Support varieties II

Recall: Λ : fin. dim. k-alg. $k = \bar{k}$: field

$\underline{r}$ = Jacobson radical

H : commu. Noetherian graded ring.

$$\mathrm{Ext}^*_\Lambda(M,N) = \bigoplus_{i \geq 0} \mathrm{Ext}^i_\Lambda(M,N)$$

left mod. right mod.

$$\begin{tikzcd} \mathrm{Ext}^*_\Lambda(N,N) & & \mathrm{Ext}^*_\Lambda(M,M) \\ & H \arrow[ul, "\varphi_N"] \arrow[ur, "\varphi_M"'] & \end{tikzcd}$$

$$V_H(M,N) \overset{\text{def}}{:=} \{ \underline{m} \in \mathrm{Max\,Spec}\, H \mid \mathrm{Ann}_H \mathrm{Ext}^*_\Lambda(M,M) \subset \underline{m} \}$$

Properties (1) $V_H(M, \Lambda/\underline{r}) = V_H(M,M) = V_H(\Lambda/\underline{r}, M) =: V_H(M)$

(2) $\exists\ 0 \to M_1 \to M_2 \to M_3 \to 0$: exact.

$V_H(M_r) \subseteq V_H(M_s) \cup V_H(M_t)$ $\{r,s,t\} = \{1,2,3\}$

~~(3)~~ $V_H(M \oplus M') = V_H(M) \cup V_H(M')$

(Fg) ~~(3)~~ $\mathrm{Ext}^*_\Lambda(M,N)$ fin. gen. H-module $\forall M, N \in \mathrm{mod}\Lambda$

$\iff$ $\mathrm{Ext}^*_\Lambda(\Lambda/\underline{r}, \Lambda/\underline{r})$ fin. gen. H-module.

Assume (Fg) throughout.

(3) Then $\cdot$ Λ is Gorenstein i.e. $\mathrm{id}_\Lambda \Lambda = \mathrm{id}\Lambda_\Lambda < \infty$

$\cdot$ $\dim V_H(M) = \mathrm{cx}(M) =$ complexity of M

$= \inf\{t \geq 0 \mid \dim_k P_n \leq a n^{t-1}$ for some $a \in \mathbb{R}$ and $\forall n \geq 0\}$

where $\cdots \to P_i \to P_{i-1} \to \cdots \to P_0 \to M \to 0$: min. proj. res. of M

Note : $\mathrm{cx}(M) = 0 \iff \mathrm{pd}_\Lambda M < \infty$

$\mathrm{cx}(M) = 1 \iff \dim_k P_i \leq N \quad \forall i \geq 0.$

bounded Betti numbers

~~1) Find~~

1) Finding an $H \rightsquigarrow HH^*(\Lambda)$

2) Best choice.

3) Examples

4) Some applications

§1 Λ as above.

$\Lambda^e = \Lambda \otimes_k \Lambda^{op}$ $\qquad \bigoplus_{i \geq 0} \mathrm{Ext}^i_{\Lambda^e}(\Lambda, \Lambda)$

$HH^*(\Lambda) = \mathrm{Ext}^*_{\Lambda^e}(\Lambda, \Lambda) =$ the Hochschild cohomology ring of Λ

Facts (1) graded ring via Yoneda product.

(2) $HH^0(\Lambda) = \mathrm{Hom}_{\Lambda^e}(\Lambda, \Lambda) = Z(\Lambda) =$ Center of Λ

(3) graded commutative :

$xy = (-1)^{|x||y|} yx$. $\quad \forall x, y$ hom. in $HH^*(\Lambda)$

$|z| =$ degree of z.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_1 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & B_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_1 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$$\left(\begin{tikzcd}[column sep=small]
0 \arrow[r] & A_1 \arrow[r] & A_2 \arrow[rr] \arrow[dr] & & B_3 \arrow[r] & C_3 \arrow[r] & 0 \\
 & & & A_3 \arrow[ur] & & &
\end{tikzcd} \right) = \eta$$

$$\left(\begin{tikzcd}[column sep=small]
0 \arrow[r] & A_1 \arrow[r] & B_1 \arrow[rr] \arrow[dr] & & C_2 \arrow[r] & C_3 \arrow[r] & 0 \\
 & & & C_1 \arrow[ur] & & &
\end{tikzcd} \right) = -\eta$$

in $\mathrm{Ext}^2(C_3, A_1)$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Lambda \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, "\wr\, E\otimes\Lambda"] & \Lambda \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E' \arrow[r] \arrow[d] & E\otimes E' \arrow[r] \arrow[d] & E' \arrow[r] \arrow[d, "\wr\, \Lambda\otimes E'"] & 0 \\
0 \arrow[r] & \Lambda \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & \Lambda \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

により $\quad 0 \to \Lambda \to E \to \Lambda \to 0 \in HH^1(\Lambda)$

$0 \to \Lambda \to E' \to \Lambda \to 0 \in$

の積は graded commu.

