Representation rings of Dynkin Quivers II.

Recall $\mathbb{k}$: field. Q: quiver. V, W: rep. of Q.

$V \otimes W$ $\quad V_x \otimes W_x \xrightarrow{V_\alpha \otimes W_\alpha} V_y \otimes W_y$

$\alpha : x \longrightarrow y$ in Q

$R(Q)$: freely generated as an ~~algebr~~ abelian group by

$\{ [V] \mid V : \text{indec. rep of } Q \}$

$[V][W] = \sum_{i \in I} [U_i]$ where $V \otimes W \simeq \bigoplus_{i \in I} U_i$ (indec. U_i)

Let $P \subset Q$ be a subquiver. Define the rep.

χ_P by $\chi_P(x) = \begin{cases} \mathbb{k} & x \in P_0 \\ 0 & x \notin P_0 \end{cases}$ $\quad \chi_P(\alpha) = \begin{cases} 1 & \alpha \in P_1 \\ 0 & \alpha \notin P_1 \end{cases}$

Properties 1) χ_P : ind $\Leftrightarrow$ P is connected.

2) $\chi_P \otimes \chi_{P'} \simeq \chi_{P \cap P'}$

3) $[\chi_Q] = 1_{R(Q)}$

4) $[\chi_P]^2 = [\chi_P]$

Let Q be Dynkin and $\{Q^i\}_{i \in I}$ the set of connected subquivers of Q

Def $e_i = [\chi_{Q^i}]$ $\quad \forall i \in I$.

$j \le i \overset{\text{def}}{\Longleftrightarrow} Q^j \subset Q^i$ $\quad (\Rightarrow e_i e_j = e_j)$

$e_i' := e_i - \sum_{j < i} e_j'$

Thm 1) $R(Q) = \prod_{i \in I} e_i' R(Q)$

2) $e_i' R(Q)$ has the $\mathbb{Z}$-basis $\{ e_i'[V] \mid V : \text{ind.}, \ \mathrm{Supp}(V) = Q^i \}$

3) $e_i'[V] e_i'[W] = \sum_{k=1}^{n} e_i' [U_k]$

$V \otimes W \simeq \bigoplus_{k=1}^{n} U_k \oplus \bigoplus_{l=1}^{m} V_l$

U_k : ind. with $\mathrm{Supp}(U_k) = Q^i$

V_l : ——————— $\subsetneq Q^i$

4) $e_i' R(Q)$ is indep. of Q. It is determined by Q^i.

Rem If Q^i is of type A_1 then $e_i' R(Q) \simeq \mathbb{Z}$.

In particular, if Q is of type A_n, then $R(Q) = \mathbb{Z}^{\frac{n(n+1)}{2}}$

Aim If Q is Dynkin of type ADE_6 then

$$R(Q) \simeq \prod_{i=1}^{n} R_{k_i} \qquad R_k = \mathbb{Z}[T_1, \cdots, T_k] / (T_i T_j)$$

Rem $e' R(Q)$ decomposes into prod. of R_k's.

Inducing representations

Let Q be a quiver. Denote by $\Bbbk Q$ the path category of Q.

$\mathrm{Ob}\, \Bbbk Q = Q_0$

$\Bbbk Q(x, y) :=$ vector space having all paths from x to y as basis

A $\Bbbk Q$-module is a $\Bbbk$-linear functor. $m : \Bbbk Q \longrightarrow \mathrm{Vec}\,\Bbbk$ (resp $\mathrm{vec}\,\Bbbk$)

cat. of vec. sp / $\Bbbk$.

(resp ——— f.d. ———)

The category of $\Bbbk Q$-modules is denoted $\mathrm{Mod}\,\Bbbk Q$ (resp. $\mathrm{mod}\,\Bbbk Q$).

Let P, Q be quivers and $F : \Bbbk P \longrightarrow \Bbbk Q$ a $\Bbbk$-linear functor

Define $F^* : \mathrm{Mod}\,\Bbbk Q \longrightarrow \mathrm{Mod}\,\Bbbk P$.

