d-Calabi-Yau algebras and d-cluster tilting subcategories

Osamu Iyama

Recently, the the concept of cluster tilting object played an important role in representation theory [BMRRT]. The concept of d-cluster tilting subcategories (maximal $(d-1)$-orthogonal subcategories) was introduced in [I1,2][KR1]. They were used to study higher-dimensional analogy of Auslander-Reiten theory, and to classify rigid Cohen-Macaulay modules over certain quotient singularities [IY].

Let $\mathcal{A}$ be an abelian or triangulated category. Put $\mathrm{Ext}^i_{\mathcal{A}}(X,Y) := \mathrm{Hom}_{\mathcal{A}}(X,Y[i])$ for triangulated case. A full subcategory $\mathcal{C}$ of $\mathcal{A}$ is called *functorially finite* if for any $X \in \mathcal{A}$, there exist morphisms $f \in \mathcal{A}(C,X)$ and $g \in \mathcal{A}(X,C')$ such that $C, C' \in \mathcal{C}$, and $\mathcal{A}(-,C) \xrightarrow{\cdot f} \mathcal{A}(-,X) \to 0$ and $\mathcal{A}(C',-) \xrightarrow{g\cdot} \mathcal{A}(X,-) \to 0$ are exact on $\mathcal{C}$.

A functorially finite subcategory $\mathcal{C}$ of $\mathcal{A}$ is called *d-cluster tilting* if

$$\begin{aligned} \mathcal{C} &= \{X \in \mathcal{A} \mid \mathrm{Ext}^i_{\mathcal{A}}(\mathcal{C},X) = 0 \text{ for any } 0 < i < d\} \\ &= \{X \in \mathcal{A} \mid \mathrm{Ext}^i_{\mathcal{A}}(X,\mathcal{C}) = 0 \text{ for any } 0 < i < d\}. \end{aligned}$$

Let R be a complete Gorenstein local ring, and Λ an R-algebra which is a finitely generated R-module. We denote by $\mathrm{mod}\,\Lambda$ (resp. $\mathrm{fl}\,\Lambda$) the category of finitely generated (resp. finite length) left Λ-modules, by $\mathcal{D}^b(\mathrm{fl}\,\Lambda)$ the bounded derived category of $\mathrm{fl}\,\Lambda$, and by $D : \mathrm{fl}\,R \to \mathrm{fl}\,R$ the Matlis duality of R.

We say that an R-linear triangulated category $\mathcal{A}$ *d-Calabi-Yau* if $\mathcal{A}(X,T) \in \mathrm{fl}\,R$ and there exists a functorial isomorphism

$$\mathcal{A}(X,Y) \simeq D\,\mathcal{A}(Y,X[d])$$

for any $X, Y \in \mathcal{A}$. We call Λ *d-Calabi-Yau* if $\mathcal{D}^b(\mathrm{fl}\,\Lambda)$ is d-Calabi-Yau.

1 Main results

Let Λ be a basic d-Calabi-Yau algebra ($d \geq 2$) with a complete set $\{e_1, \cdots, e_n\}$ of orthogonal primitive idempotents. Put

$$\begin{aligned} I_i &:= \Lambda e_i \Lambda \\ S_i &:= \Lambda/(I_i + J_\Lambda) \end{aligned}$$

for each i. *We assume that*

$$\mathrm{Ext}^l_\Lambda(S_i,S_i) = 0 \quad \textit{for any } 1 \leq i \leq n,\ 0 < l < d.$$

A *(left) tilting chain* is a (finite or infinite) decreasing sequence

$$\Lambda = T_0 \supset T_1 \supset T_2 \supset \cdots$$

of two-sided ideals of Λ satisfying

$$\begin{aligned} T_{i+1} = T_i I_{a_i} &\quad \text{for some } 1 \leq a_i \leq n, \\ \mathrm{Tor}^\Lambda_l(T_i, \Lambda/I_{a_i}) = 0 &\quad \text{for any } l > 0, \end{aligned}$$

for any $i \geq 0$. Put

$$\Lambda_i := \Lambda/T_i.$$

Then we have a sequence of surjections $\cdots \to \Lambda_3 \to \Lambda_2 \to \Lambda_1$ of algebras. Put

