Introduction to support varieties

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Λ: fin. dim k-alg. $\quad k = \bar{k}$. $\quad \underline{r}$ = Jacobson radical of Λ

Suppose $H = k[x_1, \cdots, x_t]$ (polynomial ring in $\{x_i\}_{i=1}^{t}$

$\deg x_i \geq 1$) and we have a hom. of graded rings

$$H \xrightarrow{\varphi_M} \operatorname{Ext}^*_\Lambda(M,M) = \bigoplus_{i \geq 0} \operatorname{Ext}^i_\Lambda(M,M)$$

(Yoneda product for all $M \in \operatorname{mod}\Lambda$)

such that the following holds.

$$\begin{tikzcd}
\operatorname{Ext}^*_\Lambda(N,N) & \operatorname{Ext}^*_\Lambda(M,N) \arrow[d, phantom, "\text{actions commutes}"] & \operatorname{Ext}^*(M,M) \\
& H \arrow[ul, "\varphi_N"] \arrow[ur, "\varphi_M"'] &
\end{tikzcd}$$

Consider $\quad A(M,N) = \operatorname{Ann}_H \operatorname{Ext}^*_\Lambda(M, \cancel{M}\,N) \subseteq H$

graded ideal in H. depends on M and N

$V_H(M) \overset{\text{def}}{:=} \{\underline{m} \in \operatorname{Max}\operatorname{Spec} H \mid A(M,M) \subseteq \underline{m}\}$

$V_H(M,N) := \{ \quad\quad A(M,N) \text{ ---- } |$

Properties of $V_H(M,N)$

(1) $V_H(M \oplus M', N) = V_H(M,N) \cup V_H(M',N')$

$\qquad\qquad \cancel{A(M,N)}$
$A(M \oplus M', N) = A(M,N) \cap A(M',N)$

(2) $0 \to N_1 \to N_2 \to N_3 \to 0$ exact in $\operatorname{mod}\Lambda$

$\operatorname{Ext}^*_\Lambda(M,N_1) \to \operatorname{Ext}^*_\Lambda(M,N_2) \to \operatorname{Ext}^*_\Lambda(M,N_3)$: exact

Claim $\quad A(M,N_1)A(M,N_3) \subseteq A(M,N_2)$
$\qquad\qquad \eta' \in A(M,N_1), \quad \eta \in A(M,N_3)$

$$\left(\begin{array}{l} x' \longmapsto x\eta \longmapsto 0 \\ 0 = \eta x' \longmapsto x\eta\eta' \end{array}\right)$$

$$V_H(M,N_2) \subseteq V_H(M,N_1) \cup V_H(M,N_3)$$

In general, $\quad V_H(M,N_r) \subseteq V_H(M,N_s) \cup V_H(M,N_t) \quad \{r,s,t\} = \{1,2,3\}$

$$V_H(M,N) \subseteq V_H(M, \underset{\substack{\parallel \\ 0}}{\underline{r}^n N}) \cup V_H(M, \underline{r}^{n-1}N/\underline{r}^n N) \cup \cdots \cup V_H(M, N/\underline{r}N)$$

$$= V_H\Big(M, \bigoplus_{i=1}^{n-1} \underline{r}^i N/\underline{r}^{i+1} N\Big) \subset V_H(M, \Lambda/\underline{r})$$

Claim $\quad V_H(M, \Lambda/\underline{r}) \subseteq V_H(M,M)$

(∵ Let $\eta \in A(M,M)$, then for $\theta \in \mathrm{Ext}^n_\Lambda(M,N)$

$\eta \cdot \theta = \varphi_{\Lambda/\underline{r}}(\eta)\cdot\theta = \theta\,\varphi_M(\eta) = \theta(\eta \cdot 1_M) = 0$

$\therefore\ A(M,M) \subseteq A(M,\Lambda/\underline{r})$ □)

$$\boxed{V_H(M) = V_H(M, \Lambda/\underline{r}) = V_H(\Lambda/r, M) = V_H(M,M)}$$

(3) $\quad (x_1,\cdots,x_t) \in V_H(M) \quad$ Hの graded maximal ideal は 必ず $V_H(M)$ にある