$m \longmapsto m \circ F$.

Proposition 1 If $F : \Bbbk P \longrightarrow \Bbbk Q$ sends paths to paths,

then $F^* : \mathrm{mod}\,\Bbbk Q \longrightarrow \mathrm{mod}\,\Bbbk P$ satisfies

$$F^*(m \oplus n) = (F^* m) \oplus (F^* n)$$
$$F^*(m \otimes n) = (F^* m) \otimes (F^* n)$$

proof. $x \in P_0$, $\alpha \in P_1$, then

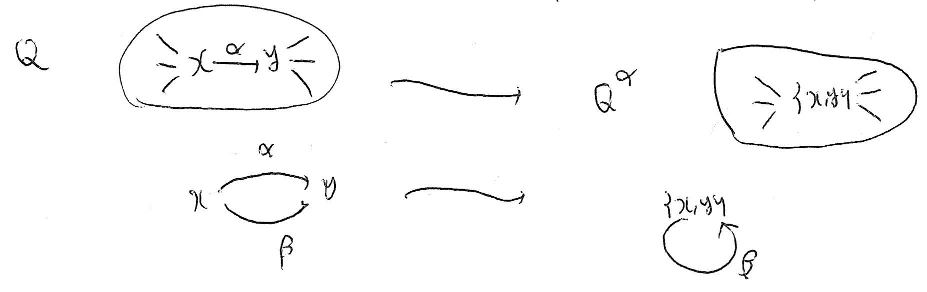
$$F^*(m \otimes n)\underset{\alpha}{(x)} = m \otimes n \underset{\alpha}{(Fx)} \overset{!}{=} m\underset{\alpha}{(Fx)} \otimes n\underset{\alpha}{(Fx)} = (F^* m \otimes F^* n)\underset{\alpha}{(x)}$$

$\square$

Let Q be a quiver and $x \xrightarrow{\alpha} y$ an arrow in Q.

Def Define Q^α by $Q_0^\alpha = (Q_0 \setminus \{x, y\}) \cup \{\{x, y\}\}$.

$Q_1^\alpha = Q_1 \setminus \{\alpha\}$



$$\begin{pmatrix} k & k \oplus kx \\ 0 & k \end{pmatrix} \subset \begin{pmatrix} k[x] & k[x] \\ k[x] & k[x] \end{pmatrix}$$

and $I_\alpha : \Bbbk Q \longrightarrow \Bbbk Q^\alpha$

$$I_\alpha(z) = \begin{cases} \{x,y\} & z \in \{x,y\} \\ z & z \notin \{x,y\} \end{cases}$$

$$I_\alpha(\gamma) = \begin{cases} 1_{\{x,y\}} & \gamma = \alpha \\ \gamma & \gamma \neq \alpha \end{cases}$$

<u>Prop 2</u> The functor $I_\alpha^* : \mathrm{mod}\, \Bbbk Q^\alpha \longrightarrow \mathrm{mod}\, \Bbbk Q$ is full, faithful and satisfies

$$I_\alpha^*(m)(z) = \begin{cases} m(\{x,y\}) & z \in \{x,y\} \\ m(z) & z \notin \{x,y\} \end{cases}$$

$$I_\alpha^*(m)(\gamma) = \begin{cases} 1 & \gamma = \alpha \\ m(\gamma) & \gamma \neq \alpha \end{cases}$$

$I_\alpha^* m$ $\quad \rightrightarrows m(\{x,y\}) \xrightarrow{1} m(\{x,y\}) \leftleftarrows \qquad$ m $\quad \rightrightarrows m(\{x,y\}) \leftleftarrows$

<u>Rem</u> I_α^* reflects isomorphism and preserves indecomposables

<u>Def</u> For $m \in \mathrm{mod}\, \Bbbk Q$, <u>$\underline{\dim}\, m$</u> $= (\dim m(x))_{x \in Q_0}$.

