Stable categories of Cohen-Macaulay modules and cluster categories

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(joint work with Claire Amiot, Idun Reiten)

Let $\mathcal{T}$ be a k-linear triangulated category with the suspension functor [1] over a field k. For an integer n, we say that $\mathcal{T}$ is *n-Calabi-Yau* (*n-CY*) if there exists a functorial isomorphism $\mathrm{Hom}_{\mathcal{T}}(X,Y) \simeq D\mathrm{Hom}_{\mathcal{T}}(Y,X[n])$ for any $X,Y \in \mathcal{T}$, where $D = \mathrm{Hom}_k(-,k)$ is the k-dual. In representation theory, there are two important classes of n-CY triangulated categories. One is the *generalized n-cluster categories* [BMRRT, Am, G] appearing in study of Fomin-Zelevinsky cluster algebras. The other is the *stable categories* of Cohen-Macaulay modules over Gorenstein isolated singularities [Au1]. The aim of this paper is to compare these two classes of categories. We will show that the stable categories of Cohen-Macaulay modules over certain Gorenstein isolated singularities are triangle equivalent to generalized n-cluster categories (Theorem 1).

1. Preliminaries

Let $n \geq 1$. A key notion in n-CY triangulated categories $\mathcal{T}$ is *n-cluster tilting* objects $M \in \mathcal{T}$ defined by $\mathrm{add}M = \{X \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(M,X[i]) = 0 \ (0 < i < n)\}$. They are certain analogue of tilting objects, and 1-cluster tilting objects are nothing but additive generators of $\mathcal{T}$.

1.1. **Cluster categories.** Let $n \geq 2$, and let A be a finite dimensional k-algebra with $\mathrm{gl.dim}A \leq n$. We denote by $\mathcal{D}_A$ the bounded derived category of the category $\mathrm{mod}A$ of finitely generated A-modules, and by $\nu := - \overset{\mathbf{L}}{\otimes}_A DA : \mathcal{D}_A \to \mathcal{D}_A$ the Nakayama functor. We have Auslander-Reiten-Serre duality $\mathrm{Hom}_{\mathcal{D}_A}(X,Y) \simeq D\mathrm{Hom}_{\mathcal{D}_A}(Y,\nu X)$ for any $X,Y \in \mathcal{D}_A$ [Ha]. Let $\nu_n := \nu \circ [-n] : \mathcal{D}_A \to \mathcal{D}_A$. If $\mathrm{gl.dim}A \leq 1$, then the orbit category $\mathcal{C}_A^{(n)} := \mathcal{D}_A/\nu_n$ forms an n-CY triangulated category called the *n-cluster category* [BMRRT, K1]. This is not the case for $\mathrm{gl.dim}A \geq 2$, and the *generalized n-cluster category* $\mathcal{C}_A^{(n)}$ is defined in [K1, Am, G] as a 'triangulated hull' of the orbit category $\mathcal{D}_A/\nu_n$ under the assumption that the functor $H^0(\nu_n) : \mathrm{mod}A \to \mathrm{mod}A$ is nilpotent. This is an n-CY triangulated category with a triangle functor $\pi : \mathcal{D}_A \to \mathcal{C}_A^{(n)}$ satisfying a certain universal property and has an n-cluster tilting object $\pi A \in \mathcal{C}_A^{(n)}$.

1.2. **Stable categories.** Let R be a complete local Gorenstein ring of Krull dimension d. We denote by $\mathrm{CM}(R) := \{X \in \mathrm{mod}R \mid \mathrm{Ext}^i_R(X,R) = 0 \ (0 < i)\}$ the category of maximal Cohen-Macaulay R-modules, and by $\underline{\mathrm{CM}}(R)$ its stable category. It is known that $\underline{\mathrm{CM}}(R)$ forms a triangulated category [Ha], and is triangle equivalent to $\mathcal{D}_R/\mathrm{per}R$ [B]. Assume that R is an isolated singularity. Then $\underline{\mathrm{CM}}(R)$ forms a $(d-1)$-CY triangulated category by a classical result due to Auslander [Au1]. If $M \in \underline{\mathrm{CM}}(R)$ is $(d-1)$-cluster tilting, then $\Gamma := \mathrm{End}_R(R \oplus M)$ satisfies $\mathrm{gl.dim}\Gamma = d$ and $\Gamma \in \mathrm{CM}(R)$ [I2]. In particular Γ is a non-commutative crepant resolution in the sense of Van den Bergh [V]. The existence of a $(d-1)$-cluster tilting object in $\underline{\mathrm{CM}}(R)$ is closely related to the geometry of resolutions of the singularity Spec$R$.

Let $S := k[[x_1, \cdots, x_d]]$ be the formal power series ring over a field k of characteristic zero, and let G be a finite subgroup of $\mathrm{SL}_d(k)$. If the quotient singularity $R := S^G$ is isolated, then $S \in \underline{\mathrm{CM}}(R)$ is $(d-1)$-cluster tilting [I1]. In particular, if $d = 2$, we have $\underline{\mathrm{CM}}(R) = \mathrm{add}S$ and so R is representation-finite [Au2, He].

