Herschend The Clebsch-Gordan problem of quiver representation

quiver. $Q = (Q_0, Q_1)$ Q_0 vertices, Q_1 arrows

Each arrow $\alpha \in Q_1$ has a starting vertex $x \in Q_0$, terminal $y \in Q_0$

$$x \xrightarrow{\alpha} y.$$

k : field. Q : quiver

A rep. V of Q consists of a k-vector space V_x for each $x \in Q_0$

and a k-linear map $V_x \xrightarrow{V(\alpha)} V_y$ for each arrow $x \xrightarrow{\alpha} y$

Ex $x \xrightarrow{\alpha} y \xrightarrow{\beta} z$

V: $V_x \xrightarrow{V(\alpha)} V_y \xrightarrow{V(\beta)} V_z$

W: $W_x \xrightarrow{W(\alpha)} W_y \xrightarrow{W(\beta)} W_z$.

$V \oplus W$ $V_x \oplus W_x \xrightarrow{V(\alpha)\oplus W(\alpha)} V_y \oplus W_y \xrightarrow{V(\beta)\oplus W(\beta)} V_z \oplus W_z$

$\otimes$ $\otimes$ $\otimes$ $\otimes$ $\otimes$ $\otimes$

U is indecomposable if $U = V \oplus W \Rightarrow V = 0$ or $W = 0$ but not both

Assume $\dim V_x < \infty$ for all $x \in Q_0$

Krull-Schmidt Theorem

Let V be a representation of finite length

Then V decomposes uniquely into a direct sum of indecomposables

$$V \simeq \bigoplus_{i=1}^{m} V_i$$

Classification problem

Classify all indecomposable representations of Q

$$V \simeq \bigoplus_{i=1}^{m} V_i$$

$$W \simeq \bigoplus_{j=1}^{n} W_j.$$

KS- decomposition

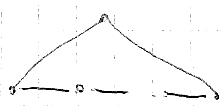
$$V \otimes W \simeq \bigoplus_{i=1}^{m,}\bigoplus_{j=1}^{n} V_i \otimes V_j$$

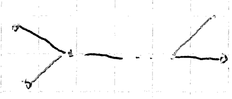
Clebsch-Gordan problem

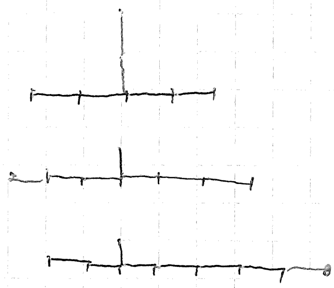
Find the Krull-Schmidt decomposition of $V \otimes W$ for all indecomposable rep's V, W of Q

Known Classifications

extended Dynkin
$\Leftrightarrow$ fin. type
tame

$\widetilde{A}_n$  $n \geq 1$

$\widetilde{D}_n$  $n \geq 4$

$\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$ 

~~extended Dynkin~~

$\Leftrightarrow$

Known solutions to the Clebsch-Gordan problem

Loop $\circlearrowright$

- Aitken 1935, Huppert, Martsinkovsky-Vlassov } $k = \bar{k}$, $\mathrm{char}(k) = 0$
- Iima-Iwamatsu 2006 } $k = \bar{k}$, $\mathrm{char}(k) = p > 0$

Kronecker quiver $\circ \rightrightarrows \circ$ M-thesis 2003

Dynkin type A, D, E_6
extended Dynkin $\widetilde{A}$ } Ph. D. thesis

$\alpha \circlearrowleft \circ \circlearrowright \beta$ $\quad \alpha\beta = \beta\alpha = \alpha^n = \beta^n = 0$

relationが全て commutative or zero relation なら ⊗ は well defined

逆はどうか ?

$k = \bar{k}$ で OK.

Representation ring

Q : quiver

$S(Q) :=$ set of isoclasses of rep. of Q

For $[V], [W] \in S(Q)$ $\quad [V] + [W] = [V \oplus W]$

$\qquad [V] \cdot [W] = [V \otimes W]$

$S(Q)$: commutative semiring

($\sqsubset$ $\mathbb{N}$ のようなもの)

The representation ring is

$R(Q) := K(S(Q))$ Grothendieck ring associated to $S(Q)$

Rem By KS-Theorem. $R(Q)$ is freely generated by the isoclasses of indecomposables as an abelian group

$$[V][W] = \sum_{i=1}^{r} [U_i] \qquad V \otimes W \simeq \bigoplus_{i=1}^{r} U_i \quad \leftarrow \text{indec}$$

Prop If Q is Dynkin, then $R(Q)$ is a $\mathbb{Z}$-order.
(free $\mathbb{Z}$-module of fin. rank)

Def For $k \in \mathbb{N}$, set $R_k = \mathbb{Z}[T_1, \cdots, T_k] / (T_i T_j \mid 1 \leq i, j \leq k)$

$$R_k = \mathbb{Z} \oplus I \qquad \text{rank}(I) = k, \qquad I^2 = \{0\}.$$

Fact If Q is of type A, D or E_6.
then $R(Q) \simeq \prod_{r=1}^{n} R_{k_r}$. (nも $k_1 \sim k_n$も orientationによってかわる)

branching point のであいによってきまる.