$q_Q : \mathbb{Z}^{Q_0} \longrightarrow \mathbb{Z}$

$$q_Q(d) = \sum_{x \in Q_0} d(x)^2 - \sum_{\alpha \in Q_1} d(t\alpha)\, d(h\alpha)$$

A vector $d \in \mathbb{Z}^{Q_0}$ is called a <u>root</u> if $q_Q(d) = 1$

<u>positive</u> if $d(x) \geq 0 \ \forall x$, $d \neq 0$.

<u>Gabriel's Thm</u> A quiver Q is of finite type if and only if Q is Dynkin.

In that case, the isoclasses of ind. are in bijection with the positive roots of q_Q. via $[m] \longmapsto \underline{\dim}\, m$.

<u>Prop. 3</u> Let Q be Dynkin and $d \in \mathbb{Z}^{Q_0}$ a positive root of q_Q such that there is an arrow $x \xrightarrow{\alpha} y$ such that $d(x) = d(y)$

Then there is a $\Bbbk Q^\alpha$-module m such that the isoclass corresp. to d contains $I_\alpha^*(m)$.

proof. Def $d^\alpha \in \mathbb{Z}^{Q_0^\alpha}$ by $d^\alpha(z) = \begin{cases} d(x) & z = x \alpha y \\ d(z) & z \neq x, y \end{cases}$

Then $$q_{Q^\alpha}(d^\alpha) = \sum_{z\in Q_0\setminus\{x,y\}} d(z)^2 + d(x)^2 - \sum_{\alpha\in Q_1\setminus\alpha} d(t\alpha)\,d(h\alpha)$$

$$= q_Q(d) \qquad \square$$

Use Gabriel's thm for Q^α and I_α^* preserves indec

Let m,n be ind. $\mathbb{k}Q$-modules for some Dynkin quiver Q s.t there is an arrow $x \xrightarrow{\alpha} y$ satisfying $\dim m(x) = \dim m(y)$, $\dim n(x) = \dim n(y)$

Then $\exists\, m^\alpha, n^\alpha \in \mathrm{mod}\,\mathbb{k}Q^\alpha$ s.t. $I_\alpha^*(m^\alpha)\simeq m$, $I_\alpha^*(n^\alpha)\simeq n$

$$\boxed{m\otimes n \simeq (I_\alpha^* m^\alpha)\otimes(I_\alpha^*(n^\alpha)) \simeq I_\alpha^*(m^\alpha\otimes n^\alpha)}$$

Assume Q is a quiver of type $\mathbb{D}_n$. i.e.

$$\begin{array}{l} a \\ \quad\rangle 1 - 2 - \cdots - n\text{-}2 \\ b \end{array}$$

Apart from the subquivers

$$\mathbb{D}_i \quad \begin{array}{l} a \\ \quad\rangle 1 - 2 - \cdots - i \\ b \end{array} \qquad 1\le i\le n-2.$$

There are $\frac{(n+2)(n-1)}{2}$ more connected subquivers all of which are of type $\mathbb{A}$, so
$$R(Q)\simeq \mathbb{Z}^{\frac{(n+2)(n-1)}{2}} \times \prod_{i=1}^{n-1} e_i R(Q) \qquad e_i = [X_{\mathbb{D}_i}]$$

Let $d\in\mathbb{Z}^{Q_0}$ be a positive root of q_Q

— If $d(x)\le 1$ $\forall x$, then we take as representative X_P for some suitable subquiver P

— Otherwise d is of the form $\begin{smallmatrix}1 & & & & & & & & & \\ & 2 & 2 & 2\cdot & 1 & 1 & 1 & 0 & \cdots & 0. \\ 1 & & & & & & & & & \end{smallmatrix}$

By our reduction, it suffices to find the ~~[illegible]~~ omnipresent summand

U of $\begin{pmatrix}1 & & \\ & 2 & 1\\ 1 & & \end{pmatrix} \otimes \begin{pmatrix}1 & & \\ & 2 & 1\\ 1 & & \end{pmatrix}$