(4) $\quad M = P \neq 0 \quad$ projective $\Rightarrow \mathrm{Ext}^*_\Lambda(M,M) = \mathrm{Hom}_\Lambda(M,M) \neq (0)$

$\Rightarrow (x_1,\cdots,x_t) \subseteq A(M,M) \subset (x_1,\cdots,x_t)$

$\Rightarrow V_H(\mathrm{proj}) = \{(x_1,\cdots,x_t)\}$ $\quad\therefore$ M has trivial variety

Moreover, $\quad \mathrm{pd}_\Lambda M < \infty \Rightarrow \mathrm{Ext}^*_\Lambda(M,M) = \mathrm{Ext}_\Lambda^{\leq \mathrm{pd}M}(M,M)$

$\Rightarrow (x_1,\cdots,x_t)^{\mathrm{pd}M+1} \subseteq A(M,M)$

$\Rightarrow V_H(M) = \text{trivial}$

Similarly $\quad \mathrm{id}_\Lambda M < \infty$, then $V_H(M) = $ trivial.

Hense, if $\mathrm{gl.dim}\,\Lambda < \infty$, then all varieties of all modules in modΛ are trivial.

(5) $\quad V_H(M) = V_H(\Omega^1_\Lambda(M))$ $\quad$ whenever $\Omega^{\mp 1}_\Lambda(M) \neq 0$ $\quad$ (by (2), $N_2 = \frac{P}{I}$)

$\quad\quad \|$

$\quad V_H(\Omega^{-1}_\Lambda(M))$

> (Fg) $\quad$ Assume that $\mathrm{Ext}^*_\Lambda(M,N)$ are finitely generated modules over H for all $M, N \in \mathrm{mod}\Lambda$

Then $\quad \underline{m} \in V_H(M) \Leftrightarrow \mathrm{Ext}^*_\Lambda(M,M)_{\underline{m}} \neq 0 \ \left(\overset{\text{def}}{\Leftrightarrow} \mathrm{Supp}_H\, \mathrm{Ext}^*_\Lambda(M,M) \ni \underline{m}\right)$

Prop $\quad$ TFAE

(a) $\quad \mathrm{Ext}^*_\Lambda(M,N) \in \mathrm{mod}H \quad \forall M,N \in \mathrm{mod}\Lambda$

(b) $\quad \mathrm{Ext}^*_\Lambda(\Lambda/\underline{r}, \Lambda/\underline{r}) \in \mathrm{mod}H$

proof

(a) $\Rightarrow$ (b) clear.

(b) $\Rightarrow$ (a) Induction on Loewy length.

(semisimple, semisimple) OK.

$$0 \to \underline{r}M \to M \to M/\underline{r}M \to 0$$

$$\begin{tikzcd}[column sep=small]
\mathrm{Ext}^*_\Lambda(M/\underline{r}M, \Lambda/\underline{r}) \arrow[rr] \arrow[dr] & & \mathrm{Ext}^*_\Lambda(M, \Lambda/\underline{r}) \arrow[rr] \arrow[dr] & & \mathrm{Ext}^*_\Lambda(M/\underline{r}M, \Lambda/\underline{r}) \\
& X \arrow[ur] \arrow[dr] & & Z \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & 0 & 0 \arrow[ur] & & 0
\end{tikzcd}$$

exact

$\mathrm{Ext}^*_\Lambda(M/\underline{r}M, \Lambda/\underline{r}) \in \mathrm{mod}\, H$, $\quad \mathrm{Ext}^*_\Lambda(M/\underline{r}M, \Lambda/\underline{r}) \in \mathrm{mod}\, H$

$X, Z \in \mathrm{mod}\, H \Rightarrow \mathrm{mod}\, H$.

