REPRESENTATION THEORY OF ORDERS

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Introduction

Orders over complete discrete valuation rings give fruitful examples of non-commutative noetherian rings of Krull dimension one. They often play important roles in number theory as well. Some results about their representation and relation to finite dimensinal algebras and higher dimensional orders will be explained. Main objects are: Bass orders (§4), AR quivers (§5,§6,§7), finite type (§7) and tame type (§8,§9). Each topics are closely related to one another.

$$\begin{tikzcd}
\S1,\S2 \arrow[r, dash] \arrow[dr, dash] & \S5 \arrow[r, dash] & \S6 \arrow[r, dash] & \S7 \\
\S3 \arrow[r, dash] & \S4 \arrow[ur, dash] \arrow[r, dash] & \S8 \arrow[r, dash] & \S9
\end{tikzcd}$$

Throughout this paper, let R be a complete regular local ring of dimension $d \geq 0$ and K a quotient field of R. An R-algebra Λ is called an R-**order** if it is finitely generated free as an R-module. For an R-order Λ, a left Λ-module L is called a Λ-**lattice** if it is finitely generated free as an R-module. We denote by $\operatorname{lat}\Lambda$ (resp. $\operatorname{pr}\Lambda$) the category of Λ-lattices (resp. projective Λ-lattices). Then $(\ )^* = \operatorname{Hom}_R(\ ,R)$ gives a duality between $\operatorname{lat}\Lambda$ and $\operatorname{lat}\Lambda^{op}$. Let $\operatorname{rin}\Lambda := (\operatorname{pr}\Lambda^{op})^*$ be the category of **relative injective** Λ-lattices. We denote by J_Λ the Jacobson radical of Λ. For $d = 0$ (R is a field), an R-algebra is an R-order iff it is finite dimensional over R. In this case, $\operatorname{lat}\Lambda$ is denoted by $\operatorname{mod}\Lambda$, which is the category of finite dimensional Λ-modules.

K.W. Roggenkamp and M. Ştefănescu (eds.), Algebra - Representation Theory, 63–96.

An additive category $\mathcal{C}$ is called **Krull-Schmidt** if any object is isomorphic to a finite direct sum of objects whose endomorphism rings are local. Then any object can be written uniquely as a finite direct sum of indecomposable objects. We denote by $\mathfrak{I}(\mathcal{C})$ the set of isomorphism classes of indecomposable objects in $\mathcal{C}$. It is a basic result that $\operatorname{lat}\Lambda$ *is a Krull-Schmidt category for an R-order* Λ.

For an R-order Λ and $L \in \operatorname{lat}\Lambda$, we put $\widetilde{L} := L \otimes_R K$. The finite dimensional K-algebra $\widetilde{\Lambda}$ is called the **ambient algebra** of Λ, and Λ is called an order **in** $\widetilde{\Lambda}$. An R-order Λ is called of **finite type** if $\mathfrak{I}(\operatorname{lat}\Lambda)$ is a finite set, and called of **bounded type** if $\sup_{L \in \mathfrak{I}(\operatorname{lat}\Lambda)} l_{\widetilde{\Lambda}}(\widetilde{L}) < \infty$, where $l_A(X)$ is the length of an A-module X.

An R-order Λ is called an **isolated singularity** if $\operatorname{gl.dim}\Lambda_{\wp} = \operatorname{ht}\wp$ holds for any non-maximal prime ideal $\wp$ of R and $\Lambda_{\wp} := \Lambda \otimes_R R_{\wp}$. It is well known that any order of finite type is an isolated singularity [**A2**], and any isolated singularity of bounded type is of finite type (Brauer-Thrall I)[**Y**]. For $d = 1$, Λ is an isolated singularity iff the ambient algebra $\widetilde{\Lambda}$ is semisimple.

In this paper, *assume* $\dim R = 1$ *unless explicitly stated otherwise.*

1 Overorders and Overrings

Let $\dim R$ be arbitrary. An R-order Γ is called an **overring** (resp. **overorder**) of Λ if $\widetilde{\Lambda}/I \supset \Gamma \supseteq (\Lambda + I)/I$ holds for some ideal I of $\widetilde{\Lambda}$ (resp. $\widetilde{\Lambda} \supset \Gamma \supseteq \Lambda$ holds). Then the natural morphism $\Lambda \to \Gamma$ induces a full faithful functor $\operatorname{lat}\Gamma \to \operatorname{lat}\Lambda$. Thus $\operatorname{lat}\Gamma$ (resp. $\mathfrak{I}(\operatorname{lat}\Lambda) - \mathfrak{I}(\operatorname{lat}\Gamma)$) can be regarded as a full subcategory of $\operatorname{lat}\Lambda$ (resp. subset of $\mathfrak{I}(\operatorname{lat}\Lambda)$). We call such a full subcategory of $\operatorname{lat}\Lambda$ (resp. subset of $\mathfrak{I}(\operatorname{lat}\Lambda)$) **rejective** (resp. **rejectable**). Naturally, we have bijections between (overrings of Λ), (rejective subcategory of $\operatorname{lat}\Lambda$) and (rejectable subsets of $\mathfrak{I}(\operatorname{lat}\Lambda)$).

It is a basic method in representation theory of orders to compare $\operatorname{lat}\Lambda$ with $\operatorname{lat}\Gamma$. To study rejective subcategories, first notice that the inclusion functor $\operatorname{lat}\Gamma \to \operatorname{lat}\Lambda$ has a right adjoint $(\ )^- := \operatorname{Hom}_\Lambda(\Gamma, \ )$ and a left adjoint $(\ )^+ := (\Gamma \otimes_\Lambda \ )^{**}$. The following shows that these properties characterize rejective subcategories of $\operatorname{lat}\Lambda$. It will be used to define rejective subcategories for τ-categories in §6.

1.1 (Adjoint functors) *Let Λ be an R-order and* $\dim R \leq 2$. *A subcategory $\mathcal{C}'$ of $\mathcal{C} := \operatorname{lat}\Lambda$ is rejective iff the natural inclusion $\mathcal{C}' \to \mathcal{C}$ has a right adjoint $(\ )^- : \mathcal{C} \to \mathcal{C}'$ with a counit ϵ^- such that ϵ^-_X is a monomorphism for any $X \in \mathcal{C}$, and a left adjoint $(\ )^+ : \mathcal{C} \to \mathcal{C}'$ with a unit ϵ^+ such that ϵ^+_X is an epimorphism for any $X \in \mathcal{C}$.*

1.2 For rejectability of a singleton set, there is the following criterion due to Drozd-Kirichencko [**DK1**]. In §6, we will study its generalization.

(DK Rejection Lemma) *Let Λ be an R-order and* $\dim R \leq 1$. *Then a singleton set $\{X\}$ is rejectable iff $X \in \operatorname{pr}\Lambda \cap \operatorname{rin}\Lambda$.*

In this case, assume that $\dim R = 1$ *and Λ is a non-maximal (1.3) local order. Then $\Gamma := O_l(J_\Lambda)$ satisfies $\mathfrak{I}(\operatorname{lat}\Lambda) - \mathfrak{I}(\operatorname{lat}\Gamma) = \{X\}$, where we put $O_l(L) := \{x \in \widetilde{\Lambda} \mid xL \subseteq L\}$ for $L \in \operatorname{lat}\Lambda$.*

1.3 Maximal orders and hereditary orders Assume $\dim R = 1$. An R-order Λ is called **maximal** if Λ has no proper overorder, and called **hereditary** if $\operatorname{gl.dim}\Lambda = 1$. Then the following results are known [**CR**].

(1) Any order in a semisimple algebra has a maximal overorder since any increasing chain of orders in a semisimple algebra stops.

(2) Any maximal order is hereditary, and the ambient algebra of any hereditary order is semisimple.

(3) **(Structure theorem)** In any division K-algebra D, there exists a unique maximal R-order $\mathcal{O}_D$. An R-order is hereditary (resp. maximal) iff it is Morita equivalent to a direct product of $\mathrm{T}_l(\mathcal{O}_D)$ (resp. $\mathcal{O}_D$), where D is a division K-algebra and $\mathrm{T}_l(\mathcal{O}_D)$ is the subring of $\mathrm{M}_l(\mathcal{O}_D)$ consisting of $(x_{ij})_{i,j}$ such that $x_{ij} \in J_{\mathcal{O}_D}$ for any $i < j$.

2 AR sequences and AR quivers

Let $\dim R$ be arbitrary. We will explain AR sequences whose meaning we can understand essentially by considering functor categories.

2.1 Modules over additive categories Let $\mathcal{C}$ be a Krull-Schmidt category which is skeletally small, namely isomorphism classes of objects form a set. We denote by $\mathcal{C}(X,Y)$ the set of morphisms from X to Y, and by $fg \in \mathcal{C}(X,Z)$ the composition of $f \in \mathcal{C}(X,Y)$ and $g \in \mathcal{C}(Y,Z)$. A **$\mathcal{C}$-module** is a contravariant additive functor from $\mathcal{C}$ to the category $\mathcal{A}b$ of abelian groups. For $\mathcal{C}$-modules M and M', we denote by $\mathrm{Hom}(M,M')$ the set of natural transformations from M to M'. Thus we obtain the abelian category $\mathrm{Mod}\,\mathcal{C}$ of $\mathcal{C}$-modules [**A1**].

Define functors $H^{\mathcal{C}} : \mathcal{C} \to \mathrm{Mod}\,\mathcal{C}$ and $H_{\mathcal{C}} : \mathcal{C} \to \mathrm{Mod}\,\mathcal{C}^{op}$ by $H_X^{\mathcal{C}} := \mathcal{C}(\ ,X)$ and $H_{\mathcal{C}}^X := \mathcal{C}(X,\ )$. We say that a $\mathcal{C}$-module M is **finitely generated** if there exists an exact sequence $H_X^{\mathcal{C}} \to M \to 0$ for some $X \in \mathcal{C}$. Then Yoneda's Lemma shows that the functor $H^{\mathcal{C}}$ (resp. $H_{\mathcal{C}}$) induces an equivalence from $\mathcal{C}$ to the category of finitely generated projective $\mathcal{C}$-modules (resp. $\mathcal{C}^{op}$-modules).

We denote by $\mathcal{J}_{\mathcal{C}}$ the **Jacobson radical** of $\mathcal{C}$, which is an ideal of $\mathcal{C}$ such that $\mathcal{J}_{\mathcal{C}}(\ ,X)$ (resp. $\mathcal{J}_{\mathcal{C}}(X,\ )$) is the **radical** (=intersection of all maximal submodules) of $H_X^{\mathcal{C}}$ (resp. $H_{\mathcal{C}}^X$) for any $X \in \mathcal{C}$. Define functors
$S^{\mathcal{C}} : \mathcal{C} \to \mathrm{Mod}\,\mathcal{C}$ and $S_{\mathcal{C}} : \mathcal{C} \to \mathrm{Mod}\,\mathcal{C}^{op}$ by $S_X^{\mathcal{C}} := H_X^{\mathcal{C}} \,/\, \mathcal{J}_{\mathcal{C}}(\ ,X)$ and $S_{\mathcal{C}}^X := H_X^{\mathcal{C}} \,/\, \mathcal{J}_{\mathcal{C}}(X,\ )$. Then $S^{\mathcal{C}}$ (resp. $S_{\mathcal{C}}$) induces a bijection from $\mathfrak{I}(\mathcal{C})$ to the set of isomorphism classes of simple $\mathcal{C}$-modules (resp. $\mathcal{C}^{op}$-modules). We simply denote $H_X^{\mathcal{C}}$ (resp. $H_{\mathcal{C}}^X$, $S_X^{\mathcal{C}}$, $S_{\mathcal{C}}^X$) by H_X (resp. H^X, S_X, S^X).

We denote by $\mathrm{pd}\,M$ the projective dimension of $\mathcal{C}$-module M. For $n \geq 0$, we denote by $\mathfrak{I}_n^+(\mathcal{C})$ (resp. $\mathfrak{I}_n^-(\mathcal{C})$) the subset of $\mathfrak{I}(\mathcal{C})$ consisting of X such that $\mathrm{pd}\,S_X \leq n$ (resp. $\mathrm{pd}\,S^X \leq n$).

$\mathcal{C}$ is called **left artinian** (resp. **right artinian**) if H_X (resp. H^X) has a finite length for any $X \in \mathcal{C}$, and called **artinian** if $\mathcal{C}$ is left and right artinian.

2.2 AR sequences It is an impotant result that *an R-order Λ is an isolated singularity iff* $\mathrm{lat}\,\Lambda$ *has AR sequences* [**A2**]. This means the second part of the following theorem.

2.2.1 Theorem *Let Λ be an R-order and $d := \dim R$.*

(1) For any $X \in \mathfrak{I}(\mathrm{pr}\,\Lambda)$ (resp. $X \in \mathfrak{I}(\mathrm{rin}\,\Lambda)$), $\mathrm{pd}\,S_X = d$ (resp. $\mathrm{pd}\,S^X = d$) holds if $d > 0$, and $\mathrm{pd}\,S_X \leq 1$ (resp. $\mathrm{pd}\,S^X \leq 1$) holds if $d = 0$. Moreover, S_X (resp. S^X) has a minimal projective

resolution whose terms are finitely generated.

(2) Λ is an isolated singularity iff, for any $X \in \mathfrak{I}(\mathrm{lat}\,\Lambda) - \mathfrak{I}(\mathrm{pr}\,\Lambda)$ (resp. $X \in \mathfrak{I}(\mathrm{lat}\,\Lambda) - \mathfrak{I}(\mathrm{rin}\,\Lambda)$), $\mathrm{pd}\,S_X = 2$ (resp. $\mathrm{pd}\,S^X = 2$) holds and S_X (resp. S^X) has a minimal projective resolution whose terms are finitely generated.

