Stable categories of preprojective algebras and cluster categories

Osamu Iyama

Let K be an algebraically closed field. For an integer n, we say that a Hom-finite K-linear triangulated category $\mathcal{T}$ is *n-Calabi-Yau* (*n-CY*) if there exists a functorial isomorphism $\mathrm{Hom}_{\mathcal{T}}(X,Y) \simeq D\mathrm{Hom}_{\mathcal{T}}(Y,X[n])$ for any $X,Y \in \mathcal{T}$, where $D = \mathrm{Hom}_K(-,K)$ is the K-dual. There are many important triangulated categories in representation theory, in particular cluster categories played an important role in categorification of cluster algebras.

1. Background (I) For an acyclic quiver Q, we denote by KQ the path algebra and by $\Pi = \Pi(KQ)$ the preprojective algebra of Q. The following dichotomies of representation theory of KQ and structure theory of Π are well known.

Q	KQ	Π
Dynkin	representation finite	finite dimensional selfinjective
non-Dynkin	representation infinite	infinite dimensional

The stable category $\underline{\mathrm{mod}}\,\Pi$ is 2-CY for Dynkin case, and the bounded derived category $\mathcal{D}^{\mathrm{b}}(\Pi)$ of finite dimensional Π-modules is 2-CY for non-Dynkin case.

(II) Let Q be an extended Dynkin quiver with the extending vertex e. Then $R := e\Pi e$ is a Kleinian singularity, and in particular the stable category $\underline{\mathrm{CM}}(R)$ of maximal Cohen-Macaulay R-modules is 1-CY [19].

Recently higher analogue of preprojective algebras are introduced in representation theory [10, 11, 13] and non-commutative algebraic geometry [15, 16]:

Definition 1 Let n be a positive integer and Λ be a finite dimensional K-algebra with $\mathrm{gl.dim}\,\Lambda \le n$. The *$(n+1)$-preprojective algebra* of Λ is defined as the tensor algebra of the Λ-bimodule $\mathrm{Ext}^n_\Lambda(D\Lambda,\Lambda)$:

$$\Pi = \Pi_{n+1}(\Lambda) := \mathrm{T}_\Lambda \mathrm{Ext}^n_\Lambda(D\Lambda,\Lambda).$$

For the case $n=1$, this is a well-known description of preprojective algebras. For the case $n=2$, this gives a description of cluster tilted algebras [3].

We will generalize CY properties in (I) and (II) above to higher cases.

The above stable categories have realizations as cluster categories defined as follows: Let n be a positive integer and Λ be a finite dimensional K-algebra with $\mathrm{gl.dim}\,\Lambda \le n$. Let $\mathcal{D}^{\mathrm{b}}(\Lambda)$ be the bounded derived category of finite dimensional Λ-modules, ν be the Nakayama functor of $\mathcal{D}^{\mathrm{b}}(\Lambda)$ and $\nu_n := \nu\circ[-n]$. The triangulated hull $\mathcal{C}_n(\Lambda)$ of the orbit category $\mathcal{D}^{\mathrm{b}}(\Lambda)/\nu_n$ is called the *n-cluster category* [4, 12, 1, 18, 5]. If $\mathcal{C}_n(\Lambda)$ is Hom-finite, then it is n-CY. Notice that $\Pi_{n+1}(\Lambda)$ is the endomorphism algebra $\mathrm{End}_{\mathcal{C}_n(\Lambda)}(\Lambda)$ of Λ in $\mathcal{C}_n(\Lambda)$.

We have the equivalences between stable categories and cluster categories:

Theorem 2 (a) [1] For a Dynkin quiver Q, we have a triangle equivalence $\underline{\mathrm{mod}}\,\Pi(KQ) \simeq \mathcal{C}_2(\underline{\Gamma})$ for the stable Auslander algebra $\underline{\Gamma}$ of KQ.

(b) [17] In (II) above, we have an equivalence $\underline{\mathrm{CM}}(R) \simeq \mathcal{C}_1(KQ')$, where Q' is the Dynkin quiver obtained by removing e from Q.

We will generalize these equivalences to higher cases.

2. Our results Throughout let n be a positive integer and Λ be a finite dimensional K-algebra with $\mathrm{gl.dim}\,\Lambda \le n$. In general, the homological behaviour of $\Pi_{n+1}(\Lambda)$ is not as nice as the case $n = 1$. So we have to restrict to the following.