$$\begin{aligned}
\mathcal{C} &:= \operatorname{add}\{\Lambda_i \mid 0 \leq i\} \quad \subset \quad \operatorname{fl}\Lambda, \\
\mathcal{A}_m &:= \{X \in \operatorname{mod}\Lambda_m \mid \operatorname{Ext}^l_{\Lambda_m}(X, \Lambda_m) = 0 \text{ for any } l > 0\} \quad \subset \quad \operatorname{mod}\Lambda_m, \\
\underline{\mathcal{A}}_m &:= \mathcal{A}_m/[\Lambda_m], \\
\mathcal{C}_m &:= \operatorname{add}\{\Lambda_i \mid 0 \leq i \leq m\} \quad \subset \quad \operatorname{mod}\Lambda_m, \\
\underline{\mathcal{C}}_m &:= \mathcal{C}_m/[\Lambda_m],
\end{aligned}$$

where we denote by $[\Lambda_m]$ the ideal of $\mathcal{A}_m$ (resp. $\mathcal{C}_m$) consisting of morphisms which factor through projective Λ_m-modules. Now we can state our main result.

1.1 Theorem *(1) For any $m \geq 0$, $\underline{\mathcal{A}}_m$ forms a d-Calabi-Yau triangulated category and $\underline{\mathcal{C}}_m$ forms a d-cluster tilting subcategory of $\underline{\mathcal{A}}_m$.*

(2) Assume $\bigcap_{i\geq 0} T_i = 0$. Then $\mathcal{C}$ forms a d-cluster tilting subcategory of $\operatorname{fl}\Lambda$.

2 Tilting ideals

In this section, we give a proof of our main theorem. A key role is played by tilting ideals over Calabi-Yau algebras.

We call $T \in \operatorname{mod}\Lambda$ *tilting* if $\operatorname{pd}_\Lambda T < \infty$, $\operatorname{Ext}^i_\Lambda(T, T) = 0$ for any $i > 0$, and there exists an exact sequence $0 \to \Lambda \to T_0 \to \cdots \to T_n \to 0$ with $T_i \in \operatorname{add}_\Lambda T$.

For tilting Λ-modules T and U, we write $T \leq U$ if $\operatorname{Ext}^i_\Lambda(T, U) = 0$ for any $i > 0$. It is well-known that basic tilting Λ-modules forms a partially ordered set.

We call a two-sided ideal T of Λ *tilting* if T is a tilting Λ-module, and *cofinite* if Λ/T has finite length as a Λ-module.

2.1 Proposition *(1) I_i is a cofinite tilting ideal of Λ.*

(2) T_m is a cofinite tilting ideal of Λ.

(3) We have $T_0 < T_1 < T_2 < \cdots$.

Proof (1) This is shown in [IR].

(2) Assume that T_i is a cofinite tilting ideal of Λ. Then $T_i \overset{\mathbf{L}}{\otimes}_\Lambda I_{a_i}$ is a tilting complex over Λ (e.g. [Y]). Since $\operatorname{Tor}^\Lambda_l(T_i, I_{a_i}) = 0$ for any $l > 0$, we have $T_i \overset{\mathbf{L}}{\otimes}_\Lambda I_{a_i} = T_i \otimes_\Lambda I_{a_i}$. Since $\operatorname{Tor}^\Lambda_1(T_i, \Lambda/I_{a_i}) = 0$, we have $T_i \otimes_\Lambda I_{a_i} = T_i I_{a_i} = T_{i+1}$. Thus T_{i+1} is a tilting Λ-module. Obviously it is a cofinite ideal.

(3) Take a projective resolution $0 \to P_d \to \cdots \to P_0 \to I_{a_i} \to 0$ of the Λ-module I_{a_i}. Applying $T_i \otimes_\Lambda I_{a_i}$, we have an exact sequence

$$0 \to T_i \otimes_\Lambda P_d \to \cdots \to T_i \otimes_\Lambda P_0 \to T_{i+1} \to 0.$$

Applying $\operatorname{Hom}_\Lambda(T_i, -)$, we have $\operatorname{Ext}^l_\Lambda(T_i, T_{i+1}) = 0$ for any $l > 0$. ∎

2.2 Lemma *Let T and U be cofinite tilting ideals of Λ. If $T \leq U$, then $\operatorname{Ext}^i_\Lambda(\Lambda/T, \Lambda/U) = \operatorname{Ext}^i_\Lambda(\Lambda/U, \Lambda/T) = 0$ for any i $(0 < i < d)$.*