Assume that Λ is an isolated singularity. For any $X \in \mathrm{lat}\,\Lambda$, we can take a complex $(X] = (\tau^+X \overset{\nu_X^+}{\to} \theta^+X \overset{\mu_X^+}{\to} X)$ (resp. $[X) = (X \overset{\mu_X^-}{\to} \theta^-X \overset{\nu_X^-}{\to} \tau^-X)$) in $\mathrm{lat}\,\Lambda$ such that $H_{\tau^+X} \to H_{\theta^+X} \to H_X \to S_X \to 0$ (resp. $H^{\tau^-X} \to H^{\theta^-X} \to H^X \to S^X \to 0$) gives the first part of a minimal projective resolution. Then we can show the following fact easily.

2.2.2 Proposition *Let Λ be an isolated singularity and $d := \dim R$.*

(1) If $X \in \mathfrak{I}(\mathrm{lat}\,\Lambda)$ satisfies $\mathrm{pd}\,S_X = 2$ (resp. $\mathrm{pd}\,S^X = 2$), then $(X] = [\tau^+X)$ (resp. $[X) = (\tau^-X]$) holds. We call $(X]$ (resp. $[X)$) an **AR sequence**.

(2) τ^+ and τ^- give mutually inverse bijections between $\mathfrak{I}(\mathrm{lat}\,\Lambda) - \mathfrak{I}(\mathrm{pr}\,\Lambda)$ and $\mathfrak{I}(\mathrm{lat}\,\Lambda) - \mathfrak{I}(\mathrm{rin}\,\Lambda)$. Moreover, if $d = 2$, then τ^+ and τ^- give mutually inverse bijections between $\mathfrak{I}(\mathrm{pr}\,\Lambda)$ and $\mathfrak{I}(\mathrm{rin}\,\Lambda)$.

2.3 AR quivers (1) $Q = (Q, Q^p, Q^i, \tau^+, d, d')$ is called a **translation quiver** if Q is a set, Q^p and Q^i are subsets of Q, τ^+ is a bijection $Q - Q^p \to Q - Q^i$, and d and d' are maps $Q \times Q \to \mathbb{N}_{\geq 0}$ such that $d(Y, X) = d'(\tau^+X, Y)$ holds for any $X \in Q - Q^p$ and $Y \in Q$.

Usually, we draw Q as a directed graph: Q is the set of vertices, and we draw valued arrows $X \xrightarrow{(d(X,Y),d'(X,Y))} Y$ for any $X, Y \in Q$ such that $d(X,Y) \neq 0$, and dotted arrows from X to τ^+X for any $X \in Q - Q^p$. We usually omit a valuation $(1,1)$.

Moreover, a translation quiver Q is called **admissible** if there exists a map $c : Q \to \mathbb{N}_{>0}$ such that $c(X)d(X,Y) = d'(X,Y)c(Y)$ holds for any X, $Y \in Q$.

(2) For a set Q, we denote by $\mathbb{Z}Q$ (resp. $\mathbb{N}Q$) the free $\mathbb{Z}$-module (resp. free abelian monoid) generated by Q. We introduce an inner

product $\langle\ ,\ \rangle$ on $\mathbb{Z}Q$ by taking Q as an orthonormal base. For $X \in \mathbb{Z}Q$, put $\operatorname{supp} X := \{Y \in Q \mid \langle X, Y\rangle \neq 0\}$, and write $X = X_+ - X_-$ where $X_+, X_- \in \mathbb{N}Q$ and $\operatorname{supp} X_+ \cap \operatorname{supp} X_- = \emptyset$.

For example, the set of isomorphism classes of objects in a Krull-Schmidt category $\mathcal{C}$ is given by $\mathbb{N}\mathfrak{I}(\mathcal{C})$.

(3) Let Λ be an isolated singularity, $\dim R \leq 2$ and $\mathcal{C} := \operatorname{lat}\Lambda$. Then a translation quiver $\mathbb{A}(\mathcal{C}) = (Q, Q^p, Q^i, \tau^+, d, d')$ called the **AR quiver** of Λ is defined by $Q := \mathfrak{I}(\mathcal{C})$, $Q^p := \mathfrak{I}_1^+(\mathcal{C})$, $Q^i := \mathfrak{I}_1^-(\mathcal{C})$, $d(X,Y) := \langle \theta^+ Y, X\rangle$ and $d'(X,Y) := \langle \theta^- X, Y\rangle$. Then $\mathbb{A}(\mathcal{C})$ is admissible by $k := R/J_R$ and $c(X) := \dim_k \operatorname{End}_\Lambda(X)/J_{\operatorname{End}_\Lambda(X)}$.

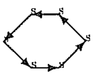
$\mathbb{A}(\operatorname{lat}\Lambda)$ displays terms of each $(X]$ and $[X)$ diagrammatically. We will give a characterization of $\mathbb{A}(\operatorname{lat}\Lambda)$ for orders of finite type in §7.

(4) For a translation quiver $Q = (Q, Q^p, Q^i, \tau^+, d, d')$, define elements θ^+, θ^-, τ^+ and τ^- of $\operatorname{End}_{\mathbb{Z}}(\mathbb{Z}Q)$ as follows. Put $\theta^+ X := \sum_{Y\in Q} d(Y,X)Y$ and $\theta^- X := \sum_{Y\in Q} d'(X,Y)Y$ for $X \in Q$. Put $\tau^+ X := 0$ for $X \in Q^p$, $\tau^- X := (\tau^+)^{-1}(X)$ for $X \in Q - Q^i$ and $\tau^- X := 0$ for $X \in Q^i$.

For $f \in \operatorname{End}_{\mathbb{Z}}(\mathbb{Z}Q)$ and a subset S of Q, define $f_{Q/S} \in \operatorname{End}_{\mathbb{Z}}(\mathbb{Z}(Q - S))$ by $f_{Q/S}(X) := f(X)|_{Q-S}$ where $|_{Q-S} : \mathbb{Z}Q \to \mathbb{Z}(Q - S)$ is the natural projection.

2.3.1 Example Let Λ be an isolated singularity, $d := \dim R \leq 2$ and $Q := \mathbb{A}(\operatorname{lat}\Lambda)$.

(1) If $d \leq 1$, then $Q^p = \mathfrak{I}(\operatorname{pr}\Lambda)$ and $Q^i = \mathfrak{I}(\operatorname{rin}\Lambda)$. If $d = 2$, then $Q^p = Q^i = \emptyset$.

(2) For $\Lambda := \mathrm{T}_l(\mathcal{O}_D)$ in 1.3(3), Q is a cycle  with l-vertices such that $Q = Q^p = Q^i$. Hence the AR quiver of a hereditary order is a disjoint union of such cycles, and the AR quiver of a maximal order is a disjoint union of loops ↻.

3 Quadratic extensions

As we will see in §4 and §8, some kind of quadratic extension often appears in representation theory of orders [**I4**]. In this section, any modules and algebras are assumed to be finitely generated R-modules.

3.1 Definition (1) An (A, B)-bimodule M is called a **Nakayama bimodule** if the set of sub A-modules of Mf is totally ordered by inclusion for any primitive idempotent f of B, and the set of sub B^{op}-modules of eM is totally ordered by inclusion for any primitive idempotent e of A. An algebra A is called a **Nakayama algebra** if A is a Nakayama (A, A)-bimodule.

(2) Let $\mathcal{C}_l$ and $\mathcal{C}_r$ be Krull-Schmidt categories. A $(\mathcal{C}_l, \mathcal{C}_r)$**-bimodule** is a covariant additive functor $M : \mathcal{C}_r \to \operatorname{Mod} \mathcal{C}_l$. In other word, a multiliner map $\mathcal{C}_l(W, X) \times M(X, Y) \times \mathcal{C}_r(Y, Z) \to M(W, Z)$ is given for any $W, X \in \mathcal{C}_l$ and $Y, Z \in \mathcal{C}_r$ such that the associativity holds.

M is called a **Nakayama bimodule** if $M(X, Y)$ is a Nakayama $(\mathcal{C}_l(X, X), \mathcal{C}_r(Y, Y))$-bimodule for any $X \in \mathcal{C}_l$ and $Y \in \mathcal{C}_r$.

3.2 Definition Let $f : B \to A$ be an algebra monomorphism.

(1) We call f a **minimal quadratic extension** if f is Morita equivalent to one of the following (O)–(V), where D is a division algebra and D' is a sub-division algebra of D of index 2.

(O): Identity map $D \to D$.

(I·II): Regular representation $D \to \mathrm{M}_2(D')$.

(III): Diagonal embedding $D \to D \times D$.

(IV): Injection $D' \to D$.

(V): $D \times D \to \mathrm{M}_2(D)$, $(x, y) \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$.

Assume that R is a valuation field. A morhism of type(I·II) is called of **type(I)** (resp. **type(II)**) if it is unramified (resp. ramified). A morhism of type(IV) is called of **type(IVa)** (resp. **type(IVb)**) if it is unramified (resp. ramified).

(2) Assume that A is semisimple. Then we call f a **normal quadratic extension** (resp. **triangular extension**) if f is Morita equivalent to a direct product of minimal quadratic extensions (resp. injections $\mathrm{T}_l(D) \to \mathrm{M}_l(D)$, where D is a division algebra and $\mathrm{T}_l(D)$ is the upper triangular matrix ring). We call f a **quadratic extension** if f is a composition $B \xrightarrow{g} C \xrightarrow{h} A$ of a normal quadratic extension g and a triangular extension h.

For $k := R/J_R$, we can show that f *is a quadratic extension iff* $2 \dim_k eBe \geq \dim_k eAe$ *holds for any idempotent* e *of* B.

In general, we call f a **quadratic extension** (resp. **normal quadratic extension, triangular extension**) if A is a Nakayama algebra, $J_A \subset B$ and $B/J_A \to A/J_A$ is a quadratic extension (resp. normal quadratic extension, triangular extension) above.

(3) An additive functor $\mathbb{I} : \mathcal{C} \to \mathcal{D}$ of Krull-Schmidt categories is called a **quadratic extension** (resp. **normal quadratic extension, triangular extension**) if $\mathbb{I}_{X,X} : \mathcal{C}(X,X) \to \mathcal{D}(\mathbb{I}(X), \mathbb{I}(X))$ is a quadratic extension (resp. normal quadratic extension, triangular extension) for any $X \in \mathcal{C}$. Then any quadratic extension $\mathbb{I}$ is uniquely written as a composition $\mathcal{C} \xrightarrow{\mathbb{J}} \mathcal{E} \xrightarrow{\mathbb{K}} \mathcal{D}$ of a normal quadratic extension $\mathbb{J}$ and a triangular extension $\mathbb{K}$. This decomposition is called a **normalization** of $\mathbb{I}$.

(4) For a normal quadratic extension $\mathbb{I} : \mathcal{C} \to \mathcal{D}$, define a graph $G(\mathbb{I})$ as follows. The set of vertices is $\mathfrak{I}(\mathcal{D})$. For any $X \in \mathfrak{I}(\mathcal{D})$, put $Y := \bigoplus_{Z \in \mathfrak{I}(\mathcal{C}), X \leq \mathbb{I}(Z)} Z \in \mathcal{C}$. Then the induced morphism $\mathbb{I}_{Y,Y} : \mathcal{C} / \mathcal{J}_{\mathcal{C}}(Y,Y) \to \mathcal{D} / \mathcal{J}_{\mathcal{D}}(\mathbb{I}(Y), \mathbb{I}(Y))$ is a minimal quadratic extension. If it is of type(I·II)(IV) or (V), then we draw an edge X—X. If it is of type(III), then $\mathbb{I}(Y) = X \oplus X'$ holds for some $X \neq X' \in \mathfrak{I}(\mathcal{D})$, and we draw an edge X—X'.

3.3 Example Let Λ be an R-order and $\dim R = 1$.

Then Λ is a hereditary order iff Λ is a Nakayama algebra iff the injection $\Lambda \to \Gamma$ is a triangular extension for some (any) overorder Γ [**HN2**]. On the other hand, we call Λ a **quadratic order** if the injection $\Lambda \to \Gamma$ is a quadratic extension for some overorder Γ.

Recall that Λ is called **Bäckström** if there exists a hereditary overorder Γ such that $\Lambda \supset J_\Lambda = J_\Gamma$ [**RR**]. Then any quadratic order is Bäckström.

4 Bass orders

In this section, assume $\dim R = 1$. An R-order Λ is called **Gorenstein** if $\Lambda^* = \mathrm{Hom}_R(\Lambda, R) \in \mathrm{pr}\,\Lambda$, and called **Bass** if any overorder of Λ is Gorenstein. It is easily shown that any hereditary order is Bass.

4.1 For simplicity, let $\Lambda^{(0)} := \Lambda$ be a local Bass order. If Λ is not hereditary, then there exists an overorder $\Lambda^{(1)}$ such that $\mathfrak{I}(\mathrm{lat}\,\Lambda^{(0)}) - \mathfrak{I}(\mathrm{lat}\,\Lambda^{(1)}) = \{\Lambda^{(0)}\}$ by DK Rejection Lemma (1.2). Again $\Lambda^{(1)}$ is Bass by definition, and we can show that $\Lambda^{(1)}$ is local if $\Lambda^{(1)}$ is not hereditary. Thus we obtain a unique (finite or infinite) sequence $\Lambda^{(0)} \subset \Lambda^{(1)} \subset \Lambda^{(2)} \subset \cdots$ satisfying $\mathfrak{I}(\operatorname{lat}\Lambda^{(i)}) - \mathfrak{I}(\operatorname{lat}\Lambda^{(i+1)}) = \{\Lambda^{(i)}\}$, where the last term $\Lambda^{(s)}$ is hereditary if the sequence is finite, and other terms are non-hereditary local Bass orders.

This sequence is fundumental for our study of Bass orders. As an application, we can obtain the following result [**DKR**][**Ro**][**HN2**], where we call an algebra **purely non-semisimple** if it does not have a simple ring direct summand.

Theorem *(1) A Gorenstein order Λ is Bass iff any minimal overorder of Λ is Gorenstein.*

(2) Any Bass order is Morita equivalent to a direct product of the following orders.

(i) A hereditary order.