Definition 3 [8] We say that Λ is *n-representation controlled* if $H^{\ell}(\nu_n^i(\Lambda)) = 0$ for any $i \in \mathbb{Z}$ and $\ell \in \mathbb{Z} - n\mathbb{Z}$.

We have the following dichotomy of n-representation controlled algebras, where $M \in \mathrm{mod}\,\Lambda$ is *n-cluster tilting* if $\mathrm{add}\,M$ coincides with the following subcategories:
- $\{X \in \mathrm{mod}\,\Lambda \mid \mathrm{Ext}^i_{\Lambda}(M, X) = 0$ for any $0 < i < n\}$.
- $\{X \in \mathrm{mod}\,\Lambda \mid \mathrm{Ext}^i_{\Lambda}(X, M) = 0$ for any $0 < i < n\}$.

Proposition 4 (Dichotomy) Λ is n-representation controlled if and only if precisely one of the following conditions holds.

(a) Λ has an n-cluster tilting module M. (*n-representation finite* [9, 10, 6])

(b) $\nu_n^{-i}(\Lambda) \in \mathrm{mod}\,\Lambda$ for any $i \ge 0$. (*n-representation infinite* [8])

For the case (a), the basic part of M is unique. We call $\mathrm{End}_{\Lambda}(M)$ and $\underline{\mathrm{End}}_{\Lambda}(M)$ the *n-Auslander algebra* and the *stable n-Auslander algebra* of Λ respectively.

Example 5 (a) It is clear from definition that the path algebra of an acyclic quiver is always 1-representation controlled. Moreover it is easy to check that 1-representation (in)finiteness coincides with representation (in)finiteness.

(b) [6] The tensor product $KQ_1 \otimes_K \cdots \otimes_K KQ_n$ for non-Dynkin quivers Q_i is n-representation infinite. The tensor product $KQ_1 \otimes_K \cdots \otimes_K KQ_n$ for Dynkin quivers Q_i is n-representation finite if each Q_i is stable under the canonical involution of the underlying graph and the Coxeter numbers of all Q_i's are equal.

Notice that n-representation infinite algebras are studied in non-commutative algebraic geometry [15, 16] under the name 'n-Fano algebra'.

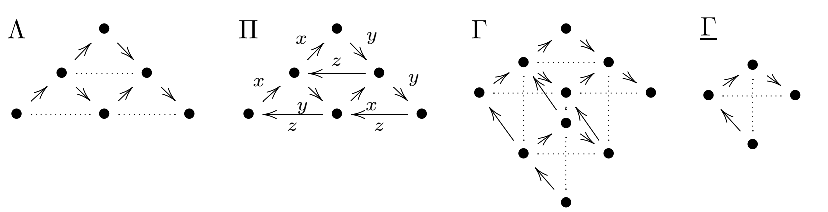
2.1. Finite case We have the results for n-representation finite algebras:

Theorem 6 [11] Let Λ be an n-representation finite algebra and $\Pi = \Pi_{n+1}(\Lambda)$.

(a) Π is a finite dimensional selfinjective algebra and $\underline{\mathrm{mod}}\,\Pi$ is $(n+1)$-CY.

(b) We have a triangle equivalence $\underline{\mathrm{mod}}\,\Pi \simeq \mathcal{C}_{n+1}(\underline{\Gamma})$ for the stable n-Auslander algebra $\underline{\Gamma}$ of Λ (e.g. Theorem 2 (a)).

Example 7 [9, 10, 11] Let $n = 2$ and Λ be an Auslander algebra of the path algebra of type A_3. Then Λ is 2-representation finite and $\Pi = \Pi_3(\Lambda)$ is the Jacobian algebra of the quiver below with potential $\sum xyz - zyx$. The 2-Auslander algebra Γ and the stable 2-Auslander algebra $\underline{\Gamma}$ are the following:



There is a general structure theorem of 2-representation finite algebras in terms of 'selfinjective quivers with potential' and their 'cuts' [7].

2.2. Infinite case We have the results for n-representation infinite algebras:

Theorem 8 Let Λ be an n-representation infinite algebra and $\Pi = \Pi_{n+1}(\Lambda)$.