Proof Consider exact sequences $0 \to \Omega^i(\Lambda/T) \xrightarrow{a} P_{i-1} \to \cdots \to P_1 \to P_0 \to \Lambda/T \to 0$ with projective Λ-modules P_j and $0 \to U \to \Lambda \xrightarrow{b} \Lambda/U \to 0$. We have a commutative diagram

$$\begin{array}{ccccccc}
 & & \operatorname{Ext}^1_\Lambda(\Omega^i(\Lambda/T), U) = \operatorname{Ext}^i_\Lambda(T, U) = 0 & & & & \\
 & & \uparrow & & & & \\
\operatorname{Hom}_\Lambda(P_{i-1}, \Lambda/U) & \xrightarrow{a\cdot} & \operatorname{Hom}_\Lambda(\Omega^i(\Lambda/T), \Lambda/U) & \longrightarrow & \operatorname{Ext}^i_\Lambda(\Lambda/T, \Lambda/U) & \longrightarrow & 0 \\
\uparrow{\cdot b} & & \uparrow{\cdot b} & & & & \\
\operatorname{Hom}_\Lambda(P_{i-1}, \Lambda) & \xrightarrow{a\cdot} & \operatorname{Hom}_\Lambda(\Omega^i(\Lambda/T), \Lambda) & \longrightarrow & \operatorname{Ext}^i_\Lambda(\Lambda/T, \Lambda) = 0 & &
\end{array}$$

of exact sequences. Thus we have $\operatorname{Ext}^i_\Lambda(\Lambda/T, \Lambda/U) = 0$. Since Λ is d-Calabi-Yau, we have the assertion.$\blacksquare$

2.3 Proposition *For any cofinite tilting ideal T, we have $\operatorname{id}_{\Lambda/T}(\Lambda/T) \le d-1$ and $\operatorname{id}_{(\Lambda/T)^{op}}(\Lambda/T) \le d-1$.*

Proof We only show $\operatorname{id}_{(\Lambda/T)^{op}}(\Lambda/T) \le d-1$. Let $0 \to \Omega_{d-1} \to P_{d-2} \to \cdots \to P_0 \to D(\Lambda/T) \to 0$ be an exact sequence with with projective Λ-modules P_j. We have $\operatorname{Tor}_i^\Lambda(\Lambda/T, D(\Lambda/T)) = D\operatorname{Ext}^i_{\Lambda^{op}}(\Lambda/T, \Lambda/T) = 0$ for any i $(0 < i < d)$ by 2.2. Applying $\Lambda/T \otimes_\Lambda -$, we have an exact sequence

$$0 \to \Lambda/T \otimes_\Lambda \Omega_{d-1} \to \Lambda/T \otimes_\Lambda P_{d-2} \to \cdots \to \Lambda/T \otimes_\Lambda P_0 \to D(\Lambda/T) \to 0.$$

Thus we only have to show that $\Lambda/T \otimes_\Lambda \Omega_{d-1}$ is a projective Λ/T-module, or equivalently, the functor $\operatorname{Hom}_{\Lambda/T}(\Lambda/T \otimes_\Lambda \Omega_{d-1}, -) = \operatorname{Hom}_\Lambda(\Omega_{d-1}, -)$ is an exact functor on $\operatorname{mod}\Lambda/T$. This is equivalent to that the functor $\operatorname{Ext}^1_\Lambda(\Omega_{d-1}, -)$ preserves monomorphisms in $\operatorname{mod}\Lambda/T$. This follows from functorial isomorphisms

$$\operatorname{Ext}^1_\Lambda(\Omega_{d-1}, -) = \operatorname{Ext}^d_\Lambda(D(\Lambda/T), -) = D\operatorname{Hom}_\Lambda(-, D(\Lambda/T)) = D\operatorname{Hom}_{\Lambda/T}(-, D(\Lambda/T)) = 1$$

on $\operatorname{mod}\Lambda/T$ since $D(\Lambda/T)$ is an injective Λ/T-module.$\blacksquare$

To give a proof of 1.1(2), we shall need the following easy observaiton which relates extensions in $\underline{\mathcal{A}}_m$ and $\operatorname{fl}\Lambda$.

