Cluster tilting in 2-Calabi-Yau categories II

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This is the second part in a series of two lectures with Idun Reiten. We shall show that cluster tilting mutation is compatible with quiver mutation and QP mutation. Throughout let K be an algebraically closed field, and let $\mathcal{C}$ be a Hom-finite 2-Calabi-Yau triangulated category over K with the suspension functor Σ. Let T be a basic cluster tilting object in $\mathcal{C}$ with an indecomposable decomposition $T = T_1 \oplus \cdots \oplus T_n$, and let $1 \leq k \leq n$. The following result [BMRRT, IY] is fundamental.

Theorem 1 (cluster tilting mutation)

(a) *There exists a unique indecomposable object* $T_k^* \in \mathcal{C}$ *such that* $T_k^* \not\simeq T_k$ *and* $\mu_k(T) := (T/T_k) \oplus T_k^*$ *is a basic cluster tilting object in* $\mathcal{C}$.

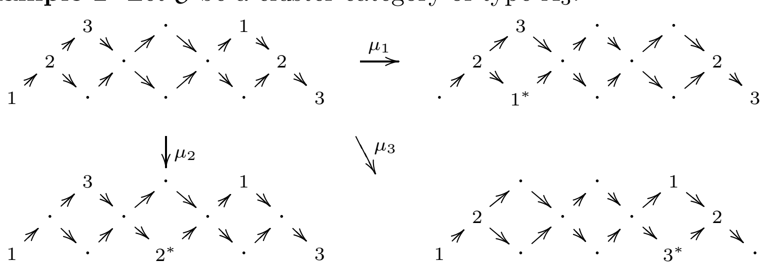
(b) *There exist triangles (called* exchange sequences*)*

$$T_k^* \xrightarrow{g} U_k \xrightarrow{f} T_k \to \Sigma T_k^* \quad \textit{and} \quad T_k \xrightarrow{g'} U'_k \xrightarrow{f'} T_k^* \to \Sigma T_k$$

such that f *and* f' *are right* $\mathrm{add}(T/T_k)$*-approximations and* g *and* g' *are left* $\mathrm{add}(T/T_k)$*-approximations.*

Clearly we have $\mu_k \circ \mu_k(T) \simeq T$.

Example 2 Let $\mathcal{C}$ be a cluster category of type A_3.



Following [FZ], we introduce mutation of quivers.

Definition 3 (*quiver mutation*) Let Q be a quiver[1] without loops. Assume that $k \in Q_0$ is not contained in 2-cycles. Define a quiver $\widetilde{\mu}_k(Q)$ by applying the following (i)-(iii) to Q.

(i) For each pair (a, b) of arrows in Q with $e(a) = k = s(b)$, add a new arrow $[ab] : s(a) \to e(b)$.

(ii) Replace each arrow $a \in Q_1$ with $e(a) = k$ by a new arrow $a^* : k \to s(a)$.

(iii) Replace each arrow $b \in Q_1$ with $s(b) = k$ by a new arrow $b^* : e(b) \to k$.

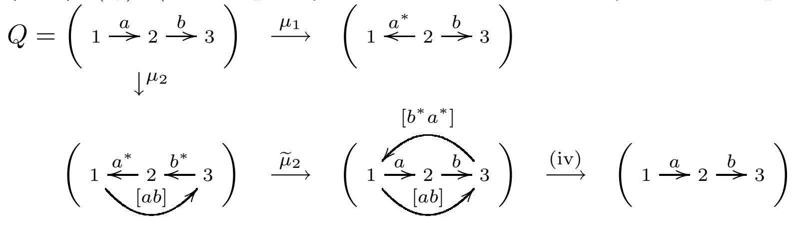
Define a quiver $\mu_k(Q)$ by applying the following (iv) to $\widetilde{\mu}_k(Q)$.

(iv) Remove a maximal disjoint collection of 2-cycles.

[1] We use the convention $a : s(a) \to e(a)$ for each $a \in Q_1$.

Then $\mu_k(Q)$ has no loops, k is not contained in 2-cycles in $\mu_k(Q)$, and $\mu_k \circ \mu_k(Q) \simeq Q$ holds.

Example 4 For the following quiver Q of type A_3, we calculate $\mu_1(Q)$, $\mu_2(Q)$ and $\mu_2 \circ \mu_2(Q)$. (For simplicity we denote a^{**} and b^{**} by a and b respectively.)



From now on, we assume that $\mathcal{C}$ has a *cluster structure* [BIRSc]. This means that the quiver Q_T of the endomorphism algebra $\mathrm{End}_{\mathcal{C}}(T)$ of any cluster tilting object T in Q has no loops and 2-cycles. In this case we have the following.