(4) char $k \neq 2$ and $|x|$ odd $\Rightarrow x^2 = 0$. $\quad$!

Want : $\varphi_M : HH^*(\Lambda) \longrightarrow \mathrm{Ext}^*_\Lambda(M, M)$ hom of graded ring.

Fact $\ {}_\Lambda P_\Lambda$: proj. left Λ^e-mod $\Rightarrow P \otimes_\Lambda M$: proj. Λ-mod. $\forall M$

$(\Lambda \otimes_k \Lambda \otimes_\Lambda M \simeq \Lambda \otimes_k M \simeq \Lambda^{\dim_k M})$

Consequence Let $\cdots \to P^n \to P^{n-1} \to \cdots \to P^0 \to \Lambda \to 0$

be a min. proj. res. of Λ as a Λ^e-module.

This splits as an exact sequence of right Λ-modules

$\Rightarrow \cdots \to P^n \otimes_\Lambda M \to P^{n-1} \otimes_\Lambda M \to \cdots \to P^0 \otimes_\Lambda M \to \Lambda \otimes_\Lambda M \to 0$, $\quad \Lambda \otimes_\Lambda M \cong M$

is a proj. res. of M (not neces. minimal)

Represent an element $\eta \in HH^n(\Lambda)$ as a map $\eta : \Omega^n_{\Lambda^e}(\Lambda) \longrightarrow \Lambda$

This induces a map

$\eta \otimes 1_M : \Omega^n_{\Lambda^e}(\Lambda) \otimes_\Lambda M \longrightarrow \Lambda \otimes_\Lambda M \simeq M.$

which we can interpret as an element in $\mathrm{Ext}^n_\Lambda(M,M)$,

hence define $\varphi_M(\eta) = \eta \otimes 1_M \in \mathrm{Ext}^n_\Lambda(M,M)$

$$\begin{tikzcd}
0 \arrow[r] & \Omega^n_{\Lambda^e}(\Lambda) \arrow[r] \arrow[d, "\eta"'] & P^{n-1} \arrow[r] \arrow[d] & \Omega^{n-1}_{\Lambda^e}(\Lambda) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r, dash] & \Lambda \arrow[r, dash] & M_\eta \arrow[r, dash] & \Omega^{n-1}_{\Lambda^e}(\Lambda) \arrow[r, dash] & 0
\end{tikzcd}$$

$- \otimes_\Lambda M$

$$\begin{tikzcd}[column sep=small]
0 \arrow[r, dash] & \Omega^n_{\Lambda^e}(\Lambda) \otimes_\Lambda M \arrow[r, dash] \arrow[d, "\eta \otimes 1_M"'] & P^{n-1} \otimes_\Lambda M \arrow[rr, dash] \arrow[d] \arrow[dr, dash] & & P^{n-2} \otimes_\Lambda M \arrow[r, dash] & \cdots \arrow[r, dash] & P^0 \otimes_\Lambda M \arrow[r, dash] & M \arrow[r] & 0 \\
& & & \Omega^{n-1}_{\Lambda^e}(\Lambda) \otimes_\Lambda M \arrow[ur, dash] \arrow[d, equal] & & & & & \\
0 \arrow[r, dash] & M \arrow[r, dash] & M_\eta \otimes_\Lambda M \arrow[dr, dash] & & & & & & \\
& & & \Omega^{n-1}_{\Lambda^e}(\Lambda) \otimes_\Lambda M \arrow[r] & 0 & & & &
\end{tikzcd}$$

Can show: φ_M : hom of graded rings.

$HH^*(\Lambda)$ acts on $\mathrm{Ext}^*_\Lambda(M,N)$ on the right and on the left via φ_N and φ_M.

$\theta \in \mathrm{Ext}^*_\Lambda(M,N)$: homogeneous

Let $\eta \in HH^*(\Lambda)$ homog.