(a) [13] $\mathcal{D}^{\mathrm{b}}(\Pi)$ is $(n+1)$-CY.

(b) [2] Let $e \in \Lambda$ be an idempotent. Assume $\dim_K(\Pi/(e)) < \infty$, $e\Lambda(1-e) = 0$ and that Π is noetherian. Then $\underline{\mathrm{CM}}(\Pi)$ is n-CY and we have a triangle equivalence $\underline{\mathrm{CM}}(e\Pi e) \simeq \mathcal{C}_n(\Lambda/(e))$ (e.g. Theorem 2 (b)).

Example 9 [2, 8] Let $n = 2$ and Λ be a Beilinson algebra of dimension 2. Then Λ is 2-representation infinite and $\Pi = \Pi_3(\Lambda)$ is the Jacobian algebra of the quiver below with potential $\sum xyz - zyx$.

Moreover $R = e\Pi e$ is the subring of $K[x, y, z]$ generated by all monomials whose degrees are multiples of 3. In particular we recover the equivalence $\underline{\mathrm{CM}}(R) \simeq \mathcal{C}_2(KQ)$ for $Q\ \bullet \rightrightarrows\!\!\!\!\rightarrow \bullet$ given in [14]. See [2] for more examples.

There is a general structure theorem of 2-representation infinite algebras in terms of 'good quivers with potential' and their 'cuts' [8].

References

[1] C. Amiot, *Cluster categories for algebras of global dimension 2 and quivers with potential*, Ann. Inst. Fourier (Grenoble) 59 (2009), no. 6, pp 2525–2590.

[2] C. Amiot, O. Iyama, I. Reiten, *Stable categories of Cohen-Macaulay modules and cluster categories*, arXiv:1104.3658.

[3] I. Assem, T. Brüestle, R. Schiffler, *Cluster tilted algebras as trivial extensions.* Bull. Lond. Math. Soc. 20(2008), no. 1, 151–162.

[4] A. B. Buan, R. Marsh, M. Reineke, I. Reiten, and G. Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. 204 (2006), no.2, 572–618.

[5] L. Guo, *Cluster tilting objects in generalized higher cluster categories*, J. Pure Appl. Algebra 215 (2011), no. 9, 2055–2071.

[6] M. Herschend, O. Iyama, *n-representation-finite algebras and twisted fractionally Calabi-Yau algebras*, Bull. Lond. Math. Soc. 43 (2011), no. 3, 449–466.

[7] M. Herschend, O. Iyama, *Selfinjective quivers with potential and 2-representation-finite algebras*, to appear in Compos. Math., arXiv:1006.1917.

[8] M. Herschend, O. Iyama, S. Oppermann, *n-representation infinite algebras*, in preparation.

[9] O. Iyama, *Cluster tilting for higher Auslander algebras*, Adv. Math. 226 (2011), no. 1, 1–61.

[10] O. Iyama, S. Oppermann, *n-representation-finite algebras and n-APR tilting*, to appear in Trans. Amer. Math. Soc., arXiv:0909.0593.

[11] O. Iyama, S. Oppermann, *Stable categories of higher preprojective algebras*, arXiv:0912.3412.

[12] B. Keller, *On triangulated orbit categories*, Doc. Math. 10 (2005), 551–581.

[13] B. Keller, *Deformed Calabi-Yau completions*, arXiv:0908.3499.

[14] B. Keller, I. Reiten, *Acyclic Calabi-Yau categories*, Compos. Math. 144 (2008), no. 5, 1332–1348, With an appendix by Michel Van den Bergh.

[15] Y. Minamoto, *Ampleness of two-sided tilting complexes*, to appear in Int. Math. Res. Not.

[16] Y. Minamoto, I. Mori, *Structures of AS-regular algebras*, Adv. Math . 226 (2011) 4061–4095.

[17] I. Reiten, M. Van den Bergh, *Two-dimensional tame and maximal orders of finite representation type*, Mem. Amer. Math. Soc. 80 (1989), no. 408.

[18] H. Thomas, *Defining an m-cluster category*, J. Algebra 318 (2007), no. 1, 37–46.

[19] Y. Yoshino, *Cohen-Macaulay modules over Cohen-Macaulay rings*, London Mathematical Society Lecture Note Series, 146. Cambridge University Press, Cambridge, 1990.