Observation 5 Combining the exchange sequences in Theorem 1, we have a complex[2]

$$T_k \xrightarrow{g'} U'_k \xrightarrow{f'g} U_k \xrightarrow{f} T_k$$

such that the following sequences are exact for the Jacobson radical $J_{\mathcal{C}}$ of $\mathcal{C}$.

$$(T, U'_k) \xrightarrow{f'g} (T, U_k) \xrightarrow{f} J_{\mathcal{C}}(T, T_k) \to 0,$$
$$(U_k, T) \xrightarrow{f'g} (U'_k, T) \xrightarrow{g'} J_{\mathcal{C}}(T_k, T) \to 0.$$

Thus the quiver and relations of $\mathrm{End}_{\mathcal{C}}(T)$ can be controlled by exchange sequences.

Using Observation 5, we have the following result [BMR, BIRSc] which asserts that cluster tilting mutation is compatible with quiver mutation.

Theorem 6 $Q_{\mu_k(T)} \simeq \mu_k(Q_T)$.

Using Theorem 6, we can show the following result [BIRSm].

Corollary 7 *Cluster tilted algebras are determined by their quivers.*

Following [DWZ], we introduce quivers with potentials.

Definition 8 Let Q be a quiver. We denote by A_i the K-vector space with the basis consisting of paths of length i, and by $A_{i,\mathrm{cyc}}$ the subspace of A_i spanned by all cycles. We denote by $\widehat{KQ} := \prod_{i \geq 0} A_i$ the complete path algebra. Its Jacobson radical is given by $J_{\widehat{KQ}} = \prod_{i \geq 1} A_i$.

A *quiver with a potential* (or *QP*) is a pair (Q, W) consisting of a quiver Q without loops and an element $W \in \prod_{i \geq 1} A_{i,\mathrm{cyc}}$ (called a *potential*). It is called *reduced* if $W \in \prod_{i \geq 3} A_{i,\mathrm{cyc}}$. Define $\partial_a W \in \widehat{KQ}$ by

$$\partial_a(a_1 \cdots a_\ell) := \sum_{a_i = a} a_{i+1} \cdots a_\ell a_1 \cdots a_{i-1}$$

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[2]Such a complex is called a *2-almost split sequence* in [I] and an *AR 4-angle* in [IY].

and extend linearly and continuously. The *Jacobian algebra* is defined by

$$\mathcal{P}(Q,W) := \widehat{KQ}/\overline{\langle \partial_a W \mid a \in Q_1 \rangle}$$

where $\overline{I}$ is the closure of I with respect to the $(J_{\widehat{KQ}})$-adic topology on $\widehat{KQ}$.

Two potentials W and W' are called *cyclically equivalent* if $W - W' \in \overline{[KQ, KQ]}$. Two QP's (Q,W) and (Q',W') are called *right-equivalent* if $Q_0 = Q'_0$ and there exists a continuous K-algebra isomorphism $\phi : \widehat{KQ} \to \widehat{KQ'}$ such that $\phi|_{Q_0} = \mathrm{id}$ and $\phi(W)$ and W' are cyclically equivalent. In this case ϕ induces an isomorphism $\mathcal{P}(Q,W) \simeq \mathcal{P}(Q',W')$.

It was shown in [DWZ] that for any QP (Q,W), there exists a reduced QP (Q',W') such that $\mathcal{P}(Q,W) \simeq \mathcal{P}(Q',W')$, and such (Q',W') is uniquely determined up to right-equivalence. We call (Q',W') a *reduced part* of (Q,W).

Example 9 Let (Q,W) be the QP below. Its reduced part is given by the QP (Q',W') below.

$$(Q,W) = \left(1 \xrightarrow{a} 2 \xrightarrow{b} 3 \text{ with } d: 3 \to 1,\ c: 1 \to 3 \ ,\ cd + abd \right) \qquad (Q',W') = \left(1 \xrightarrow{a} 2 \xrightarrow{b} 3 \ ,\ 0 \right)$$

Definition 10 (*QP mutation*) Let (Q,W) be a QP. Assume that $k \in Q_0$ is not contained in 2-cycles. Replacing W by a cyclically equivalent potential, we assume that no cycles in W start at k. Define a QP $\widetilde{\mu}_k(Q,P) := (\widetilde{\mu}_k(Q), [W] + \Delta)$ as follows:

- $\widetilde{\mu}_k(Q)$ is given in Definition 3.
- $[W]$ is obtained by substituting $[ab]$ for each factor ab in W with $e(a) = k = s(b)$.
- $\Delta := \sum_{a,b \in Q_1,\ e(a)=k=s(b)} a^*[ab]b^*$.

Define a QP $\mu_k(Q,P)$ as a reduced part of $\widetilde{\mu}_k(Q,P)$.