The loop

Let $k = \overline{k}$ and $\text{char}(k) = 0$

$$Q \qquad \circlearrowleft \alpha$$

A representation of Q consists of a linear operator $V \circlearrowleft A$

$$\phi : V \circlearrowleft A \longrightarrow V' \circlearrowleft A'$$

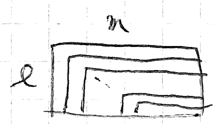
$$\begin{tikzcd} V \arrow[r, "A"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowleft"] & V' \arrow[d, "\phi'"] \\ V \arrow[r, "A'"] & V' \end{tikzcd}$$

If ϕ is an isomorphism, then $A' = \phi A \phi^{-1}$

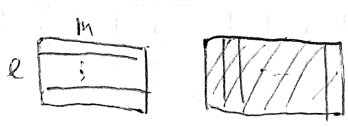
Thm (Jordan Normal Form)
The indecomposable rep's of Q are classified by

$$k^t \circlearrowleft J_\lambda(t) = \begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix} \in M_{t \times t}$$

Thm For all $\lambda, \mu \in k \setminus \{0\}$ and $l, m \in \mathbb{N} \setminus \{0\}$.
the following formula hold.



1) $J_\lambda(l) \otimes J_\mu(m) \simeq \bigoplus_{i=0}^{l-1} J_{\lambda\mu}(l+m-2i-1) \qquad l \leq m$



2) $J_\lambda(l) \otimes J_0(m) \simeq l J_0(m)$

3) $J_0(\ell) \otimes J_0(m) \xrightarrow{\sim} (m-\ell+1) J_0(\ell) \oplus \left(\bigoplus_{i=1}^{\ell-1} 2 J_0(i) \right)$ $\quad \ell \leq m$.

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proof.

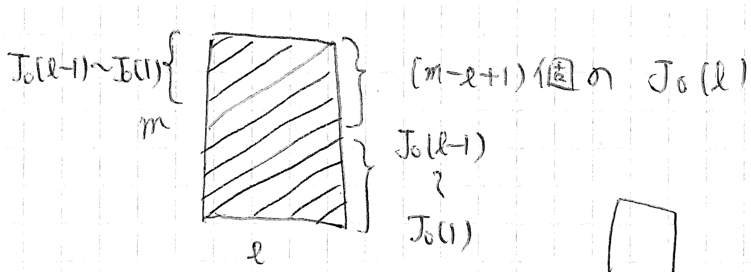
3) $V = k^{\ell} \otimes k^{m}$ has the basis $(e_i \otimes e_j)_{i=1}^{\ell}{}_{j=1}^{m}$

set $e_i \otimes e_j = 0$ if $i > \ell$ or $j > m$

$$J_0(\ell) \otimes J_0(m) \ (e_i \otimes e_j) = e_{i+1} \otimes e_{j+1}.$$

For each $r \in \mathbb{Z}$. let $U_r = \mathrm{span}_k \{ e_i \otimes e_j \mid i - j = r \}$.

$$V = \bigoplus_{r \in \mathbb{Z}} U_r.$$



$\square$

2) $U_r = \mathrm{span} \{ (J_\lambda^j(\ell) e_r) \otimes e_j \mid 1 \leq j \leq m \}$

$\Rightarrow V = \bigoplus_{r \in \mathbb{Z}} U_r$. , $U_r \simeq J_0(m)$

1) $J_1(2) \otimes J_\lambda(\ell) = \left(\begin{array}{ccc|ccc} \lambda & 1 & 0 & \lambda & 1 & 0 \\ & \ddots & 1 & & \ddots & 1 \\ 0 & & \lambda & 0 & & \lambda \\ \hline & & & \lambda & 1 & 0 \\ & 0 & & & \ddots & 1 \\ & & & 0 & & \lambda \end{array} \right) \sim \cdots \sim J_\lambda(\ell+1) \oplus J_\lambda(\ell-1)$

char $k = 0$ でなければ $2J_\lambda(\ell)$ になったりする

を実際に示す

$[J_\lambda(\ell+1)] = [J_1(2)][J_\lambda(\ell)] - [J_\lambda(\ell-1)]$.

これをつかって. induction で 示される

$\square$

$R' = \mathrm{span}_{\mathbb{Z}} \{ J_1(\ell) \mid \ell \in \mathbb{N} \setminus \{0\} \} \subset R(Q) \qquad Q = (\cdot \circlearrowleft)$

Assume $\mathrm{char}(k) = p > 0$

Let $G_\alpha = \langle \sigma_\alpha \rangle$: cyclic group of order $p^\alpha =: q$

$\sigma_\alpha - 1 \in kG_\alpha, \qquad (\sigma_\alpha - 1)^q = \sigma_\alpha^q - 1^q = 0.$

$kG_\alpha = k[\sigma_\alpha - 1] \xrightarrow{\sim} k[T]/T^q$

<u>Thm</u> The modules $V_s = kG_\alpha / (\sigma_\alpha - 1)^s \quad 1 \leq s \leq q$ classify all indecomposable kG_α-modules

V_s has the basis $(\sigma_\alpha - 1)^i$ $\quad 0 \leq i \leq s-1$

$$\sigma_\alpha (\sigma_\alpha - 1)^i = (\sigma_\alpha - 1)^{i+1} + (\sigma_\alpha - 1)^i.$$

σ_α has the matrix $J_1(s)$

普通の 表現 の テンサー積

$A_\alpha = R(kG_\alpha)$ 積は. (σ_α を arrow とおしごくに 相当)

(Green J. A.)

$$A_0 \subset A_1 \subset \cdots \subset A = \bigcup_\alpha A_\alpha \xrightarrow{\sim} R^1$$