(ii) $\begin{pmatrix} \Omega & \Omega \\ J_\Omega^{s+1} & \Omega \end{pmatrix}$ where Ω is a local maximal order and $s > 0$.

(iii) A local Bass order in a semisimple algebra.

(iv) A Bass order in a purely non-semisimple algebra.

In 1.3, we already have detailed description for (i) and (ii). In 4.2 (resp. 4.3), we will review some results for (iii) (resp. (iv)).

4.2 Local Bass orders in semisimple algebras A chain $\{\Lambda_i\}_{0 \le i \le s}$ $(0 \le s \le \infty)$ is called a **primary Bass chain** of length s if Λ_i is a unique minimal overorder of a local Bass order Λ_{i+1} for any i $(0 \le i < s)$ and Λ_0 is hereditary. Then a pair $(\Lambda_0, J_{\Lambda_1})$ is called the **dominating pair**. We can show that it is one of the following (I)–(IVb) where Ω and Ω' are local maximal orders.

(I) $(\mathrm{M}_2(\Omega), \mathrm{M}_2(J_\Omega))$. (II) $\left(\begin{pmatrix} \Omega & \Omega \\ J_\Omega & \Omega \end{pmatrix}, \begin{pmatrix} J_\Omega & \Omega \\ J_\Omega & J_\Omega \end{pmatrix}\right)$.

(III) $(\Omega \times \Omega', J_\Omega \times J_{\Omega'})$. (IVa) (Ω, J_Ω). (IVb) (Ω, J_Ω^2).

The argument in 4.1 and 1.3(1) show that, *for any local Bass order Λ in a semisimple algebra, there exists a unique primary Bass chain $\{\Lambda_i\}_{0 \le i \le s}$ of length $s(< \infty)$ such that $\Lambda = \Lambda_s$*. In this case, s is called the **rank** of Λ, and Λ is called of **type(T)**$_s$ where $T \in \{$I,II,III,IVa,IVb$\}$ is determined by the dominating pair. Moreover, an order in 4.1(2)(ii) is called of **type(V)**$_s$.

Describing AR quivers, we obtain the following good correspondence between types of Bass orders and classical Dynkin diagrams with an involution, where τ^+ is the identity map except $\tau^+X = Y$ and $\tau^+Y = X$ in type(III)$_s$.

$$
\begin{array}{lll}
(\mathrm{I})_s & B_{s+1} & \overset{\Lambda_0}{\bullet} \underset{(1\ 2)}{\overset{(2\ 1)}{\rightleftarrows}} \overset{\Lambda_1}{\bullet} \rightleftarrows \overset{\Lambda_2}{\bullet} \rightleftarrows \cdots \rightleftarrows \overset{\Lambda_s}{\bullet} \\
(\mathrm{II})_s & D_{s+2} & \begin{array}{c} \bullet \rightleftarrows \overset{\Lambda_1}{\bullet} \rightleftarrows \overset{\Lambda_2}{\bullet} \rightleftarrows \cdots \rightleftarrows \overset{\Lambda_s}{\bullet} \\ \downarrow\uparrow \\ \bullet \end{array} \\
(\mathrm{III})_s & D'_{s+2} & \begin{array}{c} \overset{X}{\bullet} \rightleftarrows \overset{\Lambda_1}{\bullet} \rightleftarrows \overset{\Lambda_2}{\bullet} \rightleftarrows \cdots \rightleftarrows \overset{\Lambda_s}{\bullet} \\ \downarrow\uparrow \\ Y\bullet \end{array} \\
(\mathrm{IVa})_s & C_{s+1} & \overset{\Lambda_0}{\bullet} \underset{(2\ 1)}{\overset{(1\ 2)}{\rightleftarrows}} \overset{\Lambda_1}{\bullet} \rightleftarrows \overset{\Lambda_2}{\bullet} \rightleftarrows \cdots \rightleftarrows \overset{\Lambda_s}{\bullet} \\
(\mathrm{IVb})_s & A'_{2s+2} & \circlearrowleft \overset{\Lambda_0}{\bullet} \rightleftarrows \overset{\Lambda_1}{\bullet} \rightleftarrows \overset{\Lambda_2}{\bullet} \rightleftarrows \cdots \rightleftarrows \overset{\Lambda_s}{\bullet} \\
(\mathrm{V})_s & A_{s+2} & \overset{\binom{\Omega}{\Omega}}{\bullet} \rightleftarrows \overset{\binom{\Omega}{J_\Omega}}{\bullet} \rightleftarrows \overset{\binom{\Omega}{J_\Omega^2}}{\bullet} \rightleftarrows \cdots \rightleftarrows \overset{\binom{\Omega}{J_\Omega^{s+1}}}{\bullet}
\end{array}
$$

In general, it is a very difficult problem to classify local Bass orders completely since it depends heavily on the arithmetic of a basic ring R. But, under weak assumptions on R and A, a classification is given in [**HN1**] which is rather number theoretic. On the other hand, there is the following very clear description of primary Bass chains of infinite length [**HN1**].

4.2.1 (primary Bass chain of infinite length) *Let A be a semisimple K-algebra. Then there exists a bijection from (the set of primary Bass chain $\{\Lambda_i\}_{0\le i}$ in A of infinite length) to (the set of division subalgebra B of A of index 2), which is given by $\{\Lambda_i\}_{0\le i} \mapsto B := K(\bigcap_{0\le i}\Lambda_i)$. Moreover, the type of a primary Bass chain of infinite length defined above coincides with the type of the corresponding injection $B \to A$ defined in 3.2(1).*

4.3 Bass orders in purely non-semisimple algebras In general, for an order Λ, it is effective to consider the ambient algebra $\widetilde{\Lambda}$ and an order $(\Lambda + J_A)/J_A$ in a semisimple algebra A/J_A. It is quite remarkable that we obtain the following clear description of Bass orders in purely non-semisimple algebras [**HN2**].

Theorem *Let Λ be an R-order in a purely non-semisimple algebra A. Then Λ is Bass iff ($l_A(Ae) = l_A(eA) = 2$ for any primitive idempotent e of A, and $(\Lambda + J_A)/J_A$ is a maximal order in A/J_A).*

Notice that, for a purely non-semisimple algebra A, $l_A(Ae) = l_A(eA) = 2$ for any primitive idempotent e of A iff A is a quasi-Frobenius algebra satisfying $J_A^2 = 0$.

4.4 Almost Bass orders An R-order Λ is called **almost Bass** if Λ and $O_l(J_\Lambda)$ are Gorenstein. By 1.2 and 4.1(1), *a local order is almost Bass iff it is Bass.*

Let $\Lambda^{(0)} := \Lambda$ be a ring-indecomposable almost Bass order. Using DK Rejection repeatedly, we obtain a unique (finite or infinite) sequence $\Lambda^{(0)} \subset \Lambda^{(1)} \subset \Lambda^{(2)} \subset \cdots$ satisfying $\mathfrak{I}(\operatorname{lat}\Lambda^{(i)}) - \mathfrak{I}(\operatorname{lat}\Lambda^{(i+1)}) = \mathfrak{I}(\operatorname{pr}\Lambda^{(i)})$, where the last term $\Lambda^{(s)}$ is hereditary if the sequence is finite, and other terms are ring-indecomposable non-hereditary almost Bass orders (cf. 4.1).

If $\widetilde{\Lambda}$ is seimsimple, then this sequence is finite by 1.3(1), and s is called the **rank** of Λ. We can obtain the following structure theorem of almost Bass orders [**H**].

4.4.1 Theorem *Let Λ be a ring-indecomposable order in a semisimple algebra.*

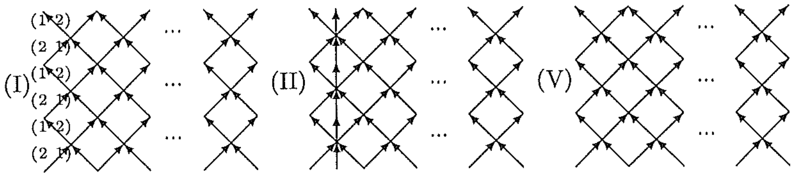
(1) Λ is an almost Bass order of rank 1 iff there exists a hereditary overorder Γ of Λ such that $J_\Lambda = J_\Gamma$ and the injection $\Lambda/J_\Lambda \to \Gamma/J_\Gamma$ is Morita equivalent to a direct product of morphisms of type (I-II), (III), (IV) and (V) (in 3.2(1)).

(2) Λ is an almost Bass order of rank $s \geq 2$ iff it is Morita equivalent to one of the following orders.

$$\begin{pmatrix}
\Delta_i & \Delta_i & \cdots & \Delta_i & \Delta_{i-1} & \cdots & \Delta_{i-1} \\
J_{\Delta_i} & \ddots & \ddots & & \ddots & \ddots & \vdots \\
\vdots & \ddots & & & & \ddots & \substack{\Delta_{i-1} \\ \Delta_i} \\
J_{\Delta_i} & & & & & & \\
J_{\Delta_{i+1}} & \ddots & & & & \ddots & \vdots \\
\vdots & \ddots & \ddots & & \ddots & \ddots & \Delta_i \\
J_{\Delta_{i+1}} & \cdots & J_{\Delta_{i+1}} & J_{\Delta_i} & \cdots & J_{\Delta_i} & \Delta_i
\end{pmatrix}
\quad
\begin{pmatrix}
\Delta_{i,1} & \Delta_{i-1} & \Delta_{i-1} & \cdots & \Delta_{i-1} \\
J & \Delta_{i,2} & \ddots & \ddots & \vdots \\
J & \ddots & \ddots & \ddots & \Delta_{i-1} \\
\vdots & \ddots & \ddots & \Delta_{i,n-1} & \Delta_{i-1} \\
J & \cdots & J & J & \Delta_{i,n}
\end{pmatrix}$$

Here, for the left order, (number of rows of Δ_i)+(number of rows of J_{Δ_i})=(size of the matrix) holds, and either $\{\Delta_l\}_l$ is a primary Bass chain, $\Delta_l = \begin{pmatrix} \Omega & J_\Omega^l \\ J_\Omega^l & \Omega \end{pmatrix}$ or $\Delta_l = \begin{pmatrix} \Omega & J_\Omega^l \\ J_\Omega^{l+1} & \Omega \end{pmatrix}$ (Ω is a local maximal order). For the right order, $\{\Delta_{l,j}\}_l$ is a primary Bass chain such that $J = J_{\Delta_{i,j}}$ and $\Delta_{i-1} = \Delta_{i-1,j}$ for any j.

The AR quiver of an order Λ in (2) above is determined by the rank s, the size n of the matrix and the type of the corresponding Bass orders $\{\Delta_l\}_l$. For example, $\mathbb{A}(\operatorname{lat}\Lambda)$ is the following if $n = 3$ and the type of Δ_l is either (I)(II) or (V), where the bottom vertices are identified with the top vertices and τ^+ is the downward shift.



4.4.2 As we have seen in 4.2.1, Bass orders have strong relationship to quadratic extensions. The following result shows that almost Bass orders have another relationship to quadratic extensions [**I4**]. It will be used in §9.

Theorem *Let Λ be an almost Bass order and* $\mathbb{G} := (\Lambda/J_\Lambda)\otimes_\Lambda \; : \operatorname{lat}\Lambda \to \operatorname{mod}\Lambda/J_\Lambda$. *Then the induced functor* $\operatorname{lat}\Lambda/\ker\mathbb{G} \to \operatorname{mod}\Lambda/J_\Lambda$ *is a quadratic extension (in 3.2(3)).*

4.5 Remark (1) Using Bass orders, we can obtain a classification of local orders of finite type [**DK2**][**HN3**]. Describing AR quivers, we obtain a good correspondense between "types" of local orders of finite type and Dynkin diagrams.

(2) Let Λ be a ring-indecomposable finite dimensional algebra. It is well known that Λ is local Nakayama (resp. quasi-Frobenius Nakayama) iff Λ/I is quasi-Frobenius for any ideal I of Λ (resp. Λ and Λ/J_Λ are quasi-Frobenius). Thus Bass orders (resp. almost Bass orders) can be regarded as one-dimensional analogy of local Nakayama algebras (resp. quasi-Frobenius Nakayama algebras).

5 τ-categories

Although our first aim is to study $\operatorname{lat}\Lambda$, it is often much better to study a wider class of additive categories called **τ-categories** [**I3**]. Results in this section will be used in §7, where our general treatment works essentially. It is also interesting that τ-categories are closely related to Auslander-Gorenstein rings (5.5, 7.4).

5.1 Definition A Krull-Schmidt category $\mathcal{C}$ is called a **τ-category** if, for any $X \in \mathcal{C}$, there exist complexes $(X] = (\tau^+X \overset{\nu_X^+}{\to} \theta^+X \overset{\mu_X^+}{\to} X)$ and $[X) = (X \overset{\mu_X^-}{\to} \theta^-X \overset{\nu_X^-}{\to} \tau^-X)$ such that $H_{\tau^+X} \to H_{\theta^+X} \to H_X \to S_X \to 0$ and $H^{\tau^-X} \to H^{\theta^-X} \to H^X \to S^X \to 0$ are minimal projective resolutions, and $H^{\theta^+X} \to H^{\tau^+X} \to S^{\tau^+X} \to 0$ and $H_{\theta^-X} \to H_{\tau^-X} \to S_{\tau^-X} \to 0$ are exact. We sometimes denote $(X]$ (resp. $[X)$, $\theta^\pm$, $\tau^\pm$) by $(X]_\mathcal{C}$ (resp. $[X)_\mathcal{C}$, $\theta^\pm_\mathcal{C}$, $\tau^\pm_\mathcal{C}$).