2. Main results

Let k be a field of characteristic zero. Let $G = \frac{1}{n}(a_1, \cdots, a_d)$ be a cyclic subgroup of $\mathrm{SL}_d(k)$ generated by a diagonal matrix $g = \mathrm{diag}(\zeta^{a_1}, \cdots, \zeta^{a_d})$ with a primitive n-th root ζ of unity and integers a_i satisfying $0 < a_i < n$, $(n, a_i) = 1$ and $\sum_{i=1}^d a_i = n$. Let $S = k[x_1, \cdots, x_d]$ be a polynomial algebra of d variables. Then S has a $\frac{\mathbb{Z}}{n}$-graded algebra structure $S = \bigoplus_{i \geq 0} S_{\frac{i}{n}}$ defined by $\deg x_i := \frac{a_i}{n}$. The invariant subring $R := S^G = \bigoplus_{i \geq 0} S_i$ is a Gorenstein isolated singularity. For $0 \leq j < n$, we define a $\mathbb{Z}$-graded R-module $T^j := \bigoplus_{i \geq 0} (T^j)_i$ by $(T^j)_i := S_{i + \frac{j}{n}}$. Let $T := \bigoplus_{j=0}^{n-1} T^j$. Then $B := \mathrm{End}_R(T)$ has a $\mathbb{Z}$-graded algebra structure $B = \bigoplus_{i \geq 0} B_i$ with the degree zero part $A := B_0 = \mathrm{End}_R^{\mathbb{Z}}(T)$. Let e be the idempotent of A corresponding to the direct summand T^0 of T, and $\underline{A} := A/\langle e \rangle$. Our main result is the following [AIR]:

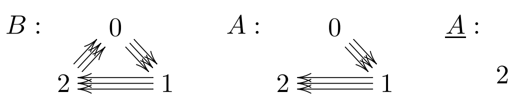
Theorem 1 We have a triangle equivalence $\underline{\mathrm{CM}}(R) \simeq \mathcal{C}_{\underline{A}}^{(d-1)}$.

Remark 2 (a) B is isomorphic to the skew group algebra $S*G$ [Au2], whose quiver is given by the McKay quiver of G. The relations are given by higher derivative of a potential [BSW].

(b) A related result is given in [DV].

(c) Theorem 1 is an analogue of Ueda's equivalence $\underline{\mathrm{CM}}^{\mathbb{Z}}(R) \simeq \mathcal{D}_{\underline{A}}$ [U].

Example 3 Let $G = \frac{1}{3}(1, 1, 1)$. The algebras B, A and $\underline{A}$ are presented by quivers



Thus $\underline{\mathrm{CM}}(R)$ is triangle equivalent to the cluster category of $2 \leftleftarrows\!\!\!\leftarrow 1$, and we recover a result by Keller and Reiten [KR].

Theorem 1 is a special case of the following result:

Let $B = \bigoplus_{i \geq 0} B_i$ be a graded k-algebra such that $\dim_k B_i < \infty$.

- B is a bimodule d-Calabi-Yau algebra of Gorenstein parameter 1, i.e. $B \in \mathrm{per}B^{\mathrm{e}}$ and $\mathbf{R}\mathrm{Hom}_{B^{\mathrm{e}}}(B, B^{\mathrm{e}})[d] \simeq B(1)$.
- $A := B_0$ has an idempotent e such that $eA(1-e) = 0$.
- B is noetherian and $\underline{B} := B/\langle e \rangle$ is a finite dimensional k-algebra.
- $C := eBe$ satisfies $\mathrm{End}_C(Be) = B$ and $\mathrm{End}_{C^{\mathrm{op}}}(eB) = B$.

Theorem 4 We have a triangle equivalence F and the commutative diagram:

$$\begin{array}{ccccc} \mathcal{D}_{\underline{A}} & \longrightarrow & \mathcal{D}_A & \xrightarrow{-\overset{\mathbf{L}}{\otimes}_A Be} & \mathcal{D}_C \\ \downarrow & & & & \downarrow \\ \mathcal{C}_{\underline{A}}^{(d-1)} & & \xrightarrow{\quad F \quad} & & \underline{\mathrm{CM}}(C) \end{array}$$

The key observation is the following.

Lemma 5 There exists a triangle in $\mathcal{D}(\mathrm{mod}^{\mathbb{Z}}(A^{\mathrm{op}} \otimes_k B))$:

$$A[-1] \to \mathbf{R}\mathrm{Hom}_{A^{\mathrm{e}}}(A, A^{\mathrm{e}}) \overset{\mathbf{L}}{\otimes}_A B(-1)[d-1] \to B \to A$$

As an application of Lemma 5, the *derived d-preprojective DG algebra* [K2] of A is B. In particular A is $(d-1)$*-representation-infinite* in the sense of [IO] or a *quasi* $(d-1)$*-Fano algebra* in the sense of [MM].

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