$\cdots\cdots$ $\square$

Assume from now on (Fg)

Consequences :

(1) $\mathrm{Ext}^*_\Lambda(\Lambda/\underline{r}, \Lambda/\underline{r}) \in \mathrm{mod}\, H \Rightarrow \mathrm{Ext}^*_\Lambda(\Lambda/\underline{r}, \Lambda/\underline{r})$ Noetherian ring

($(R, \underline{m})$ comm. local Noeth. ring

$\mathrm{Ext}^*_R(R/\underline{m}, R/\underline{m})$ Noeth. $\Leftrightarrow$ R is a complete intersection)

(2) Dimension of varieties

$$\dim V_H(M) \overset{\text{def}}{=} \dim H/A(M,M)$$

$\uparrow$ Krull dim

fact rate of growth of $H/A(M,M)$ as a graded vector space

$= \gamma(H/A(M,M)) = \inf\{ \gamma \in \mathbb{N} \mid \dim_k (H/A(M,M))_n \leq a n^{\gamma-1} \ \forall n > 0 \}$

a: constant

Have $\gamma(H/A(M,M)) = \gamma(\mathrm{Ext}^*_\Lambda(M,M))$

($\mathrm{Ext}^*_\Lambda(M,M)$ fin. gen. module over $H/A(M,M)$)

Also have $\dim H/A(M,M) = \dim H/A(M, \Lambda/\underline{r}) = \gamma(\mathrm{Ext}^*_\Lambda(M, \Lambda/\underline{r}))$

Let $\mathbb{P}_*: \ \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ ~~min. proj. resol.~~ min. proj. resol. of M

$\mathrm{Ext}^i_\Lambda(M, \Lambda/\underline{r}) \simeq \mathrm{Hom}_\Lambda(P_i, \Lambda/\underline{r}) \subseteq \mathrm{Hom}_k(P_i, \Lambda/\underline{r})$

$\dim_k \mathrm{Ext}^i_\Lambda(M, \Lambda/\underline{r}) \leq \dim_k P_i \cdot \dim_k \Lambda/\underline{r}$

$\Rightarrow \gamma(\mathrm{Ext}^*_\Lambda(M, \Lambda/\underline{r})) \leq \gamma(\mathbb{P}_*) =$ the growth of the min. proj. resol. of M.

$= \mathrm{cx}(M) =$ the complexity of M

Can show equality

Hence $\dim V_H(M) = cx(M)$

<u>Note</u> (1) $cx(M) = 0 \Leftarrow$ $\mathrm{pd}_\Lambda M < \infty$ $\underset{\Lambda:\text{ selfinj}}{\Leftarrow}$ M: projective

$\left(\dim_k P_n \le a n^{0-1} = \frac{a}{n}\right)$

(2) $cx(M) \le 1 \Leftrightarrow M$ has a bounded min. proj. resol.

$(\dim_k P_n \le a)$

<u>Prop</u> If H and Λ satisfy (Fg), then Λ is Gorenstein
$(\mathrm{id}\,\Lambda_\Lambda < \infty$ + $\mathrm{id}\,_\Lambda\Lambda < \infty)$

Proof: $V_H(D(\Lambda))$ is trivial. (1)つでも)

$\Rightarrow$ $0 = \dim V_H(D(\Lambda)) = cx(D(\Lambda))$ $\Rightarrow$ $\mathrm{pd}_\Lambda D(\Lambda) < \infty \Leftrightarrow \mathrm{id}\,\Lambda_\Lambda < \infty$.

$V_H(\Lambda)$ is trivial

$\Rightarrow$ $0 = \dim V_H(\Lambda) = \gamma(\mathrm{Ext}^*_\Lambda(\Lambda/\mathfrak{r}, \Lambda))$ $\Rightarrow \mathrm{id}_\Lambda \Lambda < \infty$

Gorenstein $\square$

<u>Group rings</u> (Carlson)

G: finite group, $k = \overline{k}$ field, $\mathrm{char}\,k \mid |G|$.

$H^*(G, k) = \mathrm{Ext}^*_{kG}(k, k)$ — graded ring via Yoneda product

$(a \in k,\ g \in G,\ ga = a)$ — $H^0(G, k) = \mathrm{Hom}_{kG}(k, k) \simeq k$

— graded commutative

$xy = (-1)^{|x||y|} yx \quad \forall x, y$: homog

— fin. gen k-alg

(Evens, Golod, Venkov)

<u>Note</u>: (1) $X, Y \in \mathrm{mod}\,kG$.

$X \otimes_k Y$ is a kG-module via $g(x \otimes y) = gx \otimes gy$

(kG: coalgebra)

(2) $Y \in \mathrm{mod}\,kG$.