2.4 Proposition *Let T be a cofinite tilting ideal of Λ and*

$$\mathcal{A} := \{X \in \operatorname{mod}\Lambda/T \mid \operatorname{Ext}^i_{\Lambda/T}(X, \Lambda/T) = 0 \text{ for any } i > 0\}.$$

(1) $\mathcal{A}$ is a Frobenius category with enough projective-injectives Λ/T.
(2) $\operatorname{Ext}^i_\Lambda(\mathcal{A}, \Lambda/T) = 0 = \operatorname{Ext}^i_\Lambda(\Lambda/T, \mathcal{A})$ for any $0 < i < d$.
(3) $\operatorname{Ext}^i_\Lambda(X, Y) = \operatorname{Ext}^i_{\Lambda/T}(X, Y)$ for any $X, Y \in \mathcal{A}$ and $0 < i < d$.
(4) The $\underline{\mathcal{A}} := \mathcal{A}/[\Lambda/T]$ is a d-Calabi-Yau triangulated category.
(5) $\mathcal{A}$ is an extension closed subcategory of $\operatorname{fl}\Lambda$.

Proof (1) The assertion follows from 2.3 and Happel's result [H].

(2) We only have to show the right equality. For any $X \in \mathcal{A}$, we can take an exact sequence $0 \to X \to P^0 \to \cdots \to P^{d-2} \to Y \to 0$ with $P^i \in \operatorname{add}\Lambda/T$ and $Y \in \mathcal{A}$. We

apply $\mathrm{Hom}_\Lambda(\Lambda/T,-) = \mathrm{Hom}_{\Lambda/T}(\Lambda/T,-)$, then the sequence does not change. Since we have $\mathrm{Ext}^i_\Lambda(\Lambda/T,\Lambda/T) = 0$ for any $0 < i < d$ by 2.2, we have the right equality.

(3) Take an exact sequence $0 \to \Omega^i_{\Lambda/T}X \to P_{i-1} \to \cdots \to P_0 \to X \to 0$ with $P_j \in \mathrm{add}\,\Lambda/T$. Applying $\mathrm{Hom}_\Lambda(-,Y)$, we obtain an exact sequence

$$\mathrm{Hom}_\Lambda(P_{i-1},Y) \to \mathrm{Hom}_\Lambda(\Omega^i_{\Lambda/T}X,Y) \to \mathrm{Ext}^i_\Lambda(X,Y) \to 0$$

since we have $\mathrm{Ext}^i_\Lambda(\Lambda/T,Y) = 0$ for any $0 < i < d$ by (2). This implies $\mathrm{Ext}^i_\Lambda(X,Y) \simeq \underline{\mathrm{Hom}}_{\Lambda/T}(\Omega^i_{\Lambda/T}X,Y) = \mathrm{Ext}^i_{\Lambda/T}(X,Y)$.

(4) Since Λ is d-Calabi-Yau, we have a functorial isomorphism

$$\underline{\mathcal{A}}(X,Y) \simeq \underline{\mathcal{A}}(X[1],Y[1]) \simeq \mathrm{Ext}^1_\Lambda(X[1],Y) \simeq D\,\mathrm{Ext}^{d-1}_\Lambda(Y,X[1]) \simeq D\underline{\mathcal{A}}(Y,X[d]).$$

(5) Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in $\mathrm{fl}\,\Lambda$ with $X, Z \in \mathcal{A}$. Then there exists a monomorphism $X \to P$ with $P \in \mathrm{add}\,\Lambda/T$. Since $\mathrm{Ext}^1_\Lambda(Z,P) = 0$ by (2), we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \cap & & \cap & & \| & & \\
0 & \longrightarrow & P & \longrightarrow & P \oplus Z & \longrightarrow & Z & \longrightarrow & 0
\end{array}$$

of exact sequences. Since Y is a submodule of $P \oplus Z \in \mathrm{mod}\,\Lambda/T$, it is a Λ/T-module. Since $\mathcal{A}$ is obviously extension closed in $\mathrm{mod}\,\Lambda/T$, we have $Y \in \mathcal{A}$. ∎

We need the following general observation.

2.5 Lemma *Let Δ be a finite dimensional algebra and $\mathcal{C} = \mathrm{add}\,M$ a full subcategory of $\mathrm{mod}\,\Delta$ containing Δ. Assume that $\Gamma := \mathrm{End}_\Delta(M)$ satisfies $\mathrm{gl.dim}\,\Gamma \le d+1$ and $\mathrm{pd}_\Gamma D(M) \le d-1$. Then, for any $X \in \mathrm{mod}\,\Delta$, there exists an exact sequence*

$$0 \to C_{d-1} \to \cdots \to C_0 \to X \to 0$$

which is a right $\mathcal{C}$-resolution of X.