We can show that $(X] = [\tau^+X)$ (resp. $[X) = (\tau^-X]$) holds for any $X \in \mathfrak{I}(\mathcal{C}) - \mathfrak{I}_1^+(\mathcal{C})$ (resp. $X \in \mathfrak{I}(\mathcal{C}) - \mathfrak{I}_1^-(\mathcal{C})$). Hence τ^+ and τ^- give mutually inverse bijections between $\mathfrak{I}(\mathcal{C}) - \mathfrak{I}_1^+(\mathcal{C})$ and $\mathfrak{I}(\mathcal{C}) - \mathfrak{I}_1^-(\mathcal{C})$. Thus we can define the **AR quiver** $\mathbb{A}(\mathcal{C})$ of $\mathcal{C}$ by 2.3(3). A τ-category $\mathcal{C}$ is called **strict** if $\operatorname{pd} S_X \le 2$ and $\operatorname{pd} S^X \le 2$ hold for any $X \in \mathcal{C}$.

For example, if Λ is an isolated singularity and $\dim R \le 2$, then $\operatorname{lat}\Lambda$ is a strict τ-category by 2.2. But our introduction of τ-categories is motivated by another important example more strongly.

Igusa-Todorov introduced **τ-species** (=modulated translation quiver in [**IT2**]), which are "algebraic realization" of translation quivers (2.3(1)) by skew fields and their bimodules. For each τ-species $\mathcal{Q}$, we can define a τ-category $\widehat{\mathbb{M}}(\mathcal{Q})$ called the **mesh category** of $\mathcal{Q}$, which is familiar in representation theory [**R**][**BG**][**IT2**]. (In 5.6, we will give a quick review of τ-species and their mesh categories.) Conversely, for each τ-category $\mathcal{C}$, we can define the **AR species** $\widehat{\mathbb{A}}(\mathcal{C})$, which is an algebraic realization of the AR quiver $\mathbb{A}(\mathcal{C})$. Then the following result 5.2 shows that τ-category is a natural domain to study mesh categories [**I3**].

5.2 Let $\mathcal{C}$ be a Krull-Schmidt category. Define the **associated completely graded category** $\widehat{\mathbb{G}}(\mathcal{C})$ of $\mathcal{C}$ by $\widehat{\mathbb{G}}(\mathcal{C}) := \prod_{i\geq 0} \mathcal{J}_{\mathcal{C}}^{i} / \mathcal{J}_{\mathcal{C}}^{i+1}$. Then $\mathcal{C}$ is called **completely graded** if $\mathcal{C}$ is equivalent to $\widehat{\mathbb{G}}(\mathcal{C})$.

Theorem *(1) $\widehat{\mathbb{G}}(\mathcal{C})$ is a τ-category for any τ-category $\mathcal{C}$.*

(2) $\widehat{\mathbb{A}}(\widehat{\mathbb{M}}(\mathcal{Q})) = \mathcal{Q}$ holds for any τ-species $\mathcal{Q}$, and $\widehat{\mathbb{M}}(\widehat{\mathbb{A}}(\mathcal{C})) = \widehat{\mathbb{G}}(\mathcal{C})$ holds for any τ-category $\mathcal{C}$. Thus $\widehat{\mathbb{A}}$ and $\widehat{\mathbb{M}}$ give mutually inverse bijections between the set of τ-species and the set of completely graded τ-categories.

5.3 Ladders A morphism $a \in \mathcal{C}(X,Y)$ is called **special** if $a+f$ is isomorphic to a as a complex for any $f \in \mathcal{J}_{\mathcal{C}}^{2}(X,Y)$. The following theorem is fundamental for τ-categories **[I3][IT1]**. In particular, it will be used to translate categorical conditions for $\mathcal{C}$ to combinatorial conditions for $\mathbb{A}(\mathcal{C})$.

Theorem *Let $\mathcal{C}$ be a τ-category, $a \in \mathcal{C}(X_0,Y_0)$ a special morphism and $L := \operatorname{Cok} H_a$.*

(1) There exist the following commutative diagram (called the **ladder** *of a), $U_i \in \mathcal{C}$ and $h_i \in \mathcal{C}(U_i, Z_{i-1})$ such that $a = \binom{b_0}{0} \in \mathcal{C}(Z_0 \oplus U_0, Y_0)$ and*

$$(Y_{i-1}] = (Z_i \oplus U_i \xrightarrow{\begin{pmatrix} b_i & -g_i \\ 0 & h_i \end{pmatrix}} Y_i \oplus Z_{i-1} \xrightarrow{\begin{pmatrix} f_i \\ b_{i-1} \end{pmatrix}} Y_{i-1})$$

for any i.

$$\begin{array}{ccccccccc}
Y_0 & \xleftarrow{f_1} & Y_1 & \xleftarrow{f_2} & Y_2 & \xleftarrow{f_3} & Y_3 & \xleftarrow{f_4} & \cdots \\
\uparrow{\scriptstyle b_0} & & \uparrow{\scriptstyle b_1} & & \uparrow{\scriptstyle b_2} & & \uparrow{\scriptstyle b_3} & & \cdots \\
Z_0 & \xleftarrow{g_1} & Z_1 & \xleftarrow{g_2} & Z_2 & \xleftarrow{g_3} & Z_3 & \xleftarrow{g_4} & \cdots
\end{array}$$

(2) The diagram in (1) induces the following commutative diagram, where each vertical complexes are minimal projective resolutions and $\mathcal{J}_{\mathcal{C}}^{n-1} L \to \mathcal{J}_{\mathcal{C}}^{n} L$ is the natural inclusion for any $n > 0$.

$$\begin{array}{ccccccccc}
L & \longleftarrow & \mathcal{J}_{\mathcal{C}} L & \longleftarrow & \mathcal{J}_{\mathcal{C}}^{2} L & \longleftarrow & \mathcal{J}_{\mathcal{C}}^{3} L & \longleftarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \cdots \\
H_{Y_0} & \xleftarrow{H_{f_1}} & H_{Y_1} & \xleftarrow{H_{f_2}} & H_{Y_2} & \xleftarrow{H_{f_3}} & H_{Y_3} & \xleftarrow{H_{f_4}} & \cdots \\
\uparrow{\scriptstyle H_{b_0}} & & \uparrow{\scriptstyle H_{b_1}} & & \uparrow{\scriptstyle H_{b_2}} & & \uparrow{\scriptstyle H_{b_3}} & & \cdots \\
H_{Z_0} & \xleftarrow{H_{g_1}} & H_{Z_1} & \xleftarrow{H_{g_2}} & H_{Z_2} & \xleftarrow{H_{g_3}} & H_{Z_3} & \xleftarrow{H_{g_4}} & \cdots
\end{array}$$

(3) Assume that L is semisimple. Then we can obtain terms Y_i and Z_i by $Z_0 = X_0 - (\theta^+ Y_0 - X_0)_-$ and the following **recursion fomula**, *where we use notations in 2.3(2). In particular, we can compute them from (X_0, Y_0) and $\mathbb{A}(\mathcal{C})$.*

$$Y_i = \theta^+ Y_{i-1} - Z_{i-1}, \quad Z_i = \tau^+ Y_{i-1} - (\theta^+ Y_i - \tau^+ Y_{i-1})_- \quad (i > 0)$$

As a conclusion, we obtain the following corollaries.

(4) $\mathcal{C}$ is left artinian iff the following condition holds.

For any $X \in \mathfrak{I}(\mathcal{C})$, put $Y_0 := 0$, $Y_1 := X$ and $Y_i := (\theta^+ Y_{i-1} - \tau^+ Y_{i-2})_+$ for $i \geq 2$. Then $Y_i = 0$ holds for sufficiently large i.

(5) Assume $\bigcap_{i \geq 0} \mathcal{J}_{\mathcal{C}}^i = 0$. Then $\mathcal{C}$ is a strict τ-category iff the following condition holds.

For any $X \in \mathfrak{I}(\mathcal{C})$, put $Y_0 := 0$, $Y_1 := X$ and $Y_i := \theta^+ Y_{i-1} - \tau^+ Y_{i-2}$ for $i \geq 2$. Then $Y_i \geq 0$ holds for any $i \geq 0$.

5.4 Quotient categories In the rest of this paper, any subcategory $\mathcal{C}'$ of a Krull-Schmidt category $\mathcal{C}$ is assumed to be *full, closed under isomorphisms, direct sums and direct summands*. Then the correspondence $\mathcal{C}' \mapsto \mathfrak{I}(\mathcal{C}')$ gives a bijection between subcategories of $\mathcal{C}$ and subsets of $\mathfrak{I}(\mathcal{C})$. We denote by $S \mapsto \operatorname{add} S$ the inverse of this correspondence.

In this case, we denote by $I_{\mathcal{C}'}$ the ideal of $\mathcal{C}$ consisting of morphisms which factor through some object in $\mathcal{C}'$. Define the quotient category $\overline{\mathcal{C}} := \mathcal{C} / \mathcal{C}'$ by $Ob(\overline{\mathcal{C}}) := Ob(\mathcal{C})$ and $\overline{\mathcal{C}}(X,Y) := \mathcal{C}(X,Y)/I_{\mathcal{C}'}(X,Y)$ for $X, Y \in \mathcal{C}$.

We call $\mathcal{C}'$ **coartinian** if $\overline{\mathcal{C}} := \mathcal{C} / \mathcal{C}'$ is artinian, namely $H_X^{\overline{\mathcal{C}}}$ and $H_{\overline{\mathcal{C}}}^X$ have finite length for any $X \in \mathcal{C}$. We call a subset S of $\mathfrak{I}(\mathcal{C})$ **artinian** if $\mathcal{C}' := \operatorname{add}(\mathfrak{I}(\mathcal{C}) - S)$ is a coartinian subcategory of $\mathcal{C}$.

Proposition *Let $\mathcal{C}'$ be a subcategory of a τ-category $\mathcal{C}$ and $\overline{\mathcal{C}} := \mathcal{C}/\mathcal{C}'$. Then $\overline{\mathcal{C}}$ is also a τ-category. For any $X \in \mathfrak{I}(\mathcal{C}) - \mathfrak{I}(\mathcal{C}')$, $(\overline{X}]_{\overline{\mathcal{C}}} = \overline{(X]_{\mathcal{C}}}$ holds if $\overline{\theta^+ X} \neq 0$, and $(\overline{X}]_{\overline{\mathcal{C}}} = (0 \to 0 \to \overline{X})$ holds if $\overline{\theta^+ X} = 0$.*

5.5 τ-categories with an order structure and Auslander orders For an additive category $\mathcal{C}$, we say that $\mathcal{C}$ has an **order structure** if there exists a complete discrete valuation ring R and an R-order Λ such that $\mathcal{C}$ is equivalent to $\mathrm{pr}\,\Lambda$.

As we have seen in 5.1, $\mathrm{lat}\,\Delta$ is a τ-category for any R-order Δ. Conversely, the following theorem 5.5.2 shows that a τ-category $\mathcal{C}$ is equivalent to $\mathrm{lat}\,\Delta$ for an order Δ if $\mathcal{C}$ has an order structure and the ambient category is semisimple [**I3**]. This result is closely related to **Auslander orders** and **Auslander-Gorenstein rings** [**ARo**][**FGR**].

5.5.1 Definition (1) Assume $\dim R = 1$. An R-order Λ is called an **Auslander order** if $\mathrm{gl.dim}\,\Lambda \leq 2$ and a minimal projective resolution $0 \to P_1 \to P_0 \to \Lambda^* \to 0$ of Λ^* satisfies $P_0 \in \mathrm{rin}\,\Lambda$. This definition is slightly different from that of [**ARo**] since we do not assume that $\Lambda \otimes_R K$ is semisimple.

(2) Let Λ be a noetherian ring and $0 \to \Lambda \to I^0 \to I^1 \to \cdots$ a minimal injective resolution of a Λ-module Λ. We say that Λ satisfies the (l,n)-**condition** $(l,n \geq 0)$ if $\mathrm{fd}\, I^i < l$ holds for any i $(0 \leq i < n)$. Moreover, Λ is called **Auslander regular** (resp. **Auslander-Gorenstein**) if $\mathrm{gl.dim}\,\Lambda < \infty$ (resp. $\mathrm{id}\,{}_\Lambda\Lambda < \infty$ and $\mathrm{id}\,\Lambda_\Lambda < \infty$) and Λ satisfies the (l,l)-condition for any $l > 0$.

5.5.2 Theorem *Let Λ be an R-order and $\dim R = 1$. Then the following conditions (1)–(3) are equivalent. Moreover, if $\widetilde{\Lambda}$ is seimsimple, then (4) is also equivalent.*

(1) $\mathrm{pr}\,\Lambda$ is a τ-category.

(2) Λ is an Auslander order.

(3) $\mathrm{gl.dim}\,\Lambda \leq 2$ and Λ is an Auslander regular ring.

(4) There exists an R-order Δ of finite type such that $\mathrm{pr}\,\Lambda$ is equivalent to $\mathrm{lat}\,\Delta$.

5.6 Appendix: τ-species and mesh categories In this subsection, we will give a quick review of τ-species and their mesh categories.

5.6.1 $\mathcal{Q} = (Q, D_X, {}_XM_Y)$ is called a **species** if Q is a set, D_X is a skew field for any $X \in Q$ and ${}_XM_Y$ is a (D_X, D_Y)-bimodule such that $\sum_{X \in Q} \dim_{D_X} {}_XM_Y < \infty$ and $\sum_{Y \in Q} \dim_{D_Y} {}_XM_Y < \infty$ for any $X, Y \in Q$.

For $X, Y \in Q$, put $P_0(X,Y) := 0$ if $X \neq Y$, and $P_0(X,X) := D_X$. Put

$$P_n(X,Y) := \bigoplus_{Z_1,\cdots,Z_{n-1}\in Q} {}_XM_{Z_1} \otimes_{D_{Z_1}} \cdots \otimes_{D_{Z_{n-1}}} {}_{Z_{n-1}}M_Y \text{ for } n > 0,$$

$$P_n(A,B) := \prod_{X,Y\in Q} \mathrm{Mat}_{\langle A,X\rangle,\langle B,Y\rangle}(P_n(X,Y))$$

$$\text{for } A, B \in \mathbb{N}Q \text{ and } n \geq 0.$$

We have a natural map $P_n(X,Y) \times P_m(Y,Z) \to P_{n+m}(X,Z)$, $(f,g) \mapsto fg := f \otimes g$ for any $X, Y, Z \in Q$. Using a matrix multiplication, we have a natural map $P_n(A,B) \times P_m(B,C) \to P_{n+m}(A,C)$ for any $A, B, C \in \mathbb{N}Q$.

