Compactly supported cohomology by the L^2 method

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It is known that, for any complex manifold M and for any holomorphic vector bundle E over M, $H_c^{0,1}(M,E)$, the compactly supported E-valued $\overline{\partial}$-cohomology group of type $(0,1)$, is Hausdorff. This is an immediate consequence of a combination of Siu's generalization of Malgrange's vanishing theorem and Serre duality. Similarly, $H_c^{0,n-q+1}(M,E)$ are Hausdorff if M is a connected n-dimensional q-convex manifold. In two papers written in 1980, I refined Andreotti-Grauert's finiteness theorem by extending Andreotti-Vesentini's L^2 theory and showed that the Hodge theory can be extended to a class of complete Kähler manifolds which are also q-convex. The result was about the de Rham cohomology of degree at least $n+q$, and applied recently to the study of Levi flat hypersurfaces. By the L^2 method, it will be shown that, for any n-dimensional q-convex bounded domain D with smooth boundary in a complex manifold M and for any holomorphic Hermitian vector bundle E over M, there exists a complete Hermitian metric say ω on D such that the natural homomorphism from $H_c^{0,p}(D,E)$ to the space of E-valued L^2 harmonic $(0,p)$-forms on D with respect to ω is bijective if $p < n-q+1$ and injective if $p = n-q+1$. As a result, besides the Hausdorffness, one can see a natural Hodge-type filtration in $H_c^1(M,C)$, if M is pseudoconvex and Kählerian.