V of $\begin{pmatrix}1 & & & \\ & 2 & 2 & 1\\ 1 & & & \end{pmatrix} \otimes \begin{pmatrix}1 & & & \\ & 2 & 1 & 1\\ 1 & & & \end{pmatrix}$

$$U \simeq \begin{cases} 0 & \\ \begin{pmatrix} 1 & & \\ & 2 & 1 \\ 1 & & \end{pmatrix} & \text{or} \\ & \text{else} \end{cases}$$

$$V \simeq \begin{cases} \begin{smallmatrix} 1 & & & \\ & 2 & 2 & 1 \\ 1 & & & \end{smallmatrix} & \text{if} \quad \text{or} \qquad 4 \\ \begin{smallmatrix} 1 & & & \\ & 2 & 1 & 1 \\ 1 & & & \end{smallmatrix} & \text{if} \quad \text{or} \quad \text{or opposite} \qquad 4 \\ 0 & \text{else} \qquad 8 \end{cases}$$

If — … — or — … —, then

we have a multiplicative basis of idempotent and $\boxed{e_i' R(Q) \simeq \mathbb{Z}^2}$

If $\leftarrow \cdots \leftarrow \rightarrow \leftarrow \cdots \leftarrow \rightarrow \cdots \rightarrow \leftarrow \cdots \leftarrow$

$\underbrace{\quad}_{r_1} \qquad \underbrace{\quad}_{r_2} \qquad \underbrace{\quad}_{r_l}$ # of arrows

(e.g. $\rightarrow \leftarrow \rightarrow$ $\quad r_1 = 0,\ r_2 = 1,\ r_3 = 0$)

One can show $\boxed{e_i' R(Q) \simeq \prod_{j=1}^{l} R_{r_j}}$

Let Q be of type $\mathbb{E}_6$

connected subquivers

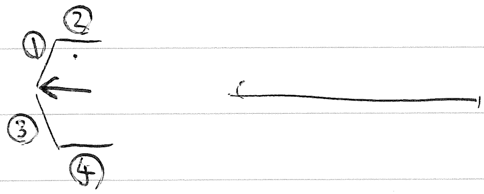
$\mathbb{A}$	21
$\mathbb{D}_4$	1
$\mathbb{D}_5$	2

Positive root. d with. $d_x \geq 1$ $(\forall x)$

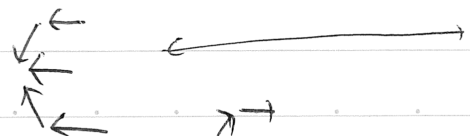
$\begin{smallmatrix} & & 1 & & \\ 1 & 1 & 1 & 1 & 1\end{smallmatrix}$, $\underbrace{\begin{smallmatrix} & & 1 & & \\ 1 & 1 & 2 & 1 & 1\end{smallmatrix},\ \begin{smallmatrix} & & 1 & & \\ 1 & 2 & 2 & 1 & 1\end{smallmatrix},\ \begin{smallmatrix} & & 1 & & \\ 1 & 1 & 2 & 2 & 1\end{smallmatrix},}_{\text{type } D \ \#S<3}$ $\begin{smallmatrix} & & 1 & & \\ 1 & 2 & 2 & 2 & 1\end{smallmatrix}$ $\begin{smallmatrix} & & 2 & & \\ 1 & 2 & 3 & 2 & 1\end{smallmatrix}$.

$\times$

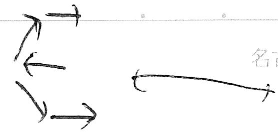
orientation $\rightsquigarrow$ 2×2 matrix with entry 0 or 1.



$\begin{pmatrix} ① & ② \\ ③ & ④ \end{pmatrix}$



$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$



$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

θ	$R(Q)$
$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$\mathbb{Z}^{21} \times R_1 \times R_2^2 \times R_6$.
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\mathbb{Z}^{24} \times R_1 \times R_2 \times R_4$
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	$\mathbb{Z}^{28} \times R_1 \times R_2^2$
$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$	$\mathbb{Z}^{26} \times R_1^5$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\mathbb{Z}^{32} \times R_1^2$
$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\mathbb{Z}^{34} \times R_1$
others	$\mathbb{Z}^{36}$