Proof Take an injective resolution $0 \to X \to I_0 \to I_1$ in $\mathrm{mod}\,\Delta$. Applying $\mathrm{Hom}_\Delta(M,-)$, we have an exact sequence $0 \to \mathrm{Hom}_\Delta(M,X) \to \mathrm{Hom}_\Delta(M,I_0) \to \mathrm{Hom}_\Delta(M,I_1)$. Since $\mathrm{pd}_\Gamma \mathrm{Hom}_\Lambda(M,I_i) \le d-1$ by our assumption, we have $\mathrm{pd}_\Gamma \mathrm{Hom}_\Lambda(M,X) \le 1$ by $\mathrm{gl.dim}\,\Gamma \le d+1$. Take a projective resolution

$$0 \to P_{d-1} \to \cdots \to P_0 \to \mathrm{Hom}_\Lambda(M,X) \to 0$$

of the Γ-module $\mathrm{Hom}_\Lambda(M,X)$. Then this is the image of a complex

$$0 \to C_{d-1} \to \cdots \to C_0 \to X \to 0$$

with $C_i \in \mathcal{C}$ under the functor $\mathrm{Hom}_\Delta(M,-)$. This complex is exact because M is a generator. ∎

2.6 Lemma *The following assertions hold for any $m \ge 0$.*

(1)$_m$ For any $X \in \operatorname{mod}\Lambda_m$, there exists an exact sequence $0 \to C_{d-1} \to \cdots \to C_0 \xrightarrow{f} X \to 0$ *with* $C_i \in \mathcal{C}_m$.

(2)$_m$ Put $M_m := \bigoplus_{i=0}^m \Lambda_i$ *and* $\Gamma_m := \operatorname{End}_\Lambda(M_m)$. *Then* $\operatorname{gl.dim}\Gamma_m \le d+1$.

PROOF We assume $(1)_{m-1}$ and $(2)_{m-1}$, and prove $(1)_m$ and $(2)_m$.

$(2)_m$ Since there are no loop for Λ, we have $T_{m-1}J_\Lambda \subseteq T_m$, so $J_\Lambda\Lambda_m$ is a Λ_{m-1}-module. For any $X \in \operatorname{ind}\mathcal{C}_m$, we take a sink map $f : C_0 \to X$ in $\mathcal{C}_m$.

First we consider the case when X is not a projective Λ_m-module. Since $\Lambda_m \in \mathcal{C}_m$, we have that f is surjective. Decompose $C_0 = C_0' \oplus P$ with $C_0' \in \mathcal{C}_{m-1}$ and a projective Λ_m-module P. Since f is right minimal, we have $\operatorname{Ker} f \subseteq C_0' \oplus J_\Lambda P$, so $\operatorname{Ker} f$ is a Λ_{m-1}-module by the above remark. It follows from $(1)_{m-1}$ that there exists an exact sequence $0 \to C_d \to \cdots \to C_1 \to \operatorname{Ker} f \to 0$. We have an exact sequence

$$0 \to C_d \to \cdots \to C_1 \to C_0 \xrightarrow{f} X \to 0.$$

Applying $\operatorname{Hom}_\Lambda(M_m, -)$, we have that the simple Γ_m-module $\operatorname{top}\operatorname{Hom}_\Lambda(M_m, X)$ has projective dimension at most $d+1$.

Next consider the case when X is not a projective Λ_m-module. Then $J_\Lambda X$ is a Λ_{m-1}-module by the above remark. By $(1)_{m-1}$, there exists an exact sequence $0 \to C_{d-1} \to \cdots \to C_0 \to J_\Lambda X \to 0$ with $C_i \in \mathcal{C}_{m-1}$. Thus we have an exact sequence

$$0 \to C_{d-1} \to \cdots \to C_0 \to X.$$

Applying $\operatorname{Hom}_\Lambda(M_m, -)$, we have that the simple Γ_m-module $\operatorname{top}\operatorname{Hom}_\Lambda(M_m, X)$ has projective dimension at most d.

Consequently, any simple Γ_m-module has projective dimension at most $d+1$, and we have shown $(2)_m$.