Define additive categories $\widehat{\mathbb{P}}(\mathcal{Q})$ and $\mathbb{P}(\mathcal{Q})$ by $Ob(\widehat{\mathbb{P}}(\mathcal{Q})) = Ob(\mathbb{P}(\mathcal{Q})) := \mathbb{N}Q$, and $\widehat{\mathbb{P}}(\mathcal{Q})(A,B) := \prod_{n\geq 0} P_n(A,B)$ and $\mathbb{P}(\mathcal{Q})(A,B) := \bigoplus_{n\geq 0} P_n(A,B)$ for $A, B \in \mathbb{N}Q$, where the composition is given by $(f_n)_{n\geq 0} \cdot (g_n)_{n\geq 0} := (\sum_{i=0}^n f_i g_{n-i})_{n\geq 0}$.

5.6.2 Proposition *Let $\mathcal{Q}$ be a species. Then $\widehat{\mathbb{P}}(\mathcal{Q})$ is a completely graded Krull-Schmidt category called the* **tensor category** *of $\mathcal{Q}$.*

5.6.3 $\mathcal{Q} = (Q, Q^p, Q^i, D_X, {}_XM_Y, \tau^+, a, b)$ is called a **τ-species** if $(Q, D_X, {}_XM_Y)$ is a species, Q^p and Q^i are subsets of Q, τ^+ is a bijection $Q - Q^p \to Q - Q^i$, $a_X : D_X \to D_{\tau^+X}$ is a ring isomorphism and $b_{X,Y} : \mathrm{Hom}_{D_Y}({}_{\tau^+X}M_Y, D_Y) \to {}_YM_X$ is a (D_Y, D_X)-isomorphism for any $X \in Q - Q^p$ and $Y \in Q$, where ${}_{\tau^+X}M_Y$ is regarded as a D_X-module through a_X.

Putting $d(X,Y) := \dim_{D_X} {}_XM_Y$ and $d'(X,Y) := \dim_{D_Y} {}_XM_Y$, we obtain a translation quiver $|\mathcal{Q}| := (Q, Q^p, Q^i, \tau^+, d, d')$ called the **underlying quiver** of $\mathcal{Q}$.

Since $\mathrm{Hom}_{D_Y}(\mathrm{Hom}_{D_Y}({}_{\tau^+X}M_Y, D_Y), {}_YM_X) = {}_{\tau^+X}M_Y \otimes_{D_Y} {}_YM_X$, we can regard $b_{X,Y}$ as an element of ${}_{\tau^+X}M_Y \otimes_{D_Y} {}_YM_X$. Put $\gamma(X) := \sum_{Y\in Q} b_{X,Y} \in P_2(\tau^+X, X)$. Then $\gamma(X)f = a_X(f)\gamma(X)$ holds for any $f \in D_X$ since $b_{X,Y}$ is a D_X^{op}-morphism. Let I be the ideal of $\mathbb{P}(\mathcal{Q})$ generated by $\{\gamma(X) | X \in Q\}$. Then we can write $I = \bigoplus_{n\geq 0} I_n$ ($I_n \subseteq P_n$). Put $\widehat{\mathbb{M}}(\mathcal{Q}) := \widehat{\mathbb{P}}(\mathcal{Q})/\widehat{I}$, where $\widehat{I} := \prod_{n\geq 0} I_n$ is an ideal of $\widehat{\mathbb{P}}(\mathcal{Q})$.

5.6.4 Proposition *Let Q be a τ-species. Then $\widehat{\mathbb{M}}(Q)$ is a completely graded τ-category called the* **mesh category** *of Q.*

6 Generalized Rejection Lemma

In this section, assume that Λ is an isolated singularity and $\dim R \leq 1$. As we have seen in §4, our theory of Bass orders are constructed on DK Rejection. In this section, we will study its generalization. In 6.1, we will characterize artinian (5.4) rejectable subsets of $\mathfrak{I}(\operatorname{lat}\Lambda)$ by a language of $\mathbb{A}(\operatorname{lat}\Lambda)$ combinatorially [**I3**].

In general, for treating AR quivers, it is important to consider relationship between (1) representation theoretic conditions for Λ, (2) categorical conditions for $\operatorname{lat}\Lambda$, and (3) combinatorial conditions for $\mathbb{A}(\operatorname{lat}\Lambda)$. We often use 5.3 and 5.4 to study relationship between (2) and (3).

We will treat not only categories of lattices but also τ-categories, and such a general treatment will be necessary in §7. Here, we call a subcategory (5.4) $\mathcal{C}'$ of a τ-category $\mathcal{C}$ **rejective** if the natural inclusion $\mathcal{C}' \to \mathcal{C}$ has a right adjoint $(\ )^- : \mathcal{C} \to \mathcal{C}'$ with a counit ϵ^- such that ϵ_X^- is a monomorphism for any $X \in \mathcal{C}$, and a left adjoint $(\ )^+ : \mathcal{C} \to \mathcal{C}'$ with a unit ϵ^+ such that ϵ_X^+ is an epimorphism for any $X \in \mathcal{C}$. Then we call a subset $\mathfrak{I}(\mathcal{C}) - \mathfrak{I}(\mathcal{C}')$ of $\mathfrak{I}(\mathcal{C})$ **rejectable**. By 1.1, this definition is compatible with that in §1.

It is quite remarkable that *rejectability of a subset S of $\mathfrak{I}(\mathcal{C})$ depends only on (i) the translation quiver obtained by the restriction of $\mathbb{A}(\mathcal{C})$ to S, and (ii) the subsets $S \cap \mathfrak{I}_1^+(\mathcal{C})$ and $S \cap \mathfrak{I}_1^-(\mathcal{C})$ of S.*

6.1 Theorem *Let $\mathcal{C}$ be a τ-category, $\mathcal{C}'$ a coartinian subcategory of $\mathcal{C}$ and $\overline{\mathcal{C}} := \mathcal{C}\,/\,\mathcal{C}'$. Then the following conditions are equivalent.*

(1)(representation theoretic condition) $\mathcal{C}'$ is a rejective subcategory of $\mathcal{C}$.

(2)(categorical condition) μ_X^- is a monomorphism in $\overline{\mathcal{C}}$ for any $X \in \mathfrak{I}(\mathcal{C}) - \mathfrak{I}_1^-(\mathcal{C})$ and μ_X^+ is an epimorphism in $\overline{\mathcal{C}}$ for any $X \in \mathfrak{I}(\mathcal{C}) - \mathfrak{I}_1^+(\mathcal{C})$.

(3)(combinatorial condition) The following (i) and (ii) hold.

(i) For any $X \in \mathfrak{I}(\mathcal{C}) - \mathfrak{I}_1^-(\mathcal{C})$, put $Y_0 := \theta_{\overline{\mathcal{C}}}^- X$, $Y_1 := \theta_{\overline{\mathcal{C}}}^+ \theta_{\overline{\mathcal{C}}}^- X - X$ and $Y_i := \theta_{\overline{\mathcal{C}}}^+ Y_{i-1} - \tau_{\overline{\mathcal{C}}}^+ Y_{i-2}$ for $i \geq 2$. Then $Y_i \geq 0$ holds for any $i \geq 0$.

(ii) For any $X \in \mathfrak{I}(\mathcal{C}) - \mathfrak{I}_1^+(\mathcal{C})$, put $Y_0 := \theta_{\overline{\mathcal{C}}}^+ X$,

$Y_1 := \theta^-_{\mathcal{C}}\theta^+_{\mathcal{C}}X - X$ *and* $Y_i := \theta^-_{\mathcal{C}}Y_{i-1} - \tau^-_{\mathcal{C}}Y_{i-2}$ *for* $i \geq 2$. *Then* $Y_i \geq 0$ *holds for any* $i \geq 0$.

6.2 It is often convenient to use the concept of **trivial** subcategories. A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is called trivial if any proper subcategory $\mathcal{C}''$ of $\mathcal{C}$ containing $\mathcal{C}'$ is not a rejective subcategory of $\mathcal{C}$. We have the following result [**I1**][**I3**].

Theorem *Let* $\mathcal{C}$ *be a* τ*-category,* $\mathcal{C}'$ *a coartinian subcategory of* $\mathcal{C}$ *and* $\overline{\mathcal{C}} := \mathcal{C}\,/\,\mathcal{C}'$. *Then the following conditions are equivalent.*

(1)(representation theoretic condition) $\mathcal{C}'$ *is a trivial subcategory of* $\mathcal{C}$.

(2)(categorical condition) $\overline{\mathcal{C}}(P,I) = 0$ *for any* $P \in \mathfrak{I}^+_1(\mathcal{C})$ *and* $I \in \mathfrak{I}^-_1(\mathcal{C})$.

(3)(combinatorial condition) For any $X \in \mathfrak{I}^-_1(\mathcal{C})$, *put* $Y_0 := X$, $Y_1 := \theta^+_{\mathcal{C}}X$ *and* $Y_i := (\theta^+_{\mathcal{C}}Y_{i-1} - \tau^+_{\mathcal{C}}Y_{i-2})_+$ *for* $i \geq 2$. *Then* $Y_i|_{\mathfrak{I}^+_1(\mathcal{C})} = 0$ *holds for any* $i \geq 0$.

6.3 Example Let Λ be an R-order, $\dim R = 1$ and S a subset of $\mathfrak{I}(\operatorname{lat}\Lambda)$ such that $\#S \leq 4$. Then S is a minimal artinian rejectable subset iff S has one of the following forms, where $S \cap \mathfrak{I}(\operatorname{pr}\Lambda) = \{P\}$ and $S \cap \mathfrak{I}(\operatorname{rin}\Lambda) = \{I\}$.

(1)	$\overset{P=I}{\bullet}$	(DK Rejection)
(2)	$\overset{P}{\bullet}\longrightarrow\overset{I}{\bullet}$	
(3)	$\overset{P}{\bullet}\xrightarrow{(a\ b)}\bullet\longrightarrow\overset{I}{\bullet}$	$ab \leq 2$
(4)	$\overset{P}{\bullet}\longrightarrow\bullet\xrightarrow{(a\ b)}\overset{I}{\bullet}$	$ab \leq 2$
(5)	$\overset{P}{\bullet}\xrightarrow{(a\ b)}\bullet\xrightarrow{(b\ a)}\overset{I}{\bullet}$	$P = \tau^+ I, ab \leq 3$
(6)	$\overset{P}{\bullet}\xrightarrow{(a\ b)}\bullet\longrightarrow\bullet\longrightarrow\overset{I}{\bullet}$	$ab \leq 2$
(7)	$\overset{P}{\bullet}\longrightarrow\bullet\xrightarrow{(a\ b)}\bullet\longrightarrow\overset{I}{\bullet}$	$ab \leq 2$
(8)	$\overset{P}{\bullet}\longrightarrow\bullet\longrightarrow\bullet\xrightarrow{(a\ b)}\overset{I}{\bullet}$	$ab \leq 2$
(9)	$\overset{P}{\bullet}\xrightarrow{(a\ b)}\bullet\xrightarrow{(b\ a)}\overset{X}{\bullet}\longrightarrow\overset{I}{\bullet}$	$P = \tau^+ X, ab \leq 3$

$$(10)\quad \overset{P}{\bullet} \longrightarrow \overset{X}{\bullet} \xrightarrow{(a\ b)} \bullet \xrightarrow{(b\ a)} \overset{I}{\bullet} \qquad X = \tau^+ I,\ ab \leq 3$$

$$(11)\quad \begin{tikzcd} \overset{P}{\bullet} \arrow[r] & \bullet \arrow[r] \arrow[d, shift right] & \overset{I}{\bullet} \\ & X\bullet \arrow[u, shift right] & \end{tikzcd} \qquad P = \tau^+ I,\ X = \tau^+ X$$

$$(12)\quad \begin{tikzcd} \bullet \arrow[r] & \overset{I}{\bullet} \\ P\bullet \arrow[u] \arrow[r] & \bullet \arrow[u] \end{tikzcd} \qquad P = \tau^+ I$$

6.4 Remark Assume $\dim R = 1$. A rejectable subset $S = \mathfrak{I}(\operatorname{lat}\Lambda) - \mathfrak{I}(\operatorname{lat}\Gamma)$ is artinian iff $\#S < \infty$ and Γ is an overorder of Λ. More generally than 6.1, we can obtain a characterization of finite rejectable subsets by summing up with a result in [**I2**], where we gave a characterization of finite rejectable subsets with a form $\mathfrak{I}(\operatorname{lat}\Lambda) - \mathfrak{I}(\operatorname{lat} e\Lambda)$ (e is a central idempotent e of $\widetilde{\Lambda}$) by using additive functions of translation quivers.

7 Orders of finite type

There are many study about orders of finite type, and classifications were given for some classes, for example, local orders (4.5(1)), Bäckström orders [**RR**], tiled orders [**ZK**] and its generalization [**S**]. But, it seems to be very difficult to give a classification without any restriction. In this section, we will give an approach to orders of finite type by using AR quivers, whose viewpoint is slightly different from classification theory. In 7.2, we will give an answer to the following problem (P_1) [**I3**] by progressing philosophy of [**ARo**] and [**A1**] which developped a connection between homological properties and representation theoretic properties.

(P_d) Give a combinatorial characterization of finite translation quivers which are realized as an AR quiver $\mathbb{A}(\operatorname{lat}\Lambda)$ of an order Λ over a complete regular local ring of dimension d.

Although answers to (P_0) and (P_2) are given by [**IT2**] and [**RV**] and partial results for (P_1) are given in [**W**], one shall meet a sheer difficulty to get a general solution for (P_1), which was not existent in other cases. We will overcome the difficulty by applying results for τ-categories in §5 and §6.