$\eta \cdot \theta := \varphi_N(\eta)\theta.$ $\qquad \theta \cdot \eta := \theta \varphi_M(\eta)$

Thm [Yoneda]

$$\eta\theta = (-1)^{|\eta||\theta|}\theta\eta.$$

Cor $\mathrm{Im}\,\varphi_M \subseteq Z_{gr}(\mathrm{Ext}^*_\Lambda(M,M))$

$$:= \langle \underset{\text{homog.}}{z \in \mathrm{Ext}^*_\Lambda(M,M)} \mid z\gamma = (-1)^{|z||\gamma|}\gamma z \quad \forall \gamma \in \mathrm{Ext}^*_\Lambda(M,M) : \text{homog} \rangle$$

Now, let $H \subseteq HH^{even}(\Lambda)$ be a commutative Noeth. graded subalg.

(with $H^0 = Z(\Lambda)$)

Define $V_H(M) = \{ \mathfrak{m} \in \mathrm{MaxSpec}\, H \mid \mathrm{Ann}_H \mathrm{Ext}^*_\Lambda(M,M) \subseteq \mathfrak{m} \}$

Then $V_H(M)$ has all the properties discussed above.

Note (1) $\underline{m}_{gr} = \langle \mathrm{rad}\, H^0, H^{\geq 1} \rangle$: unique max. graded ideal in H.

(Λ : indec を仮定 $\Longrightarrow$ $Z(\Lambda)$ local).

(2) $\mathrm{Ann}_H \mathrm{Ext}^*_\Lambda(M,M) \subseteq \underline{m}_{gr}$ $(M \neq 0)$.

$\{\underline{m}_{gr}\} \subseteq V_H(M)$.

$\Rightarrow$ $V_H(M) = \{\underline{m}_{gr}\}$ M has trivial variety.

(3) $V_H(M) = V_H(M, \Lambda/\underline{r}) \Rightarrow$

$$\sqrt{\mathrm{Ann}_H \mathrm{Ext}^*_\Lambda(M,M)} = \sqrt{\mathrm{Ann}_H \mathrm{Ext}^*_\Lambda(M,\Lambda/\underline{r})} \supseteq \sqrt{\mathrm{Ann}_H \mathrm{Ext}^*_\Lambda(\Lambda/\underline{r},\Lambda/\underline{r})}$$

$\Rightarrow$ $V_H(M) \subseteq V_H(\Lambda/\underline{r})$ $\forall M \in \mathrm{mod}\Lambda$

Hence, underlying geometric object

$$H/\underbrace{\sqrt{\mathrm{Ann}_H \mathrm{Ext}^*_\Lambda(\Lambda/\underline{r},\Lambda/\underline{r})}}_{\mathrm{Ker}\, \varphi_{\Lambda/\underline{r}}}$$

$\varphi_{\Lambda/\underline{r}} : H \to \mathrm{Ext}^*_\Lambda(\Lambda/\underline{r},\Lambda/\underline{r})$.

Prop [Snashall-S, Green-Snashall-S].

(a) $\mathrm{Ker}\, \varphi_{\Lambda/\underline{r}}$ is a nilpotent ideal with nilpotency index at most the Loewy length of Λ

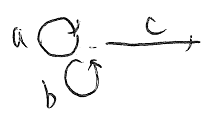
(b) $\sqrt{\mathrm{Ker}\, \varphi_{\Lambda/\underline{r}}} = \langle$ hom. nilpotent element $\rangle =: \mathcal{N}_H$.

$$H := \left\{ \begin{array}{ll} HH^{even}(\Lambda) & \text{char } k \neq 2 \\ HH^*(\Lambda) & \text{char } k = 2 \end{array} \right\} \quad \text{commu.}$$

$\Sigma := H/\mathcal{N}_H \simeq HH^*(\Lambda)/\mathcal{N}_{HH^*(\Lambda)}$

~~Conj~~ [SS] Σ is a fin. gen. k-alg.

No! Fei Xu

Q : 

$kQ/\langle a^2, b^2, ab - ba, bc \rangle$

Counterexample !