$(1)_m$ By 2.3, we can take a projective resolution $0 \to P_{d-1} \to \cdots \to P_0 \to D(\Lambda_m) \to 0$ in $\operatorname{mod}\Lambda_m$. We have $\operatorname{Ext}^i_\Lambda(M_m, \Lambda_m) = 0$ for any i $(0 < i < d)$. Applying $\operatorname{Hom}_\Lambda(M_m, -)$, we have an exact sequence $0 \to \operatorname{Hom}_\Lambda(M_m, P_{d-1}) \to \cdots \to \operatorname{Hom}_\Lambda(M_m, P_0) \to D(M_m) \to 0$. Thus we have $\operatorname{pd}_{\Gamma_m} D(M_m) \le d-1$. Applying 2.5 to $(\Delta, \mathcal{C}) := (\Lambda_m, \mathcal{C}_m)$, we proved that $(1)_m$ holds. ∎

2.7 Proof of 1.1 By 2.2, we have $\operatorname{Ext}^i_\Lambda(\mathcal{C}, \mathcal{C}) = 0$ for any $0 < i < d$.

(1) We have $\underline{\mathcal{A}}_m(X, Y[i]) = \operatorname{Ext}^i_\Lambda(X, Y)$ for any $X, Y \in \mathcal{A}_m$ and $0 < i < d$ by 2.4(3). In particular, we have $\underline{\mathcal{A}}_m(\mathcal{C}_m, \mathcal{C}_m[i]) = \operatorname{Ext}^i_\Lambda(\mathcal{C}_m, \mathcal{C}_m) = 0$ for any $0 < i < d$.

On the other hand, take any $X \in \mathcal{A}_m$ such that $\underline{\mathcal{A}}_m(X, \mathcal{C}_m[i]) = 0$ for any $0 < i < d$. By 2.6$(1)_m$, we can take an exact sequence $0 \to C_{d-1} \to \cdots \to C_0 \xrightarrow{f} X \to 0$ with $C_i \in \mathcal{C}_m$. Using $\operatorname{Ext}^i_\Lambda(X, \mathcal{C}_m) = 0$ for any $0 < i < d$, we have that f splits. Thus $X \in \mathcal{C}_m$.

Since $\underline{\mathcal{A}}_m$ is d-Calabi-Yau by 2.4(4), any $X \in \mathcal{A}_m$ satisfying $\underline{\mathcal{A}}_m(\mathcal{C}_m, X[i]) = 0$ for any $0 < i < d$ belongs to $\mathcal{C}_m$.

(2) Take any $X \in \operatorname{fl}\Lambda$ such that $\operatorname{Ext}^i_\Lambda(X, \mathcal{C}) = 0$ for any $0 < i < d$. Since $\bigcap_{i \ge 0} T_i = 0$, there exists m such that $X \in \operatorname{mod}\Lambda_m$. By 2.6$(1)_m$, we can take an exact sequence $0 \to C_{d-1} \to \cdots \to C_0 \xrightarrow{f} X \to 0$ with $C_i \in \mathcal{C}_m$. Then f splits because we have $\operatorname{Ext}^i_\Lambda(X, \mathcal{C}_m) = 0$ for any $0 < i < d$, we have that f splits. Thus $X \in \mathcal{C}_m \subset \mathcal{C}$.

Since Λ is d-Calabi-Yau, any $X \in \operatorname{fl}\Lambda$ satisfying $\operatorname{Ext}^i_\Lambda(\mathcal{C}, X) = 0$ for any $0 < i < d$ belongs to $\mathcal{C}$. ∎

3 Examples and remarks

Let Λ be a preprojective algebra of extended Dynkin type and W the associated affine Weyl group. For a primitive idempotent e_i of Λ, put $I_i := \Lambda(1-e_i)\Lambda$. The following is given in [IR].

3.1 Theorem *For any element $w \in W$, take a reduced expression $w = s_{i_1}\cdots s_{i_k}$ and put $\Lambda^w := I_{i_1}\cdots I_{i_k}$. Then Λ^w depends only on w. We have a bijection $w \mapsto \Lambda^w$ from W to the set of isoclasses of basic tilting Λ-modules.*

3.2 Corollary *Put $M_m := \bigoplus_{i=0}^m \Lambda_i$ and $\Gamma_m := \operatorname{End}_\Lambda(M_m)$. Then $\operatorname{gl.dim}\Gamma_m \le d+1$ and there exist exact sequences*

$$\begin{array}{ll} 0 \to \Gamma_m \to I_0 \to \cdots \to I_{d+1} \to 0 & \text{in } \operatorname{mod}\Gamma_m, \\ 0 \to \Gamma_m \to I'_0 \to \cdots \to I'_{d+1} \to 0 & \text{in } \operatorname{mod}\Gamma_m^{op} \end{array}$$

such that $\operatorname{pd}_{\Gamma_m} I_i \le d-1$ and $\operatorname{pd}_{\Gamma_m^{op}} I'_i \le d-1$ for any i $(0 \le i \le d)$.

