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Representation rings of Dynkin quivers I.

Recall: $\mathbb{k}$ field. Q: quiver. V, W: representation of Q.

$V \otimes W$: $\quad V_x \otimes W_x \xrightarrow{V(\alpha)\otimes W(\alpha)} V_y \otimes W_y \qquad x \xrightarrow{\alpha} y$ in Q

$R(Q)$ Freely generated by $\{[V] \mid V:$ ind. rep. of $Q\}$

as an abelian group

$$[V]\cdot[W] = \sum_{i\in I} [V_i]$$

$$V\otimes W = \bigoplus_{i\in I} V_i$$

<u>Aim</u> $\quad Q$ - Dynkin $\quad R(Q) = \prod_{i=1}^{n} R_{k_i} \qquad R_k = \mathbb{Z}[T_1, \cdots, T_k]/(T_iT_j)$

$\mathbb{A}, \mathbb{D}, \mathbb{E}_6$.

(conjecture for $\mathbb{E}_7, \mathbb{E}_8$)

<u>Def</u> 1) $\underset{\text{vec}}{\mathrm{Vec}}\ \mathbb{k} :=$ category of vector spaces over $\mathbb{k}$
fin. dim

2) A category $\mathcal{A}$ is called $\mathbb{k}$-linear if

$$\mathcal{A}(a, a') = \mathrm{Hom}_{\mathcal{A}}(a, a') \in \mathrm{Vec}\,\mathbb{k}$$

and composition is $\mathbb{k}$-bilinear

3) Let $\mathcal{A}$ be a small $\mathbb{k}$-linear category
An $\mathcal{A}$-module is a $\mathbb{k}$-linear functor (covariant を考える)

$$m : \mathcal{A} \longrightarrow \underset{\text{vec}}{\mathrm{Vec}}\,\mathbb{k}$$

$\underset{\text{mod}}{\mathrm{Mod}}\,\mathcal{A}$:= The category of $\mathcal{A}$-modules
morphisms are natural transformations

<u>Ex</u> 1) A: $\mathbb{k}$-algebra. $\quad \mathrm{Ob}\,\mathcal{A} = \{a\}$

$\mathcal{A}(a,a) = \mathrm{End}_{\mathcal{A}}(a) \cong A$

An $\underset{m}{\mathcal{A}\text{-module}}$ consists of a $\mathbb{k}$-vector space $M = m(a)$

and an algebra morphism $m: \underset{A}{\overset{\shortparallel}{\mathcal{A}(a,a)}} \longrightarrow \mathrm{End}_{\mathbb{k}}(M)$

2) Q: quiver $\quad \mathrm{Ob}\,\mathcal{A} = Q_0$

$\mathcal{A}(x, y) =$ vector space with basis $\{$ paths from x to y in $Q\}$

Denote $\mathcal{A}$ by $\mathbb{k}Q$.

$m \in \operatorname{mod} \mathbb{k}Q$: $x \in Q_0$. $m(x) = V_x$ vector space $/\mathbb{k}$.

$x \xrightarrow{\alpha} y \rightsquigarrow m(\alpha) : V_x \longrightarrow V_y$

$\operatorname{Mod} \mathbb{k}Q \simeq \operatorname{rep}_{\mathbb{k}} Q \simeq A-\operatorname{Mod}$

$\hookleftarrow$ path alg

Let $\mathcal{A}, \mathcal{B}$ be small $\mathbb{k}$-linear categories and $F : \mathcal{A} \to \mathcal{B}$ be a $\mathbb{k}$-linear functor

1) $F^* : \operatorname{mod} \mathcal{B} \longrightarrow \operatorname{mod} \mathcal{A}$

$m \longmapsto m \circ F$

$(\phi_b)_{b \in \mathcal{B}} \longmapsto (\phi_{F(a)})_{a \in \mathcal{A}}$

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[dr, "F^*(m)"'] & \mathcal{B} \arrow[d, "m"] \\ & \operatorname{Vec}\mathbb{k} \end{tikzcd}$$

2) $\mathcal{A} \otimes \mathcal{B}$ objects $\mathcal{A} \times \mathcal{B}$

$(\mathcal{A} \otimes \mathcal{B})((a,b),(a',b')) = \mathcal{A}(a,a') \otimes \mathcal{B}(b,b')$

Let $m, n \in \operatorname{mod} \mathcal{A}$

$m \otimes n : \mathcal{A} \otimes \mathcal{A} \longrightarrow \operatorname{vec}\mathbb{k}$.