§2 Best choice

Koszul algebras

Examples (1) $k[x]$.

$$0 \to k[x] \xrightarrow{x} k[x] \to k \to 0$$

(2) $k[x]/(x^2)$ $\quad \cdots \xrightarrow{x} k[x]/(x^2) \xrightarrow{x} k[x]/(x^2) \to k \to 0$

Recall : $\Lambda = \bigoplus_{i \geq 0} \Lambda_i$ graded k-alg. $\dim_k \Lambda_i < \infty$

a) M is linear module if $\exists$ an exact seq

$$\cdots \to P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

with P_i : fin. gen. (graded) projective Λ-modules gen in degree i, and f_i : degree zero maps

b) Λ : Koszul if $\Lambda/\Lambda_{\geq 1} \cong \Lambda / \bigoplus_{i \geq 1} \Lambda_i \cong \Lambda_0$ is a lin. module

(3) $k[x_1, \cdots, x_n]$
(4) $k\langle x_1, \cdots, x_n \rangle / (x_i^2, \{x_i x_j + x_j x_i\}_{i<j})$ $\Big\}$ Koszul.

Fact Λ : Koszul $\Rightarrow$ $E(\Lambda) = \mathrm{Ext}^*_\Lambda(\Lambda_0, \Lambda_0)$ Koszul

- $\Lambda = kQ/I$ Koszul $\quad I \subseteq \langle \text{arrows} \rangle^2$

$\Rightarrow$ I is quadratic, i.e. $I = \langle I_2 \rangle$.

- $I_2 \subset kQ_2$, $\quad 0 \to I^\perp \to (kQ_2)^* \to I_2^* \to 0$.

$E(\Lambda) \simeq kQ/I^\perp$

Thm [Buchweitz - Green - Snashall - S]
Λ : (fin. dim) Koszul.
$\varphi_{\Lambda/r} : HH^*(\Lambda) \to \mathrm{Ext}^*_\Lambda(\Lambda/r, \Lambda/r) = E(\Lambda)$
Then $\mathrm{Im}\, \varphi_{\Lambda/r} = Z_{gr}(E(\Lambda))$

Koszulを仮定しないと、Im は "$Z_{gr}^{A_\infty}$" になるらしい

Thm [Erdmann - S]. Λ : fin. dim. Koszul TFAE

(a) $\exists H$ s.t. Λ satisfies (Fg)

(b) $HH^*(\Lambda)$: noeth. and $E(\Lambda)$ is a fin. gen. $HH^*(\Lambda)$-module

(c) $Z_{gr}(E(\Lambda))$ Noeth and $E(\Lambda)$ is a fin. gen. $Z_{gr}(E(\Lambda))$-module.

§3. Examples

(1) Λ = exterior alg. as above $= k\langle x_1, \dots, x_n\rangle / (x_i^2,\ x_ix_j + x_jx_i)$

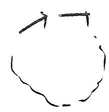
$E(\Lambda) = k\langle x_1, \dots, x_n\rangle / \langle x_ix_j - x_jx_i\rangle = k[x_1, \dots, x_n]$

$$Z_{gr}(E(\Lambda)) = \begin{cases} k[x_1, \dots, x_n] & \text{char } k = 2 \\ k[x_1^2, \dots, x_n^2] & \text{char } k \neq 2 \end{cases}$$ (!)

(2) $\Lambda = kQ / \langle \text{arrows} \rangle^2$ (Koszul)

Q : connected quiver.

$E(\Lambda) = kQ$

$Z_{gr}(E(\Lambda)) = k$ unless ~~Q contai~~ $Q =$ 

(3) $\Lambda = k\langle x_1, \dots, x_n\rangle / (x_i^2, \{x_ix_j + q_{ij}x_jx_i\}_{i<j})$ $q_{ij} \in k \setminus \{0\}$

Λ satisfies (Fg) $\underset{\text{[Erdmann–S]}}{\Longleftrightarrow}$ all q_{ij} are roots of unity [Koszul. selfinjective]

§4 Some (one) application

Thm [Erdmann–Holloway–Snashall–S–Taillefer]

Λ: fin. dim. k-alg. $H \subseteq HH^*(\Lambda)$ and (Fg)

$\underline{a} \subseteq H$ hom. ideal.

Then $\exists M \in \operatorname{mod}\Lambda$ s.t. $V_H(M) = V_H(\underline{a}) = \{\underline{m} \mid \underline{a} \subseteq \underline{m}\}$.

Prop [EHSST]

Λ: (Fg), $\dim H \geq 2 \Rightarrow \Lambda$ is of infinite rep. type

Prop [Bergh–S]

Λ: (Fg) $\underbrace{\Lambda: \text{selfinj.}}_{\text{証明にはいる}}$ $\dim H \geq 3 \Rightarrow \Lambda$ is of wild type

Λ: (Fg), selfinjective. rep.-dim $\Lambda \geq \dim HH^*(\Lambda) + 1$

[Bergh–Iyengar–Krause–Oppermann] can drop (Fg).