Proof By 2.3, Λ_m is a cotilting Λ_m-module with $\operatorname{id}_{\Lambda_m}\Lambda_m \le d-1$, and M_m is a d-cluster tilting object in $\mathcal{B}_m$. Thus Γ_m is an Auslander algebra of type $(0, d-1, d)$ in the sense of [I2, 4.1]. In particular, Γ_m satisfies $\operatorname{gl.dim}\Gamma_m \le d+1$ and the two-sided $(d, d+1)$-condition by [I2, 4.2.1]. Thus we obtain the assertion.■

In general, we show the following:

3.3 Proposition *Let $\mathcal{C}$ be a d-cluster tilting subcategory of $\operatorname{fl}\Lambda$ and*

$$\widetilde{\mathcal{C}} := \operatorname{add}\{X[dn] \mid X \in \mathcal{C},\ n \in \mathbb{Z}\}$$

Then we have

$$\begin{aligned} \widetilde{\mathcal{C}} &= \{X \in \mathcal{D}^b(\operatorname{fl}\Lambda) \mid \operatorname{Hom}_{\mathcal{D}^b(\operatorname{fl}\Lambda)}(X, \widetilde{\mathcal{C}}[i]) = 0 \text{ for any } 0 < i < d\} \\ &= \{X \in \mathcal{D}^b(\operatorname{fl}\Lambda) \mid \operatorname{Hom}_{\mathcal{D}^b(\operatorname{fl}\Lambda)}(\widetilde{\mathcal{C}}, X[i]) = 0 \text{ for any } 0 < i < d\}. \end{aligned}$$

We need the following simple observation.

3.4 Lemma *Assume $\operatorname{gl.dim}\Lambda \le d$ and $X \in \mathcal{D}^b(\operatorname{Mod}\Lambda)$ satisfies $H^i(X) = 0$ for any $i \notin d\mathbb{Z}$. Then X decomposes to a direct sum of $H^{dn}(X)[-dn]$.*

Proof Without loss of generality, we assume that X is a complex $\cdots \to C^i \to C^{i+1} \to \cdots$ of injective Λ-modules. We have an exact sequence

$$0 \to Z^{dn-d} \to C^{dn-d+1} \to \cdots \to C^{dn-1} \xrightarrow{a} Z^{dn} \xrightarrow{b} H^{dn} \to 0$$

with injective Λ-modules C^i. It follows from $\operatorname{gl.dim}\Lambda \le d$ that $\operatorname{Im} a$ is injective. Thus b splits, and H^{dn} is a direct summand of X.■

3.5 Proof of 3.3 Let us calculate $\operatorname{Hom}_{\mathcal{D}^b(\operatorname{fl}\Lambda)}(X[dn], Y[dm+i])$ for $X, Y \in \mathcal{C}$, $n, m \in \mathbb{Z}$ and i $(0 < i < d)$. If $n > m$, then this is clearly zero. If $n < m$, then this is zero by $\operatorname{gl.dim}\Lambda \le d$. If $n = m$, then this is again zero by the assumption of $\mathcal{C}$. Consequently, we have $\operatorname{Hom}_{\mathcal{D}^b(\operatorname{fl}\Lambda)}(\widetilde{\mathcal{C}}, \widetilde{\mathcal{C}}[i]) = 0$ for any i $(0 < i < d)$.

Since Λ is d-CY, we have ${}^{\perp_{d-1}}\widetilde{\mathcal{C}} = \widetilde{\mathcal{C}}^{\perp_{d-1}}$. We only have to show ${}^{\perp_{d-1}}\widetilde{\mathcal{C}} \subseteq \widetilde{\mathcal{C}}$. Fix any $X \in {}^{\perp_{d-1}}\widetilde{\mathcal{C}}$ and i $(0 < i < d)$. We show $H^i := H^i(X) = 0$. Let $C^0 \xrightarrow{g} Z^i \xrightarrow{f} H^i \to 0$ be a natural exact sequence. Assume $H^i \neq 0$ and take non-zero map $a : \Lambda \to H^i$. Then there exists $b : \Lambda \to Z^i$ such that $a = bf$. It follows form $Z^i \in \mathrm{fl}\,\Lambda$ that there exists m and $c : \Lambda_m \to Z^i$ such that b factors through c. It follows from $\Lambda_m \in \mathcal{C} \subset \widetilde{\mathcal{C}}$ that $\mathrm{Hom}_{\mathcal{D}^b(\mathrm{fl}\,\Lambda)}(\Lambda_m, X[i]) = 0$. Thus c factors through g. This implies $a = 0$, a contradiction. Thus $H^i = 0$.