$k \otimes_k Y \simeq Y$ as kG-modules

(3) Define $\varphi_M: H^*(G, k) \to \mathrm{Ext}^*_{kG}(M, M)$ by

$$\varphi_M(0 \to k \to E_1 \to \cdots \to E_n \to k \to 0) = (0 \to \underset{\underset{M}{\shortparallel}}{k \otimes_k M} \to E_1 \otimes_k M \to \cdots \to E_n \otimes_k M \to \underset{\underset{M}{\shortparallel}}{k \otimes_k M} \to 0)$$

<u>Can show</u>

- φ_M map of graded rings $H^*(G,k)$
- Given $\theta \in \mathrm{Ext}^m_{kG}(M,N)$ and $\eta \in \mathrm{Ext}^n_{kG}(k,k)$

then $\theta\varphi_M(\eta) = (-1)^{mn}\varphi_N(\eta)\theta$ $\boxed{\text{Why}}$

Hence n even, always equality $\theta\varphi_M(\eta) = \varphi_N(\eta)\theta$

- $\mathrm{Ext}^*_{kG}(M,N)$ is a fin. gen $H^*(G,k)$-module $\forall M,N \in \mathrm{mod}\,kG$ (Evens, Venkov)
- Let $H = H^{\text{even}}(G,k)$, then H is a comm. graded noeth k-alg

~~and~~ $H^*(G,k)$ is fin. gen. over H.

$\Rightarrow$ $\mathrm{Ext}^*_{kG}(M,N) \in \mathrm{mod}\,H$ for all $M,N \in \mathrm{mod}\,kG$

i.e. (Fg) is satisfied !

$\Rightarrow$ $V_G(M) = \{\underline{m} \in \mathrm{MaxSpec}\,H \mid \mathrm{Ann}_H \mathrm{Ext}^*_{kG}(M,M) \subseteq \underline{m}\}$

has all the properties "described above"

- Furthermore, given any graded ideal $\underline{a} \subseteq H$.

there exists M in $\mathrm{mod}\,kG$ s.t.

$V_G(M) = V(\underline{a}) = \{\underline{m} \mid \underline{a} \subseteq \underline{m}\}$.

M is indec.

$\dim V_G(M) = 1 \Leftarrow M \simeq \Omega^n_{kG}(M)$ for some $n \geq 1$. $\boxed{\text{Why}}$

Have a commu. diag.

$$\begin{tikzcd} H^*(G,k) \arrow[r, "\varphi_M"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Ext}^*_{kG}(M,M) \arrow[d, equal] \\ HH^*(kG) = \mathrm{Ext}^*_{(kG)^e}(kG,kG) \arrow[r] & \mathrm{Ext}^*_{kG}(M,M) \end{tikzcd}$$

$(kG)^e = kG \otimes_k (kG)^{op}$

これは $\forall$ f.d. alg でOK.

$R = k[\![x_1,\dots,x_n]\!]/(a_1,\dots,a_t)$ $\{a_1,\dots,a_t\}$ reg seq $\subseteq (x_1,\dots,x_n)^2$

$\{\chi_i\}_{i=1}^t$ operators on $\mathrm{Ext}^*_R(M,N)$

$\rightsquigarrow$ $R[\chi_1,\dots,\chi_t]$-module structure.

$\chi_i \rightsquigarrow$ elements in $HH^2(R) = \mathrm{Ext}^2_{R^e}(R,R)$

$HH^{even}(\Lambda)$ は H の canonical choice だが.

~~一般~~ $\underbrace{\quad H\text{をつねに}}_{(Fg)\text{をみたす}}$. それにできるかどうかは不明

- Nicole Snashall — S
- Erdmann — Holloway — Snashall — S — Taillefer.
- Green — Snashall — S
- Buchweitz — Green — Snashall — S
- ―――――― - Madsen — S

$\mathcal{C}$: tensor triang. cat.

$$e \otimes C \simeq C \simeq C \otimes e$$

$\mathrm{Hom}^*_{\mathcal{C}}(e, e) \simeq \bigoplus_{i \geq 0} \mathrm{Hom}_{\mathcal{C}}(e, e[i])$ は graded commutative

[Alvarez Suarez
Eckmann — Hilton]