7.1 Orderlike τ-categories

7.1.1 Definition Let $\mathcal{C}$ be a τ-category such that $\bigcap_{i\geq 0} \mathcal{J}_{\mathcal{C}}^i = 0$ and $\mathfrak{I}(\mathcal{C}) < \infty$.

(1) For A, $B \in \mathfrak{I}(\mathcal{C})$ and $n \geq 0$, we say that (A, B) is a **Nakayama pair** of distance n if there exists the following commutative diagram which satisfies $a_0 = \mu_A^-$, $a_n = \mu_B^+$, and $(Y_{i-1}] = [X_i) = (X_i \xrightarrow{(-a_i\ g_i)} Y_i \oplus X_{i-1} \xrightarrow{\left(\begin{smallmatrix} f_i \\ a_{i-1}\end{smallmatrix}\right)} Y_{i-1})$ for any i $(1 \leq i \leq n)$.

$$\begin{tikzcd}
Y_0 & Y_1 \arrow[l, "f_1"'] & \cdots \arrow[l, "f_2"'] & Y_{n-1} \arrow[l, "f_{n-1}"'] & B = Y_n \arrow[l, "f_n"'] \\
A = X_0 \arrow[u, "a_0"] & X_1 \arrow[l, "g_1"'] \arrow[u, "a_1"] & \cdots \arrow[l, "g_2"'] & X_{n-1} \arrow[l, "g_{n-1}"'] \arrow[u, "a_{n-1}"] & X_n \arrow[l, "g_n"'] \arrow[u, "a_n"]
\end{tikzcd}$$

In this case, we write $B = n^-(A)$ and $A = n^+(B)$. Put $s^-(A) := \bigcup_{i=0}^n \operatorname{supp} X_i$ and $s^+(B) := \bigcup_{i=0}^n \operatorname{supp} Y_i$.

(2) We call $\mathcal{C}$ **orderlike** if n^- induces a bijection $\mathfrak{I}_1^-(\mathcal{C}) \to \mathfrak{I}_1^+(\mathcal{C})$ and $\mathfrak{I}(\mathcal{C}) = \bigcup_{A\in\mathfrak{I}_1^-(\mathcal{C})} s^-(A)$ holds.

The following gives a reason why we call such (A, B) Nakayama pair, and such $\mathcal{C}$ orderlike.

7.1.2 Example Let Λ be an R-order of finite type, $\dim R = 1$ and $\mathcal{C} := \operatorname{lat}\Lambda$. Then $\mathcal{C}$ is an orderlike τ-category such that $n^+ : \mathfrak{I}_1^+(\mathcal{C}) \to \mathfrak{I}_1^-(\mathcal{C})$ is given by the Nakayama functor $\operatorname{Hom}_\Lambda(\ ,\Lambda)^*$. Take $A \in \mathfrak{I}_1^-(\mathcal{C})$ (resp. $B \in \mathfrak{I}_1^+(\mathcal{C})$) and $X \in \mathfrak{I}(\mathcal{C})$. Then $X \in s^-(A)$ (resp. $X \in s^+(B)$) holds iff an injective hull $X \rightarrowtail I$ (resp. projective cover $P \to X$) of X satisfies $A \leq I$ (resp. $B \leq P$). Moreover, $\mathcal{C}\,/\,\mathfrak{I}_1^-(\mathcal{C})$ and $\mathcal{C}\,/\,\mathfrak{I}_1^+(\mathcal{C})$ are artinian.

7.2 Theorem *Let Q be an admissible translation quiver with a finite number of vertices. Then the following conditions are equivalent.*

(1)(representation theoretic condition) There are a complete discrete valuation ring R and an R-order Λ such that Q is the AR quiver of Λ.

(2)(categorical condition) If a τ-category $\mathcal{C}$ satisfies $\mathbb{A}(\mathcal{C}) = Q$, then $\mathcal{C}$ is orderlike and $\mathcal{C}\,/\,\mathfrak{I}_1^+(\mathcal{C})$ is artinian.

(3)(combinatorial condition) Q is an admissible translation quiver satisfying the following conditions, where we use notations

in 2.3(2)(4).

(i) For any $X \in Q^i$, put $Y_0 := \theta^- X$, $Y_1 := \theta^+\theta^- X - X$ and $Y_i := \theta^+ Y_{i-1} - \tau^+ Y_{i-2}$ for $i \geq 2$, and $t(X) := \bigcup_{i\geq 0} \operatorname{supp} Y_i$. Then there exists $n \geq 0$ such that $Y_i \in \mathbb{N}(Q - Q^p)$ for any i $(0 \leq i < n)$, $Y_n \in Q^p$ and $Y_{n+1} = 0$. Moreover, $Q = \bigcup_{X \in Q^i} t(X)$ holds.

(ii) For any $X \in Q^i$, put $Y_0 := 0$, $Y_1 := X$ and $Y_i := (\theta^+_{Q/Q^p} Y_{i-1} - \tau^+_{Q/Q^p} Y_{i-2})_+$ for $i \geq 2$. Then $Y_i = 0$ holds for sufficiently large i.

(2)$\Leftrightarrow$(3) follows from 5.3, and (1)$\Rightarrow$(2) follows from 7.1.2. Thus we only have to show (2)$\Rightarrow$(1). We can take a τ-species $\mathcal{Q}$ such that $|\,\mathcal{Q}\,| = Q$ since Q is admissible. We only have to show that $\widehat{\mathbb{M}}(\mathcal{Q})$ is equivalent to $\operatorname{lat}\Lambda$ for an order Λ. By 5.5.2, we only have to show that $\widehat{\mathbb{M}}(\mathcal{Q})$ has an order structure. This is the most difficult part, which is essential and peculiar to the case $\dim R = 1$. We can overcome it by the following lemma, which is proved by using a reduction to rejective subcategories as an application of results in §6.

7.2.1 Lemma *Let $\mathcal{C}$ be a completely graded orderlike τ-category, k a field and $k[[x]]$ a formal power series ring. Assume that $\mathcal{C}$ is a k-category, $\mathcal{C}\,/\,\mathcal{J}_{\mathcal{C}}(X,X)$ is a finite field extension of k for any $X \in \mathfrak{I}(\mathcal{C})$ and $\mathcal{C}\,/\,\mathfrak{I}_1^+(\mathcal{C})$ is artinian. Then $\mathcal{C}$ has a $k[[x]]$-order structure.*

7.3 Remark As an application of our characterization, we can give a complete classification of Gorenstein orders of finite type whose tree types are classical Dynkin diagrams. It is given by combining with generalized Rejection in §6, and a remarkable fact is that such an order has an almost Bass overorder canonically.

In general, it is an interesting problem to give an algorithm to recover Λ from $\mathbb{A}(\operatorname{lat}\Lambda)$ using generalized Rejection since mesh categories are not crucial when $\dim R = 1$. Notice that, for $\dim R = 1$, Gabriel quivers (=presentation of rings by quivers with relations) do not seem to play as crucial role as the case in [**BGRS**] ($\dim R = 0$ and R is an algebraically closed field) since one of the most important orders in applications are orders over $R = \mathbb{Z}_p$ and they are not easily presented by quivers.

7.4 Appendix: Artinian τ-categories In this subsection, let Γ be a finite dimensional algebra over a field R and $\mathcal{C} := \operatorname{pr}\Gamma$. Assume that $\mathcal{C}$ is a strict τ-category and we will study relationship between (0) homological conditions for Γ, (1) representation theoretic conditions for $\mathcal{C}$, and (2) categorical conditions for $\mathcal{C}$. Using 5.3, we can easily translate (2) into a combinatorial conditions for $\mathbb{A}(\mathcal{C})$. In particular, the equivalence of (1-4) and (2-4) gives an answer to (P_0) in [**IT2**].

Theorem *Let Γ be a finite dimensional algebra over a field, $0 \to \Gamma \to I^0 \to I^1 \to \cdots$ a minimal injective resolution of a Γ-module Γ, and $\mathcal{C} := \operatorname{pr}\Gamma$. Then the following conditions (0-i), (1-i) and (2-i) are equivalent for each i $(1 \le i \le 4)$.*

(0-i)(homological conditions) $\operatorname{gl.dim}\Gamma \le 2$, *and*

(0-1) Γ and Γ^{op} satisfy the $(2,2)$-condition (in 5.5.1).

(0-2) Γ is Auslander regular.

(0-3) Γ is Auslander regular and I^1 does not have a non-zero projective direct summand.

(0-4) Γ satisfies the $(1,2)$-condition (We call such Γ an **Auslander algebra***).*

(1-i)(representation theoretic conditions)

(1-1): There exists a finite dimensional algebra Λ and a torsion theory $(\mathcal{T},\mathcal{F})$ on $\operatorname{mod}\Lambda$ *such that $\mathcal{C}$ is equivalent to $\mathcal{F}$.*

(1-2): (1-1) and $(\mathcal{T},\mathcal{F})$ is a hereditary torsion theory.

(1-3): (1-1) and $\mathcal{F}$ is the category of Λ-modules whose socles are projective.

(1-4): (1-1) and $\mathcal{F} = \operatorname{mod}\Lambda$.

(2-i)(categorical conditions)

(2-1): $\mathcal{C}$ is a strict τ-category.

(2-2): (2-1) and n^- gives a map $\mathfrak{I}_1^-(\mathcal{C}) - \mathfrak{I}_0^-(\mathcal{C}) \to \mathfrak{I}(\mathcal{C})$.

(2-3): (2-1) and n^- gives a map $\mathfrak{I}_1^-(\mathcal{C}) - \mathfrak{I}_0^-(\mathcal{C}) \to \mathfrak{I}_1^+(\mathcal{C})$.

(2-4): (2-1) and n^- gives a map $\mathfrak{I}_1^-(\mathcal{C}) - \mathfrak{I}_0^-(\mathcal{C}) \to \mathfrak{I}(\mathcal{C}) - \mathfrak{I}_1^+(\mathcal{C})$.

Another answer to (P_0) was given in [**B**] which uses additive functions on translation quivers. It is closely related to non-artinian rejection in 6.4 and results in [**RV**][**RVo**]. It is an open problem to give an answer to (P_1) using additive functions.

8 Orders of tame type In this section, assume $\dim R \leq 1$ unless explicitly stated otherwise. We say that (R, Λ) is in **classical situation** if ($\dim R = 0$ and R is an algebraically closed field) or ($\dim R = 1$, R is the formal power series ring $k[[x]]$ over an algebraically closed field k and the ambient algebra $\widetilde{\Lambda}$ is semisimple). For classical situation, there are well known definitions of **tame type** and **wild type**, and we have **Dichotomy Theorem**, which asserts that *any order is either tame or wild, and not the both* [**DG**][**D**].

Unfortunately, for general situation, no reasonable definitions of tame type and wild type which satisfy Dichotomy Theorem seem to be established. It will be difficult since assumptions for classical situation are essentially used in the proof of Dichotomy Theorem.

In this section, we will not study dichotomy problem. In 8.2.1, we will show that some kinds of orders have a good representation theoretic property, which seems to give a definition of tame type for general situation. We call such orders of **pseudo tame type**, which are defined by using finite dimensional hereditary algebras.

8.1 Definition (1) Let H be a finite dimensional hereditary algebra over a field k [**DR**]. Then

$$q_H(X) := \dim_k \mathrm{End}_H(X) - \dim_k \mathrm{Ext}^1_H(X, X)$$

gives a quadratic form q_H over $\mathbb{Q} \otimes_{\mathbb{Z}} K_0(H)$, where $K_0(H)$ is a Grothendieck group of H. A hereditary algebra H is called **tame** (resp. **wild**) if q_H is positive semidefinite (resp. not positive semidefinite).

(2) Let R be a complete discrete valuation ring, $k := R/J_R$ and Λ an R-order. We say that Λ is of **pseudo tame type** if there exist a countable number of full subcategories $\mathcal{C}_i$ of $\mathrm{lat}\,\Lambda$, tame hereditary k-algebras H_i and full functors $\mathbb{F}_i : \mathcal{C}_i \to \mathrm{mod}\, H_i$ such that $\Im(\mathrm{lat}\,\Lambda) = \bigcup_i \Im(\mathcal{C}_i)$ and ($\mathbb{F}_i(X) = 0$ iff $X = 0$). We say that Λ is of **pseudo wild type** if there exist a full subcategory $\mathcal{C}'$ of $\mathrm{lat}\,\Lambda$, a wild hereditary k-algebra H and a full dense functor $\mathbb{F} : \mathcal{C}' \to \mathrm{mod}\, H$.

Unfortunately, their dichotomy is an open problem.

8.2 Orders of pseudo tame type and almost Bass orders

8.2.1 Theorem *Let Λ be an R-order,* $\dim R = 1$ *and Γ an almost Bass overorder of Λ. Assume that $J_\Lambda = J_\Gamma$ and the injection $\overline{\Lambda} := \Lambda/J_\Lambda \to \overline{\Gamma} := \Gamma/J_\Gamma$ is a normal quadratic extension (in 3.2(2)). Then Λ is of pseudo tame type.*

8.2.2 Definition (1) Let Λ and Γ be R-orders in 8.2.1. Define a graph $G(\Lambda)$ with thin edges and thick edges as follows.

Since $\mathbb{F} := \overline{\Gamma}\otimes_{\overline{\Lambda}} : \operatorname{mod}\overline{\Lambda} \to \operatorname{mod}\overline{\Gamma}$ is a normal quadratic extension, we have a graph $G(\mathbb{F})$ by 3.2(4). For $\mathbb{G} := \overline{\Gamma}\otimes_{\Gamma} : \operatorname{lat}\Gamma \to \operatorname{mod}\overline{\Gamma}$, the induced functor $\operatorname{lat}\Gamma/\ker\mathbb{G} \to \operatorname{mod}\overline{\Gamma}$ is a quadratic extension by 4.4.2. Taking a normalization $\operatorname{lat}\Gamma/\ker\mathbb{G} \xrightarrow{\mathbb{J}} \mathcal{E} \xrightarrow{\mathbb{K}} \operatorname{mod}\overline{\Gamma}$, we obtain a graph $G(\mathbb{J})$ by 3.2(4).