Then k is not contained in 2-cycles in $\mu_k(Q,W)$, and it was shown in [DWZ] that $\mu_k \circ \mu_k(Q,W)$ is right-equivalent to (Q,W).

Example 11 For a QP (Q,W) below, we calculate $\mu_2(Q,W)$ and $\mu_2 \circ \mu_2(Q,W)$. (The reduced part of $\widetilde{\mu}_2 \circ \mu_2(Q,W)$ was calculated in Example 9.)

$$(Q,W) = \left(1 \xrightarrow{a} 2 \xrightarrow{b} 3 \ ,\ 0 \right) \quad \xrightarrow{\mu_2} \quad \left(1 \xleftarrow{a^*} 2 \xleftarrow{b^*} 3 \text{ with } [ab]: 1 \to 3 \ ,\ a^*[ab]b^* \right)$$

$$\xrightarrow{\widetilde{\mu}_2} \left(1 \xrightarrow{a} 2 \xrightarrow{b} 3 \text{ with } [b^*a^*]: 3 \to 1,\ [ab]: 1 \to 3 \ ,\ [ab][b^*a^*] + b[b^*a^*]a \right) \xrightarrow{\text{reduced}} \left(1 \xrightarrow{a} 2 \xrightarrow{b} 3 \ ,\ 0 \right)$$

Using Observation 5, we have the following result [BIRSm] which asserts that cluster tilting mutation is compatible with QP mutation.

Theorem 12 *If* $\mathrm{End}_{\mathcal{C}}(T) \simeq \mathcal{P}(Q,W)$*, then* $\mathrm{End}_{\mathcal{C}}(\mu_k(T)) \simeq \mathcal{P}(\mu_k(Q,W))$.

Immediately we have the following conclusion.

Corollary 13 *If* $\mathrm{End}_{\mathcal{C}}(T)$ *is a Jacobian algebra of a QP, then so is* $\mathrm{End}_{\mathcal{C}}(T')$ *for any cluster tilting object* $T' \in \mathcal{C}$ *reachable from* T *by successive mutation.*

We have the following applications [BIRSm] of Corollary 13 (see also [K]).

Example 14 (a) Cluster tilted algebras are Jacobian algebras of QP's.

(b) Let Λ be a preprojective algebra and W the corresponding Coxeter group. For any $w \in W$, we have a 2-CY triangulated category $\mathcal{C} := \underline{\mathrm{Sub}}\Lambda_w$ [BIRSc]. For any cluster tilting object $T \in \mathcal{C}$ reachable from a cluster tilting object given by a reduced expression of w by successive mutation, $\mathrm{End}_{\mathcal{C}}(T)$ is a Jacobian algebra of a QP.

We end this report by the following *nearly Morita equivalence* for Jacobian algebras [BMR2, BIRSm], where f.l. is the category of modules with finite length.

Theorem 15 *For a QP (Q, W), we have an equivalence*

$$\mathrm{f.l.}\, \mathcal{P}(Q, W)/\operatorname{add} S_k \simeq \mathrm{f.l.}\, \mathcal{P}(\mu_k(Q, W))/\operatorname{add} S'_k,$$

where S_k and S'_k are simple modules associated with the vertex k.

References

[BIRSc] A. Buan, O. Iyama, I. Reiten, J. Scott, *Cluster structures for 2-Calabi-Yau categories and unipotent groups*, arXiv:math/0701557.

[BIRSm] A. Buan, O. Iyama, I. Reiten, D. Smith, *Mutation of cluster tilting object and quiver with potentials*, in preparation.

[BMR] A. Buan, R. Marsh, I. Reiten, *Cluster mutation via quiver representations*, Comment. Math. Helv. 83 (2008), no. 1, 143–177.

[BMR2] A. Buan, R. Marsh, I. Reiten, *Cluster-tilted algebras*, Trans. Amer. Math. Soc. 359 (2007), no. 1, 323–332.

[BMRRT] A. Buan, R. Marsh, M. Reineke, I. Reiten, G. Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. 204 (2006), no. 2, 572–618

[DWZ] H. Derksen, J. Weyman, A. Zelevinsky, *Quivers with potentials and their representations I: Mutations*, arXiv:0704.0649.

[FZ] S. Fomin, A. Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. 15 (2002), no. 2, 497–529.

[I] O. Iyama, *Higher-dimensional Auslander-Reiten theory on maximal orthogonal subcategories*, Adv. Math. 210 (2007), no. 1, 22–50.

[IY] O. Iyama, Y. Yoshino, *Mutation in triangulated categories and rigid Cohen-Macaulay modules*, to appear in Invent. Math., arXiv:math/0607736.

[K] B. Keller, in preparation.