N)$

Lem 9. $M \in CM(R)$: ind

$s: 0 \to N_s \to E_s \xrightarrow{p} M \to 0$ in $S(M)$ TFAE:

(i) s: AR seq ending in M

(ii) $\forall p' : \underset{\text{ind}}{L} \to M$, $\exists f: L \to E_s$ s.t. $p' = pf$
nonsplit epi)

$$\begin{tikzcd} & L \arrow[dl, dashed, "\exists f"'] \arrow[d, "p'"] \\ E_s \arrow[r, "p"'] & M \end{tikzcd}$$

(i)' s: AR seq starting from N.

(ii)' $\forall q' : N \to \underset{\text{ind}}{L}$: nonsplit mono) $\exists g: E_s \to L$ s.t. $q' = gq$.

$$\begin{tikzcd} N \arrow[r, "q"] \arrow[d, "\forall q'"'] & E_s \arrow[dl, dashed, "\exists g"] \\ L & \end{tikzcd}$$

(proof): (i) $\Rightarrow$ (ii).

$$\begin{tikzcd} & & & L \arrow[d, "p'"] \arrow[dl] & \\ 0 \arrow[r] & N \arrow[r] \arrow[d] & E_s \arrow[r, "p"] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathrm{Ker}(p,p') \arrow[r] \arrow[d, "\wr"] & E_s \oplus L \arrow[r, "{(p,p')}"] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K_j \arrow[r] \arrow[d, "\wr"] & X_j \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N \arrow[r] & E_s \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

$\mathrm{Ker}(p,p') \cong \Sigma k_i$

$\exists j$. $\quad 0 \to K_j \to X_j \to M \to 0 \in S(M)$ nonsplit. $\quad K_j$: ind.

(ii)⇒(i).

$t<s$ となる $t\in S(M)$ をとる。このとき $t\sim s$ と示せばよい

$$\begin{array}{ccccccccc} s: & 0 \to & N_s & \xrightarrow{q} & E_s & \xrightarrow{p} & M & \to & 0 \\ & & \downarrow\uparrow g & & \downarrow\uparrow f & \circlearrowleft & \| & & \\ & 0 \to & N_t & \xrightarrow[q_t]{} & E_t & \xrightarrow[p_t]{} & M & \to & 0 \end{array}$$

∴ $s<t$ ゆえに $s\sim t$ ∴ s: AR-seq //.

Def 10. $N, M \in CM(R)$

$f: M \to N$: R-hom

Then f: irreducible morphism :⇔
(i) f: non split epi か nonsplit mono
(ii) $M \xrightarrow{f} N$, $M \xrightarrow{g} X \xrightarrow{h} N$ ⇒ g: split mono or h: split epi.

Lem 11: $M \in CM(R)$: ind.
N

$s: 0 \to N_s \to E_s \xrightarrow{p} M \to 0 \ \in S(M)$

$0 \to N \xrightarrow{q} E_s \to M_s \to 0$

Then s: AR-seq ⇒ p: irr. q: irr.

(Proof): $E_s \xrightarrow{p} M$, $E_s \xrightarrow{g} X \xrightarrow{h} M$ h: non(split epi) とする.

$$\begin{array}{ccccccccc} & 0 \to & N_s & \to & E_s & \xrightarrow{p} & M & \to & 0 \\ & & \downarrow & & \downarrow \binom{1}{g} & \circlearrowleft & \| & & \\ & 0 \to & K & \to & E_s\oplus X & \xrightarrow{(p,h)} & M & \to & 0 \\ & & \downarrow & \circlearrowleft & \downarrow (0,b) & \circlearrowleft & \| & & \\ \exists j & 0 \to & K_j & \to & G_j & \to & M & \to & 0 \quad \text{nonsplit.} \\ & & \downarrow & \circlearrowleft & \downarrow f & \circlearrowleft & \| & & \\ s: & 0 \to & N_s & \to & E_s & \to & M & \to & 0 \end{array}$$

by Lem 5. fbg: auto g: split mono. ∴ p: irr.

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Lem5. : $s: 0 \to N_s \to E_s \to M \to 0 \quad \in S(M)$.

$h \in \mathrm{End}_R(N_s)$

$\mathrm{Ext}^1_R(M, h)(s) = s \Rightarrow h$: auto.

$$\begin{array}{ccccccccc} s: 0 & \to & N_s & \to & E_s & \to & M & \to & 0 \\ & & \downarrow h & & \downarrow & & \| & & \\ s: 0 & \to & N_s & \to & E_s & \to & M & \to & 0 \end{array}$$

Cor 12. $M \in CM(R)$: ind.

$s: 0 \to N_s \to E_s \to M \to 0 \in S(M)$: AR-seq.

$f: \underset{\text{ind-CM}}{L} \to M$ $\qquad g: N_s \to \underset{\text{ind-CM}}{L'}$

TFAE: (i) f: irr $\qquad$ (i) g: irr.

(ii) L は E_s の direct summand. $\qquad$ (ii) L' は E_s の direct summand.

Lem14: $s: 0 \to \underset{\text{ind.CM.}}{N} \to E_s \to \underset{\text{ind.CM.}}{M} \to 0$ (ex) nonsplit.

TFAE: (i) s: AR-seq ending in M

(ii) s: AR-seq starting from N.

Def15: "CM(R) admits AR-seq" $\overset{\text{def}}{\iff}$ $\forall M^{\in CM(R)}$: ind. nonfree

$\exists$ AR-seq. ending in M

$\iff \forall N^{\in CM(R)}$: ind. $N \not\cong \omega_R$

$\exists$ AR-seq. starting from N.