$(a,a') \longmapsto m(a) \otimes n(a')$ $\qquad a, a' \in \mathcal{A}$

$\alpha \otimes \alpha' \longmapsto m(\alpha) \otimes n(\alpha')$ $\qquad \alpha, \alpha'$: morphisms in $\mathcal{A}$

<u>Def</u> $\Delta : \mathbb{k}Q \longrightarrow \mathbb{k}Q \otimes \mathbb{k}Q$.

$x \longmapsto (x, x)$

$\mu \longmapsto \mu \otimes \mu$ (μ : path)

<u>Rem</u> Let $m, n \in \operatorname{mod} \mathbb{k}Q$. Then $\Delta^*(m \otimes n)$ corresponds to the pointwise tensor product

Let Q be a quiver and $P \subset Q$

<u>Def</u> $\chi_P \in \operatorname{mod} \mathbb{k}Q$ by

$$\chi_P(x) = \begin{cases} \mathbb{k} & \text{if } x \in P_0 \\ 0 & \text{if } x \notin P_0 \end{cases}$$

$$\chi_P(\alpha) = \begin{cases} 1_{\mathbb{k}} & \text{if } \alpha \in P_1 \\ 0 & \text{if } \alpha \notin P_1 \end{cases}$$

Ex Q : $\underset{1}{\cdot} \longrightarrow \underset{2}{\cdot} \longrightarrow \underset{3}{\cdot} \longrightarrow \underset{4}{\cdot}$ $\qquad P$ $\bullet \longrightarrow \circ \quad \bullet \quad \times$

$$\chi_P : \ \Bbbk \xrightarrow{1} \Bbbk \xrightarrow{0} \Bbbk \longrightarrow 0$$

Prop 1) χ_P : indec $\iff$ P : connected

2) $\chi_P \otimes \chi_{P'} \xrightarrow{\sim} \chi_{P \cap P'}$

3) $m \otimes \chi_{\mathrm{supp}(m)} \xrightarrow{\sim} m$ $\qquad \mathrm{supp}(m) \subset Q.$

$\mathrm{supp}(m)_0 = \{ x \in Q_0 \mid m(x) \neq 0 \}$

$\mathrm{supp}(m)_1 = \{ \alpha \in Q_1 \mid m(\alpha) \neq 0 \}$

proof 1) $P = P^1 \cup P^2$ (disjoint)

$\Rightarrow \chi_P = \chi_{P^1} \oplus \chi_{P^2}$

2) $(\chi_P \otimes \chi_{P'})(x) = \begin{cases} \Bbbk & x \in P \cap P' \\ 0 & x \notin P \cap P' \end{cases}$

Use $\Bbbk \otimes \Bbbk \xrightarrow{\sim} \Bbbk$

$1 \otimes 1 \longmapsto 1$

3) $(m \otimes \chi_{\mathrm{supp}(m)})(x) = m(x) \otimes \Bbbk \xrightarrow{\sim} m(x)$ $\qquad \square$

Rem By 3). $m \otimes \chi_Q \xrightarrow{\sim} m$ $\qquad [\chi_Q] = 1_{R(Q)}$

By 2) $[\chi_P]^2 = [\chi_{P \cap P}] = [\chi_P]$

$\therefore [\chi_P]$: idempotent $\quad \forall P \subset Q$

Ex Q : type $\mathbb{A}$ $\quad \underset{1}{\cdot} — \cdots — \underbrace{i — i{+}1 — \cdots — j}_{P^{ij}} — \cdots — \underset{n}{\cdot}$

$\chi_{ij} = \chi_{P^{ij}}$

Let $1 \le i \le j \le n \qquad 1 \le i' \le j' \le n$

$k = \max(i, i'), \quad l = \min(j, j')$

Then $P^{ij} \cap P^{i'j'} = \begin{cases} P^{kl} & \text{if } k \le l \\ \phi & \text{else} \end{cases}$

$$\chi_{ij} \otimes \chi_{i'j'} = \begin{cases} \chi_{kl} & k \le l \\ 0 & \text{else} \end{cases}$$

Gabriel's Theorem Let Q be a quiver.

