Mutation and g-vectors in τ-tilting theory

Osamu Iyama
(joint work with Takahide Adachi, Idun Reiten)

Let A_Q be a cluster algebra associated with a quiver Q. For a potential W, the cluster category $\mathcal{C}_{Q,W}$ is defined by the Ginzburg dg algebra $\Gamma_{Q,W}$. We denote by $\text{c-tilt}\,\mathcal{C}_{Q,W}$ the set of isomorphic classes of basic cluster-tilting objects, and $\text{c-tilt}_0\,\mathcal{C}_{Q,W}$ the subset consisting of *reachable* cluster-tilting objects (i.e. cluster-tilting objects which are obtained from $\Gamma_{Q,W}$ by iterated mutation). When (Q,W) is non-degenerate, we have a bijection

$$\text{c-tilt}_0\,\mathcal{C}_{Q,W} \to \{\text{clusters in } A_Q\} \tag{1}$$

which commutes with mutation [CKLP] after works by a number of authors.

τ-tilting theory [AIR] provides another framework of categorifying cluster algebras, which is simpler since it deals with the Jacobian algebra $P_{Q,W}$ instead of the Ginzburg dg algebra $\Gamma_{Q,W}$. It is interesting from a representation theoretic viewpoint since it works for arbitrary finite dimensional algebras.

1. τ-tilting theory Let k be an algebraically closed field and Λ a basic finite dimensional k-algebra. We denote by $\mathsf{mod}\,\Lambda$ the category of finitely generated left Λ-modules, and $\mathsf{proj}\,\Lambda$ (respectively, $\mathsf{inj}\,\Lambda$) the full subcategory consisting of projective (respectively, injective) modules. We have an equivalence $\nu := (D\Lambda)\otimes_\Lambda - : \mathsf{proj}\,\Lambda \to \mathsf{inj}\,\Lambda$ called *Nakayama* functor. For $M \in \mathsf{mod}\,\Lambda$, we denote by $P_1^M \xrightarrow{f} P_0^M \to M \to 0$ a minimal projective presentation. The *AR-translation* of M is defined by an exact sequence $0 \to \tau M \to \nu(P_1^M) \xrightarrow{\nu(f)} \nu(P_0^M)$.

Clearly $\tau M = 0$ if M is projective. Moreover τ gives a bijection between the set of isomorphism classes of indecomposable non-projective Λ-modules and the set of isomorphism classes of indecomposable non-injective Λ-modules.

The notion of τ-rigid modules appeared in an old work by Auslander-Smalø:

Definition 1 Let $M \in \mathsf{mod}\,\Lambda$. We call M *τ-rigid* if $\text{Hom}_\Lambda(M, \tau M) = 0$. We call M *τ-tilting* if it is τ-rigid and $|M| = |\Lambda|$ holds, where $|M|$ is the number of non-isomorphic indecomposable direct summands of M. We call M *support τ-tilting* if there exists an idempotent e of Λ such that M is a τ-tilting $(\Lambda/\langle e\rangle)$-module.

We denote by $\text{s}\tau\text{-tilt}\,\Lambda$ the set of isomorphism classes of basic support τ-tilting Λ-modules. We give a few examples.

Example 2 (a) Λ and 0 always belong to $\text{s}\tau\text{-tilt}\,\Lambda$.
(b) If Λ is local, then $\text{s}\tau\text{-tilt}\,\Lambda = \{\Lambda, 0\}$.
(c) [M] Let Π be a preprojective algebra of a Dynkin quiver Q, W the corresponding Weyl group, and $I_i := \langle 1 - e_i\rangle$ a two-sided ideal of Π for the vertex i of Q. Then there exists a bijection $W \to \text{s}\tau\text{-tilt}\,\Pi$ sending $w \in W$ to $I_w := I_{i_1}\cdots I_{i_\ell}$, where $w = s_{i_1}\cdots s_{i_\ell}$ is an arbitrary reduced expression of w.
(d) For the path algebra kQ of the quiver Q of type A_2, $\text{s}\tau\text{-tilt}(kQ)$ consists of 5 elements kQ, $D(kQ)$, S_1, S_2 and 0. This is a special case of the next (e).