We obtain $G(\Lambda)$ by replacing each edges in the disjoint union of $G(\mathbb{F})$ and $G(\mathbb{J})$ by thick edges, and drawing a thin edge between each vertex $X \in \mathfrak{I}(\mathcal{E})$ (in $G(\mathbb{J})$) and $\mathbb{K}(X) \in \mathfrak{I}(\operatorname{mod}\overline{\Gamma})$ (in $G(\mathbb{F})$).

(2) A **cycle** on $G(\Lambda)$ is an infinite sequence $(\cdots, X_{-1}, X_0, X_1, X_2, \cdots)$ of vertices in $G(\Lambda)$ such that and X_{2i-1} and X_{2i} are connected by a thin edge for any i, X_{2i} and X_{2i+1} are connected by a thick edge for any i, and there exists an even $p > 0$ such that $X_{i+p} = X_p$ for any i.

A **walk** on $G(\Lambda)$ is a sequence $(X_1, X_2, \cdots, X_{2n})$ $(n > 0)$ of vertices in $G(\Lambda)$ such that X_{2i-1} and X_{2i} are connected by a thin edge for any i $(0 < i \leq n)$, and X_{2i} and X_{2i+1} are connected by a thick edge for any i $(0 < i < n)$. Let V be the subset of vertices in $G(\Lambda)$ consisting of $X \in G(\mathbb{J})$ such that there is a thick edge starting from X. A walk is called **good** if neither X_1 nor X_{2n} are not in V.

8.2.3 Theorem *Let Λ be an R-order in 8.2.1. For any cycle and good walk* $\mathbf{W}$ *on $G(\Lambda)$, we can define a full subcategory $\mathcal{N}^{\mathbf{W}}$ of* $\operatorname{lat}\Lambda$ *which satisfies the following (1)–(3).*

(1) $\mathfrak{I}(\operatorname{lat}\Lambda)$ *is a disjoint union of* $\mathfrak{I}(\mathcal{N}^{\mathbf{W}})$ *for any cycle and good walk* $\mathbf{W}$.

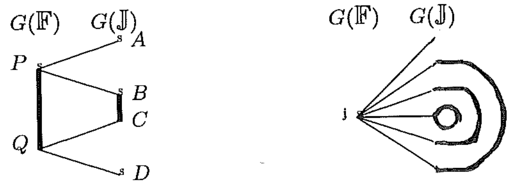
(2) $\#\mathfrak{I}(\mathcal{N}^{\mathbf{W}}) = 1$ *or* 2 *for any good walk* $\mathbf{W}$.

(3) For any cycle $\mathbf{W}$*, there exist a tame hereditary algebra* $H^{\mathbf{W}}$ *and a full functor* $\mathbb{F}^{\mathbf{W}} : \mathcal{N}^{\mathbf{W}} \to \operatorname{mod} H^{\mathbf{W}}$ *such that* ($\mathbb{F}^{\mathbf{W}}(X) = 0$ *iff* $X = 0$).

Notice that 8.2.1 follows from 8.2.3, and we can describe the image of $\mathbb{F}^{\mathbf{W}}$ in 8.2.3(3). As a conclusion, we obtain the following corollary since $\#\mathcal{N}^{\mathbf{W}} = \infty$ holds for any cycle $\mathbf{W}$.

8.2.4 Corollary *Λ in 8.2.1 is of bounded type iff $G(\Lambda)$ does not have a cycle.*

8.2.5 Example



(1) Let Ω be a local maximal order, $\Gamma := \left(\begin{smallmatrix} \Omega & J_\Omega \\ J_\Omega^s & \Omega \end{smallmatrix}\right)$ and $\Lambda := \{\left(\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right) \in \Gamma \mid w - z \in J_\Omega\}$. Then Λ and Γ satisfy the condition in 8.2.1.

Put $s = 1$ for example. Then $G(\Lambda)$ is the left diagram above, and the following list shows all good walks $\mathbf{W}$ and the corresponding element of $\mathfrak{I}(\mathcal{N}^{\mathbf{W}})$, where we put

$$(\Omega{=}\Omega) := \{(x, y) \in \Omega \times \Omega \mid x - y \in J_\Omega\}.$$

$$\begin{aligned}
&\left(\begin{smallmatrix} \Omega & \Omega & \cdots & \Omega & \Omega \\ \Omega & \Omega & \cdots & \Omega & \Omega \end{smallmatrix}\right) && (Q{-}C{=}B{-}P{=}\cdots{=}Q{-}C{=}B{-}P) \\
&\left(\begin{smallmatrix} \Omega & \Omega & \cdots & \Omega & \Omega \\ J_\Omega & \Omega & \cdots & \Omega & \Omega \end{smallmatrix}\right) && (A{-}P{=}Q{-}C{=}B{-}P{=}\cdots{=}Q{-}C{=}B{-}P) \\
&\left(\begin{smallmatrix} \Omega & \Omega & \cdots & \Omega & J_\Omega \\ \Omega & \Omega & \cdots & \Omega & \Omega \end{smallmatrix}\right) && (Q{-}C{=}B{-}P{=}\cdots{=}Q{-}C{=}B{-}P{=}Q{-}D) \\
&\left(\begin{smallmatrix} \Omega & \Omega & \cdots & \Omega & J_\Omega \\ J_\Omega & \Omega & \cdots & \Omega & \Omega \end{smallmatrix}\right) && (A{-}P{=}Q{-}C{=}B{-}P{=}\cdots{=}Q{-}D)
\end{aligned}$$

There is a unique cycle

$$\mathbf{W} = (\cdots B{-}P{=}Q{-}C{=}B{-}P{=}Q{-}C\cdots).$$

For simplicity, assume that $k := \Omega/J_\Omega$ is an algebraically closed field. Then $\mathfrak{I}(\mathcal{N}^{\mathbf{W}}) = \{L_{n,a} \mid n > 0,\ a \in k - \{0\}\}$, where $L_{n,a} := p^{-1}(V_{n,a})$, $p : \mathrm{M}_{2,n}(\Omega) \to \mathrm{M}_{2,n}(k)$ is a natural surjection and

$$V_{n,a} := \{\left(\begin{smallmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ ax_1 & x_1+ax_2 & x_2+ax_3 & \cdots & x_{n-1}+ax_n \end{smallmatrix}\right) \in \mathrm{M}_{2,n}(k) \mid x_i \in k\}.$$

(2) Let Γ be a local Bass order of rank s. Assume that Λ is an order such that $J_\Gamma = J_\Lambda \subset \Lambda \subseteq \Gamma$ and $\Lambda/J_\Lambda \to \Gamma/J_\Gamma$ is of type(I·II), (III), (IV) or (V) (in 3.2(1)). Then Λ and Γ satisfy the condition in 8.2.1. For example, if $s = 3$, then $G(\Lambda)$ is the right diagram above.

8.3 Orders in purely non-semisimple algebras Recall that we call Λ in 8.2.1 a **quadratic order** if Γ is hereditary (3.3). For representation type of orders in purely non-semisimple algebras, we obtain the following simple criterion [**I4**]. Notice that (1) follows from 8.2.1, and the condition in (1) is a generalization of that in 4.3,

8.3.1 Theorem *Let Λ be an R-order in a purely non-semisimple algebra A and* $\dim R = 1$.

(1) If $l_A(Ae) \leq 2$ and $l_A(eA) \leq 2$ for any primitive idempotent e of A and $(\Lambda + J_A)/J_A$ is a quadratic order in A/J_A, then Λ is of pseudo tame type.

(2) If Λ is not of pseudo wild type, then Λ satisfies the conditions in (1).

(3) In particular, Λ is of bounded type iff Λ satisfies the condition in (1) and $G(\Lambda)$ (in 8.2.2) does not have a cycle.

8.4 Some results for $\dim R = 0$ **or** 2 In this section, we will give results for $\dim R = 0$ or 2, which is closely related to 8.2.1. As we will see in 8.5, we can prove them by reducing to same kind of problem though each reductions are rather different. It is remarkable that quadratic orders appear in each results.

8.4.1 Theorem *Let Λ be a quadratic R-order,* $\dim R = 1$ *and I a 2-sided ideal of Λ such that $\widetilde{\Lambda} = \widetilde{I}$. Then the artin algebra Λ/I is of pseudo tame type.*

Notice that algebras above give a generalized class of special biserial algebras and clannish algebras which are shown to be of tame type in [**WW**][**CB2**].

8.4.2 Theorem *Let Λ be an R-order,* $\dim R = 2$, *Γ an overorder of Λ such that* $\mathrm{gl.dim}\,\Gamma = 2$. *Assume that I is a principal ideal of Γ contained in Λ and $S := R/R \cap I$ is a complete discrete valuation ring with a quotient field L. If Γ/I is a quadraic S-order and*

$\Lambda/I \otimes_S L \to \Gamma/I \otimes_S L$ is a quadratic extension (in 3.2(2)), then Λ is of pseudo tame type.

8.5 Sketch of proof In this subsection, we will explain proof of theorems above. First, we reduce each $\operatorname{lat}\Lambda$ to some Drozd's bimodule problem $\operatorname{Mat}(\mathcal{C}, M)$ **[D][CB1]**.

8.5.1 Definition Let $\mathcal{C}$ be an additive category $\mathcal{C}$. For a $(\mathcal{C},\mathcal{C})$-bimodule M (3.1(2)), define an additive category $\operatorname{Mat}(\mathcal{C}, M)$ as follows. An objects of $\operatorname{Mat}(\mathcal{C}, M)$ is a pair Xm consisting of an object $X \in \mathcal{C}$ and an element $m \in M(X, X)$. Put

$$\operatorname{Hom}(Xm, X'm') := \{f \in \mathcal{C}(X, X') \mid mf = fm'\},$$

and the composition is given by that of $\mathcal{C}$.

Using 8.4.1 and 8.2.1, we will explain how to reduce.

For 8.4.1, put $\overline{\Lambda} := \Lambda/I$. Then we have a functor $\operatorname{Mat}(\operatorname{pr}\overline{\Lambda} \times \operatorname{pr}\overline{\Lambda}, \mathcal{J}_{\operatorname{pr}\overline{\Lambda}}) \to \operatorname{mod}\overline{\Lambda}$, $(P_1, P_0)f \mapsto \operatorname{Cok} f$. This functor is full dense since any $L \in \operatorname{mod}\overline{\Lambda}$ has a minimal projective resolution $P_1 \xrightarrow{f} P_0 \to L \to 0$.

For 8.2.1, we use the notation in 8.2.2. By considering a canonical injection $(\overline{\Lambda} \otimes_\Lambda L \to \overline{\Gamma} \otimes_\Gamma (\Gamma L))$ for any $L \in \operatorname{lat}\Lambda$, we obtain a full functor $\operatorname{lat}\Lambda \to \operatorname{Mat}(\operatorname{mod}\overline{\Lambda} \times \operatorname{lat}\Gamma/\ker\mathbb{G}, \operatorname{mod}\overline{\Gamma})$.

For each cases, using 4.4.2 and so on, we can show that thus constructed M is a **quadratic** $(\mathcal{C},\mathcal{C})$-bimodule which we will explain in §9. We will see that $\operatorname{Mat}(\mathcal{C}, M)$ is of pseudo tame type in 9.3. Thus we finish to prove theorems in this section.

9 Quadratic bimodules

In this section, fix a commutative local ring R and put $k := R/J_R$. Any modules and algebras are assumed to be finitely generated R-modules.

9.1 Definition Let $\mathbb{I} = (\mathbb{I}_l\ \mathbb{I}_r) : \mathcal{C} \to \mathcal{C}_l \times \mathcal{C}_r$ be a quadratic extension (resp. normal quadratic extension) (3.2(3)) and M a Nakayama $(\mathcal{C}_l, \mathcal{C}_r)$-bimodule (3.1(2)). Then we call $(M; \mathcal{C}_l, \mathcal{C}_r)$ (or simply M) a **quadratic** (resp. **normal quadratic**) $(\mathcal{C},\mathcal{C})$-bimodule.

For any quadratic $(\mathcal{C},\mathcal{C})$-bimodule M, we can take $\mathcal{C}_l \times \mathcal{C}_r$ such that $(M; \mathcal{C}_l, \mathcal{C}_r)$ is a normal quadratic $(\mathcal{C},\mathcal{C})$-bimodule by taking the normalization (3.2(3)).

9.2 Definition Let $(M; \mathcal{C}_l, \mathcal{C}_r)$ be a normal quadratic $(\mathcal{C}, \mathcal{C})$-bimodule.

(1) Define a graph $G(\mathcal{C}, M)$ with thin edges and thick edges as follows. Let $G(\mathbb{I})$ be the graph of $\mathbb{I} : \mathcal{C} \to \mathcal{C}_l \times \mathcal{C}_r$ defined by 3.2(4). We obtain $G(\mathcal{C}, M)$ by replacing edges in $G(\mathbb{I})$ by thick edges, and drawing $l_{\mathcal{C}_l(X,X)}(M(X,Y))$ thin edges between each $X \in \mathfrak{I}(\mathcal{C}_l)$ and $Y \in \mathfrak{I}(\mathcal{C}_r)$.

(2) We define a **cycle** and a **walk** on $G(\mathcal{C}, M)$ by a same way as in 8.2.2(2). Notice that walks and cycles can be regarded as a generalization of Green walks [**G**] and strings [**WW**].