The Tits form of Q is the quadratic form

$$q_Q : \mathbb{Z}^{Q_0} \longrightarrow \mathbb{Z} \qquad q_Q(d) = \sum_{x \in Q_0} d(x)^2 - \sum_{\alpha \in Q_1} d(t\alpha)\, d(h\alpha)$$

$d \in \mathbb{Z}^{Q_0}$ is called a <u>root</u> if $q_Q(d) = 1$ and

<u>positive</u> if $d(x) \geq 0$ for all $x \in Q_0$

$d \neq 0 \in \mathbb{Z}^{Q_0}$

<u>Thm</u> [Gabriel] Let Q be a quiver.

Then Q is of finite type if and only if Q is Dynkin.

In that case the isoclasses of indecomposables are in bijection with the positive roots of q_Q via $[M] \longmapsto \underline{\dim} M$.

If Q is of type $\mathbb{A}$, then the positive roots are

$$(0, \cdots, 0, \underset{i}{1}, \cdots, \underset{j}{1}, 0, \cdots, 0)$$

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ここに (☆) をつかう

<u>Idempotents in $R(Q)$</u>

Let R be a commutative ring and $S \subset R$ a finite subset of idempotents closed under multiplication.

Define 1) $e \leq f \overset{\text{def}}{\iff} ef = e$ for all $e, f \in S$

2) $S \longrightarrow R$ $\quad e \longmapsto e'$

$$e' := e - \sum_{f < e} f' \quad \text{recursively}$$

<u>Prop 3</u> 1) $e = \sum f'$

2) $e' f' = \begin{cases} e' & \text{if } e = f \\ 0 & \text{if } e \neq f \end{cases}$

$\boxed{\text{$e'$ は e の中で それより小さいものに 含まれない部分}}$

$\mathbb{R}^3$ で all idempotents とすると $(1,0,0)$ $(0,1,0)$ $(0,0,1)$ 以外の $(\ )' = 0$

Let Q be Dynkin and $\{Q^i\}_{i \in I}$ the set of connected subquivers of Q.

Set $e_i = [X_{Q^i}]$ and $S = \{e_i\}_{i \in I}$.

$$e_i \leq e_j \iff \underset{\substack{\parallel \\ [X_{Q^i \cap Q^j}]}}{e_i e_j} = e_i = [X_{Q^i}] \iff Q^i \subset Q^j$$

<u>Thm 4</u> In the notation above:

1) $R(Q) = \prod_{i \in I} e'_i R(Q)$

2) $e'_i \; R(Q)$ has the $\mathbb{Z}$-basis $\{ e'_i[m] \mid m\text{-ind.}, \; Supp(m) = Q^i \}$

3) $e'_i[m] \, e'_i[n] = \sum_{k=1}^{n} e'_i[u_k]$, where (~~…~~ $Supp(m) = Q^i$ (n))

$$m \otimes n = \left(\bigoplus_{k=1}^{n} u_k \right) \oplus \left(\bigoplus_{l=1}^{m} v_l \right)$$

u_k is indec with $Supp(u_k) = Q^i$

v_l ―――――――――― $\neq Q^i$.

4) $e'_i R(Q)$ is indep. of Q.

It depends only on Q^i.

(☆) (∘ connected subquiver の ∩ も connected) であれば十分. !

・e'_i の primitive 性は case by case

Rem If Q' is of Type A, $e'_i R(Q) \simeq \mathbb{Z}$!

In particular, if Q is of type A_n, then $R(Q) \simeq \mathbb{Z}^{\frac{n(n+1)}{2}}$

1) $1_{R(Q)} = [\chi_Q] =: e_0$

$e_0 e_i = e_i \Rightarrow e_i \leq e_0 \; (\forall i)$

$3.1 \Rightarrow \quad 1 = e_0 = \sum_{i \in I} e'_i \Rightarrow 1)$

2) Since $Supp(e_i \otimes m) \subset Q^i$. $e_i R(Q)$ is gen. by

$\{ e_i[m] \mid m : \text{ind.} \; supp(m) \subset Q^i \}$

Since $e_i = \sum_{j \leq i} e'_j$ $\quad e_i R(Q) \supset e'_i R(Q)$

So $e'_i R(Q)$ is gen. by $\{ e'_i[m] \}$

If $supp(m) = Q^j \subsetneq Q^i$, then

$$e'_i[m] = e'_i[\chi_{Q^j} \otimes m] = e'_i e_j[m] = \overbrace{e'_i \sum_{k \leq j} e'_k}^{0} [m] = 0 \quad \square$$