(e) Let $\mathcal{C}$ be a 2-Calabi-Yau triangulated category with a cluster-tilting object T. Then we have a bijection

$$\text{c-tilt}\,\mathcal{C} \to \text{s}\tau\text{-tilt End}_{\mathcal{C}}(T),\ U \mapsto \text{Hom}_{\Lambda}(T, U). \tag{2}$$

When $\mathcal{C} = \mathcal{C}_{Q,W}$ and $T = \Gamma_{Q,W}$ for a non-degenerate Jacobi-finite potential (Q, W), by combining (1) and (2) we have a bijection

$$\text{s}\tau\text{-tilt}_0 P_{Q,W} \to \{\text{clusters in } A_Q\}$$

for the Jacobian algebra $P_{Q,W}$, where $\text{s}\tau\text{-tilt}_0 P_{Q,W}$ is the set of *reachable* support τ-tilting $P_{Q,W}$-modules.

To introduce mutation of support τ-tilting modules, we need to deal with pairs (M, P) with $M \in \mathsf{mod}\Lambda$ and $P \in \mathsf{proj}\Lambda$. We call (M, P) *τ-rigid* if M is τ-rigid and $\text{Hom}_{\Lambda}(P, M) = 0$. We call (M, P) *support τ-tilting* if (M, P) is τ-rigid and $|M| + |P| = |\Lambda|$.

Theorem 3 Let (M, P) be a basic τ-rigid-pair for Λ.
(a) (M, P) is a direct summand of some $(N, Q) \in \text{s}\tau\text{-tilt}\,\Lambda$ (i.e. M and P are direct summands of N and Q respectively).
(b) If $|M| + |P| = |\Lambda| - 1$, then (M, P) is a direct summand of precisely two elements $(N_i, Q_i) \in \text{s}\tau\text{-tilt}\,\Lambda$ $(i = 1, 2)$.
(c) [J] There exist finite dimensional k-algebra Γ with $|M| + |P| = |\Lambda| - |\Gamma|$ and a bijection $\{(N, Q) \in \text{s}\tau\text{-tilt}\,\Lambda \mid (N, Q) \text{ has } (M, P) \text{ as a direct summand}\} \to \text{s}\tau\text{-tilt}\,\Gamma$.

Note that (b) is a special case of (c) since $|\Gamma| = 1$ implies $\text{s}\tau\text{-tilt}\,\Gamma = \{\Gamma, 0\}$.

We call (N_1, Q_1) and (N_2, Q_2) in (b) above *mutation* of each other. The *exchange graph* of Λ has the set $\text{s}\tau\text{-tilt}\,\Lambda$ of vertices and edges correspond to mutation.

It is important to know the number of connected components of the exchange graph. The partial order gives an effective tool.

2. Partial order The partial order on tilting modules due to Riedtmann-Schofield and Happel-Unger can be extended to support τ-tilting modules.

A *torsion class* is a full subcategory $\mathcal{T}$ in $\mathsf{mod}\Lambda$ which is closed under factor modules and extensions. We call a torsion class *functorially finite* if there exists $M \in \mathsf{mod}\Lambda$ such that $\mathcal{T} = \mathsf{Fac}M$, where $\mathsf{Fac}M$ is the full subcategory of $\mathsf{mod}\Lambda$ consisting of all factor modules of finite direct sums of copies of M.

Theorem 4 (a) There exists a bijection from $\text{s}\tau\text{-tilt}\,\Lambda$ to the set of all functorially finite torsion classes in $\mathsf{mod}\Lambda$. Hence $\text{s}\tau\text{-tilt}\,\Lambda$ has a natural partial order, i.e. we define $M \geq N$ if and only if $\mathsf{Fac}M \supset \mathsf{Fac}N$. Clearly Λ is a unique maximal element and 0 is a unique minimal element.
(b) [DIJ] Let $M \in \text{s}\tau\text{-tilt}\Lambda$, and let $\mathcal{T}$ be a torsion class in $\mathsf{mod}\Lambda$. If $\mathsf{Fac}M \supsetneq \mathcal{T}$ (respectively, $\mathsf{Fac}M \subsetneq \mathcal{T}$), then there exists a mutation N of M such that $\mathsf{Fac}M \supsetneq \mathsf{Fac}N \supset \mathcal{T}$ (respectively, $\mathsf{Fac}M \subsetneq \mathsf{Fac}N \subset \mathcal{T}$).
(c) The exchange graph of $\text{s}\tau\text{-tilt}\,\Lambda$ coincides with the Hasse graph of $\text{s}\tau\text{-tilt}\,\Lambda$.