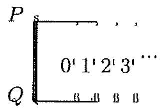
9.3 Theorem *Let $(M; \mathcal{C}_l, \mathcal{C}_r)$ be a normalized quadratic $(\mathcal{C}, \mathcal{C})$-bimodule. For any cycle and walk* $\mathbf{W}$ *on $G(\mathcal{C}, M)$, we can define a full subcategory $\mathcal{N}^{\mathbf{W}}$ of* $\mathrm{Mat}(\mathcal{C}, M)$ *which satisfies the following (1)–(3).*

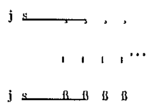
(1) $\mathfrak{I}(\mathrm{Mat}(\mathcal{C}, M)) - \mathfrak{I}(\mathrm{Mat}(\mathcal{C}, 0))$ *is a disjoint union of* $\mathfrak{I}(\mathcal{N}^{\mathbf{W}})$ *for any cycle and walk* $\mathbf{W}$.

(2) $\#\mathfrak{I}(\mathcal{N}^{\mathbf{W}}) = 1$ *or* 2 *for any walk* $\mathbf{W}$.

(3) For any cycle $\mathbf{W}$*, there exist a tame hereditary algebra $H^{\mathbf{W}}$ and a full functor* $\mathbb{F}^{\mathbf{W}} : \mathcal{N}^{\mathbf{W}} \to \mathrm{mod}\, H^{\mathbf{W}}$ *such that ($\mathbb{F}^{\mathbf{W}}(X) = 0$ iff $X = 0$).*

9.4 Example Let R be a complete discrete valuation ring and Γ a local maximal R-order.





(1) Put $\Lambda := \{(x, y) \in \Gamma \times \Gamma \mid x - y \in J_\Gamma\}$. Then $(\mathrm{pr}\,\Gamma; \mathrm{pr}\,\Gamma, \mathrm{pr}\,\Gamma)$ is a quadratic $(\mathrm{pr}\,\Lambda, \mathrm{pr}\,\Lambda)$-bimodule with the left graph above. The set of isomorphism classes of objects in $\mathrm{Mat}(\mathrm{pr}\,\Lambda, \mathrm{pr}\,\Gamma)$ is given by $\bigcup_{n \geq 0} \mathrm{Gl}_n(\Lambda) \backslash \mathrm{M}_n(\Gamma)$, where $\mathrm{Gl}_n(\Lambda)$ acts $\mathrm{M}_n(\Gamma)$ by $(x, y) \cdot m := xmy^{-1}$.

For simplicity, assume that $\Gamma = R = k[[x]]$ and k is an algebraically closed field. Then $\mathfrak{I}(\mathcal{N}^{\mathbf{W}})$ consists of the following left matrix for any walk $\mathbf{W} = (P \overset{l_1}{—} Q \textbf{—} P \overset{l_2}{—} Q \textbf{—} \cdots \textbf{—} P \overset{l_{n-1}}{—} Q)$, and $\mathfrak{I}(\mathcal{N}^{\mathbf{W}})$ consists of the following right matrix for any cycle

$\mathbf{W} = (\cdots P \overset{l_1}{-} Q = P \overset{l_2}{-} Q = \cdots = P \overset{l_n}{-} Q = P \overset{l_1}{-} Q \cdots)$, where $T \in \mathrm{M}_m(k) \subseteq \mathrm{M}_m(\Gamma)$ is a Jordan block of non-zero eigenvalue.

$$\begin{pmatrix} 0 & x^{l_1} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & x^{l_2} & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & x^{l_{n-2}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & x^{l_{n-1}} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} O & x^{l_1}I & O & \cdots & O & O & O \\ O & O & x^{l_2}I & \ddots & O & O & O \\ O & O & O & \ddots & O & O & O \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ O & O & O & \cdots & O & x^{l_{n-2}}I & O \\ O & O & O & \cdots & O & O & x^{l_{n-1}}I \\ x^{l_n}T & O & O & \cdots & O & O & O \end{pmatrix}$$

(2) Let $\Gamma \supset \Lambda_i \supset J_\Gamma$ be a local Bass order of rank 1 for $i = 1, 2$ and put $\mathcal{C} := \operatorname{pr}\Lambda_1 \times \operatorname{pr}\Lambda_2$. Then $(\operatorname{pr}\Gamma; \operatorname{pr}\Gamma, \operatorname{pr}\Gamma)$ is a quadratic $(\mathcal{C}, \mathcal{C})$-bimodule with the right graph above. The set of isomorphism classes of objects in $\mathrm{Mat}(\mathcal{C}, \operatorname{pr}\Gamma)$ is given by $\bigcup_{n\geq 0} \mathrm{Gl}_n(\Lambda_1) \backslash \mathrm{M}_n(\Gamma) / \mathrm{Gl}_n(\Lambda_2)$.

9.5 Reduction theorem In the rest, put

$$\underline{\mathrm{Mat}}(\mathcal{C}, M) := \mathrm{Mat}(\mathcal{C}, M) / \mathcal{J}_{\mathcal{C}}$$

for a $(\mathcal{C}, \mathcal{C})$-bimodule M. For an algebra H, we denote by $\mathrm{mod}_0\, H$ the category of finitely generated H-modules without simple direct summand. The following reduction is fundamental for quadratic bimodules.

9.5.1 (Quadratic reduction) *Let $(M; \mathcal{C}_l, \mathcal{C}_r)$ be a normal quadratic $(\mathcal{C}, \mathcal{C})$-bimodule and M' a maximal sub $(\mathcal{C}_l, \mathcal{C}_r)$-bimodule of M. Then the following (1) and (2) hold.*

(1) There exist an additive category $\mathcal{C}'$, a quadratic extension $\mathbb{I}' : \mathcal{C}' \to \mathcal{C}_l \times \mathcal{C}_r$ and a full dense functor (called **reduction functor***) $\mathbb{R} : \underline{\mathrm{Mat}}(\mathcal{C}, M) \to \underline{\mathrm{Mat}}(\mathcal{C}', M')$.*

(2) Define a full subcategory $\mathcal{M}$ of $\mathrm{Mat}(\mathcal{C}, M)$ by $\mathfrak{I}(\mathcal{M}) = \{X \in \mathfrak{I}(\mathrm{Mat}(\mathcal{C}, M)) - \mathfrak{I}(\mathrm{Mat}(\mathcal{C}, 0)) \mid \mathbb{R}(X) \in \mathfrak{I}(\mathrm{Mat}(\mathcal{C}', 0))\}$. Then there exist a tame hereditary algebra H and a full dense functor $\mathbb{R}' : \mathcal{M} \to \mathrm{mod}_0\, H$ such that ($\mathbb{R}'(X) = 0$ iff $X = 0$).

As a conclusion, we can reduce a quadratic $(\mathcal{C}, \mathcal{C})$-bimodule M to a quadratic $(\mathcal{C}', \mathcal{C}')$-bimodule M'. For its proof, we use a computation of certain functors for tame hereditary algebras.

9.5.2 Example Using 9.4(1), we will explain the reduction functor. For simplicity, assume that $\Gamma = R = k[[x]]$ and k is an algebraically closed field.

Take $m \in \mathrm{M}_n(\Gamma)$ and we will explain how to define $\mathbb{R}(m)$. Considering the Jordan normal form of $\overline{m} \in \mathrm{M}_n(k)$, we obtain $x \in \mathrm{Gl}_n(\Lambda)$ such that $xm = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$ satisfies that $\overline{m_2} = 0 = \overline{m_3}$, $\overline{m_1}$ is a totality of Jordan blocks of eigenvalue 0, and $\overline{m_4}$ is a totality of Jordan blocks of non-zero eigenvalue. It is easily shown that we may assume $m_2 = 0 = m_3$ by taking proper x.

For simplicity, assume $\overline{m_1} = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \end{pmatrix}$. Then it is easily shown that there exists $x' \in \mathrm{Gl}_n(\Lambda)$ such that $x'm_1 = \begin{pmatrix} a_{11}&a_{12}&0&a_{13}&0&0 \\ 0&0&1&0&0&0 \\ a_{21}&a_{22}&0&a_{23}&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ a_{31}&a_{32}&0&a_{33}&0&0 \end{pmatrix}$, where each a_{ij} is an element of J_Γ. Then we put $\mathbb{R}(m) := (a_{ij})_{1 \le i,j \le 3} \in \mathrm{M}_3(J_\Gamma)$.

Take another $l \in \mathrm{M}_n(\Gamma)$. It can be shown that $\mathrm{Gl}_n(\Lambda)m = \mathrm{Gl}_n(\Lambda)l$ holds if and only if $\overline{m_1}$ is conjugate to $\overline{l_1}$ and $(\Lambda')^\times \mathbb{R}(m) = (\Lambda')^\times \mathbb{R}(l)$ holds, where

$$\Lambda' := \{(x,y) \in \begin{pmatrix} \Gamma & \Gamma & \Gamma \\ J_\Gamma & \Gamma & \Gamma \\ J_\Gamma & J_\Gamma & \Gamma \end{pmatrix} \times \begin{pmatrix} \Gamma & J_\Gamma & J_\Gamma \\ \Gamma & \Gamma & J_\Gamma \\ \Gamma & \Gamma & \Gamma \end{pmatrix} \mid x - y \in \begin{pmatrix} J_\Gamma & \Gamma & \Gamma \\ \Gamma & J_\Gamma & \Gamma \\ \Gamma & \Gamma & J_\Gamma \end{pmatrix}\}.$$

We can apply the argument above for other m with different $\overline{m_1}$. Thus $\mathbb{R}$ gives a bijection from $\mathfrak{I}(\bigcup_{n \ge 0} \mathrm{Gl}_n(\Lambda) \backslash \mathrm{M}_n(\Gamma))$ to a disjoint union of (Jordan blocks of non-zero eigenvalue) and $\mathfrak{I}((\Lambda')^\times \backslash \mathrm{M}_3(\Gamma))$ etc.

9.5.3 Remark In general, for a $(\mathcal{C},\mathcal{C})$-bimodule M and a derivation $d : \mathcal{C} \to M$, an additive category $\mathrm{Mat}(\mathcal{C}, M; d)$ is defined [**CB1**]. For them, the following reduction is known, where $d = 0$ does not imply $d' = 0$. Thus, in study of a general $(\mathcal{C},\mathcal{C})$-bimodule M, it is more natural to consider a derivation $d : \mathcal{C} \to M$ since such derivation will appear in a process of reduction. Although 9.5.1 is in the same philosophy, it is quite peculiar phenomenon that we can exclude derivations for quadratic reduction.

Lemma Let M be a $(\mathcal{C},\mathcal{C})$-bimodule, M' a sub $(\mathcal{C},\mathcal{C})$-bimodule and $d : \mathcal{C} \to M$ a derivation. Putting $\mathcal{C}' := \mathrm{Mat}(\mathcal{C}, M/M'; \overline{d})$, we obtain a derivation $d' : \mathcal{C}' \to M'$ and an equivalence $\underline{\mathrm{Mat}}(\mathcal{C}, M; d) \to \underline{\mathrm{Mat}}(\mathcal{C}', M'; d')$.

9.6 Sketch of proof of 9.3 For any cycle and walk $\mathbf{W}$, we can obtain a **reduction sequence**, which is a "successive sequence $\Phi = (\mathbb{R}^{(0)}, \mathbb{R}^{(1)}, \cdots, \mathbb{R}^{(n-1)})$ of reduction functors (in 9.5.1)". We denote by $\mathrm{Red}(\mathcal{C}, M)$ the set of reduction sequences. Then, for any $\Phi \in \mathrm{Red}(\mathcal{C}, M)$, we have a quadratic $(\mathcal{C}^{(j)}, \mathcal{C}^{(j)})$-bimodule $M^{(j)}$ and the following diagram.

$$\begin{tikzcd}
\underline{\mathrm{Mat}}(\mathcal{C}, M) & & \underline{\mathrm{Mat}}(\mathcal{C}^{(1)}, M^{(1)}) & \cdots & \underline{\mathrm{Mat}}(\mathcal{C}^{(n-1)}, M^{(n-1)}) & & \underline{\mathrm{Mat}}(\mathcal{C}^{(n)}, M^{(n)}) \\
\bigcup & & \bigcup & \cdots & \bigcup & & \bigcup \\
\underline{\mathrm{Mat}}(\mathcal{C}^{(0)}, N^{(1)}) \arrow[uurr, "\mathbb{R}^{(0)}"] & & \underline{\mathrm{Mat}}(\mathcal{C}^{(1)}, N^{(2)}) & \cdots & \underline{\mathrm{Mat}}(\mathcal{C}^{(n-1)}, N^{(n)}) \arrow[uurr, "\mathbb{R}^{(n-1)}"] & & \underline{\mathrm{Mat}}(\mathcal{C}^{(n)}, 0)
\end{tikzcd}$$

Let $\mathcal{M}^{\Phi}$ be the full subcategory of $\underline{\mathrm{Mat}}(\mathcal{C}, M)$ such that $\mathfrak{I}(\mathcal{M}^{\Phi})$ consists of $X \in \mathfrak{I}(\underline{\mathrm{Mat}}(\mathcal{C}, M))$ which satisfies

$$\mathbb{R}^{(n-1)} \circ \cdots \circ \mathbb{R}^{(0)}(X) \in \underline{\mathrm{Mat}}(\mathcal{C}^{(n)}, 0)$$

and

$$\mathbb{R}^{(n-2)} \circ \cdots \circ \mathbb{R}^{(0)}(X) \notin \underline{\mathrm{Mat}}(\mathcal{C}^{(n-1)}, 0).$$

By 9.5.1, we obtain the following lemma, which is crucial to prove 9.3.

Lemma *(1) For any $X \in \mathfrak{I}(\underline{\mathrm{Mat}}(\mathcal{C}, M))$, there exists $\Phi \in \mathrm{Red}(\mathcal{C}, M)$ such that $X \in \mathfrak{I}(\mathcal{M}^{\Phi})$.*

(2) For any $\Phi \in \mathrm{Red}(\mathcal{C}, M)$, there exist a tame hereditary algebra H^{Φ} and a full dense functor $\mathbb{R}^{\Phi} : \mathcal{M}^{\Phi} \to \mathrm{mod}_0\, H^{\Phi}$ such that ($\mathbb{R}^{\Phi}(X) = 0$ iff $X = 0$).

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