Since $\widetilde{\mathcal{C}}$ is closed under $[dn]$ $(n \in \mathbb{Z})$, we have $H^i(X) = 0$ for any $X \in {}^{\perp_{d-1}}\widetilde{\mathcal{C}}$ and $i \notin d\mathbb{Z}$. It follows from gl.dim $\Lambda = d$ that X decomposes to a direct sum of $H^{dn}(X)[-dn]$ by 3.4. Since $H^{dn}(X) \in {}^{\perp_{d-1}}\mathcal{C} = \mathcal{C}$. Thus we have $X \in \widetilde{\mathcal{C}}$.∎

3.6 Question It seems that $\widetilde{\mathcal{C}}$ is not functorially finite. It is natural to ask whether $\mathcal{D}^b(\mathrm{fl}\,\Lambda)$ does not have a d-cluster tilting subcategory.

References

[AS] M. Auslander, S. O. Smalo: Almost split sequences in subcategories. J. Algebra 69 (1981), no. 2, 426–454.

[B] R. Bocklandt: Graded Calabi Yau Algebras of dimension 3, arXiv:math.RA/0603558.

[BMRRT] A. Buan, R. Marsh, M. Reineke, I. Reiten, G. Todorov: Tilting theory and cluster combinatorics, to appear in Adv. Math., arXiv:math.RT/0402054.

[BIRS] A. Buan, O. Iyama, I. Reiten, J. Scott: Cluster structures for 2-Calabi-Yau categories and unipotent subgroups, in preparation.

[BIKR] I. Burban, O. Iyama, B. Keller, I. Reiten: Maximal rigid and cluster tilting objects, in preparation.

[GLS] C. Geiss, B. Leclerc, J. Schröer: Rigid modules over preprojective algebras, to appear in Invent. Math., arXiv:math.RT/0503324.

[G] V. Ginzburg: Calabi-Yau algebras, arXiv:math.AG/0612139.

[H1] D. Happel: Triangulated categories in the representation theory of finite-dimensional algebras. London Mathematical Society Lecture Note Series, 119. Cambridge University Press, Cambridge, 1988.

[H2] D. Happel: On Gorenstein algebras. Representation theory of finite groups and finite-dimensional algebras (Bielefeld, 1991), 389–404, Progr. Math., 95, Birkhauser, Basel, 1991.

[HU]

[I1] O. Iyama: Higher dimensional Auslander-Reiten theory on maximal orthogonal subcategories, to appear in Adv. Math., arXiv:math.RT/0407052.

[I2] O. Iyama: Auslander correspondence, to appear in Adv. Math., arXiv:math.RT/0411631.

[IR] O. Iyama, I. Reiten: Fomin-Zelevinsky mutation and tilting modules over Calabi-Yau algebras, arXiv:math.RT/0605136.

[IY] O. Iyama, Y. Yoshino: Mutations in triangulated categories and rigid Cohen-Macaulay modules, arXiv:math.RT/0607736.

[KR1] B. Keller, I. Reiten: Cluster-tilted algebras are Gorenstein and stably Calabi-Yau, arXiv:math.RT/0512471.

[KR2] B. Keller, I. Reiten: Acyclic Calabi-Yau categories, arXiv:math.RT/0610594.

[RS] C. Riedtmann, A. Schofield: On a simplicial complex associated with tilting modules. Comment. Math. Helv. 66 (1991), no. 1, 70–78.

[V1] M. Van den Bergh: Three-dimensional flops and noncommutative rings. Duke Math. J. 122 (2004), no. 3, 423–455.

[V2] M. Van den Bergh: Non-commutative crepant resolutions. The legacy of Niels Henrik Abel, 749–770, Springer, Berlin, 2004.

[Ye] A. Yekutieli: Dualizing complexes, Morita equivalence and the derived Picard group of a ring. J. London Math. Soc. (2) 60 (1999), no. 3, 723–746.

[Yo] Y. Yoshino: Cohen-Macaulay modules over Cohen-Macaulay rings. London Mathematical Society Lecture Note Series, 146. Cambridge University Press, Cambridge, 1990.