Note that (c) is an easy consequence of (b).

We call a finite dimensional k-algebra Λ *τ-rigid-finite* if there exists only finitely many indecomposable τ-rigid Λ-modules, or equivalently, $\text{s}\tau\text{-tilt}\,\Lambda$ is a finite set

[DIJ]. For example, any representation-finite algebra and any local algebra are τ-rigid-finite. Any preprojective algebra Π of Dynkin type is also τ-rigid-finite (Example 2(c)), and the partially ordered set sτ-tilt$\,\Pi$ is isomorphic to W with respect to the opposite of weak order.

It is an interesting question to classify τ-rigid-finite algebras.

Another easy consequence of (b) above is the following result.

Corollary 5 [DIJ] A finite dimensional k-algebra Λ is τ-rigid-finite if and only if any torsion class in $\mathsf{mod}\,\Lambda$ is functorially finite. In this case, the exchange graph of sτ-tilt$\,\Lambda$ is connected.

This is an analog of a classical result: A finite dimensional k-algebra Λ is representation-finite if and only if any subcategory in $\mathsf{mod}\,\Lambda$ is functorially finite.

3. g-vectors A combinatorial invariant of τ-rigid pairs is given by g-vectors. Let Λ be a basic finite dimensional k-algebra such that $1 = e_1 + \cdots + e_n$ for primitive orthogonal idempotents $e_1, \ldots, e_n$. The Grothendieck group $K_0(\mathsf{proj}\,\Lambda)$ of an additive category $\mathsf{proj}\,\Lambda$ is a free abelian group with a basis $[e_1\Lambda], \ldots, [e_n\Lambda]$.

Theorem 6 [AIR, DIJ] (a) If (M, P) is a τ-rigid pair, then P_0^M and $P_1^M \oplus P$ have no non-zero common direct summands. We define the *g-vector* (or *index*) of (M, P) as

$$g^{(M,P)} := [P_0^M] - [P_1^M \oplus P] \in K_0(\mathsf{proj}\,\Lambda).$$

(b) τ-rigid pairs are determined by their g-vectors.
(c) Let $(M, P) \in$ sτ-tilt$\,\Lambda$. Then g-vectors of indecomposable direct summands of (M, P) give a basis of $K_0(\mathsf{proj}\,\Lambda)$. Let $C(M, P)$ be the cone in $K_0(\mathsf{proj}\,\Lambda) \otimes_{\mathbf{Z}} \mathbf{R}$ spanned by these basis elements.
(d) Different cones intersect only at their boundaries.
(e) If Λ is τ-rigid-finite, then $\bigcup_{(M,P)\in \mathrm{s}\tau\text{-tilt}\,\Lambda} C(M, P) = K_0(\mathsf{proj}\,\Lambda) \otimes_{\mathbf{Z}} \mathbf{R}$.

For the preprojective algebra Π of Dynkin type, the cones $C(M, P)$ are precisely Weyl chambers.

Note that τ-rigid pairs for Λ have a structure of a simplicial complex, and a geometric realization is given by g-vectors. If Λ is τ-rigid-finite, then it is homeomorphic to an $(n-1)$-sphere as an easy consequence of (d) and (e) above.

We conjecture that g-vectors determine the partial order on sτ-tilt$\,\Lambda$.

References

[AIR] T. Adachi, O. Iyama, I. Reiten, *τ-tilting theory*, to appear in Compos. Math., arXiv:1210.1036.

[CKLP] G. Cerulli Irelli, B. Keller, D. Labardini-Fragoso, P. Plamondon, *Linear independence of cluster monomials for skew-symmetric cluster algebras*, arXiv:1203.1307.

[DIJ] L. Demonet, O. Iyama, G. Jasso, *On g-vectors of τ-tilting modules and τ-rigid-finite algebras*, in preparation.

[J] G. Jasso, *Reduction of τ-tilting modules and torsion pairs*, arXiv:1302.2709.

[M] Y. Mizuno, *Classifying τ-tilting modules over preprojective algebras of Dynkin type*, to appear in Math. Z., arXiv:1304.0667.