微分積分学II 期末試験回答

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問題 1　配点 30

(1) $$\iint_D \frac{x}{\cos^2 xy} dxdy = \int_0^{1/4} dx \int_0^{\pi} \frac{x}{\cos^2 xy} dy = \int_0^{1/4} [\tan xy]_0^{\pi}\, dx = \int_0^{1/4} \tan \pi x dx$$
$$= \int_0^{1/4} \frac{\sin \pi x}{\cos \pi x} dx = \frac{1}{\pi} [-\log(\cos \pi x)]_0^{1/4} = \frac{1}{\pi}(-\log \frac{\sqrt{2}}{2}) = \frac{\log 2}{2\pi}$$

(2) $$\iint_D \sqrt{x^2+y}\ dxdy = \int_{-\sqrt{2}}^{\sqrt{2}} dx \int_{x^2}^{4-x^2} \sqrt{x^2+y}\ dy = 2\int_0^{\sqrt{2}} dx \left[\frac{2}{3}(x^2+y)^{3/2}\right]_{y=x^2}^{y=4-x^2}$$
$$= \frac{4}{3}\int_0^{\sqrt{2}} 4^{3/2} - (2x^2)^{3/2} dx = \frac{4}{3}\left(8\sqrt{2} - \left[2\sqrt{2}\frac{x^4}{4}\right]_0^{\sqrt{2}}\right) = \frac{4}{3}6\sqrt{2} = 8\sqrt{2}$$

注：分割せずに $\int_{-\sqrt{2}}^{\sqrt{2}}$ のままで計算すると，x^4 の寄与分が消えて，$32\sqrt{2}/3$ という間違った結果になるが，8 点を与えた．

(3) $$\iint_D x^2 dxdy = \int_{-1}^{1} x^2 dx \int_{|x|-1}^{1-|x|} dy = 4\int_0^1 x^2 dx \int_0^{1-x} dy = 4\int_0^1 x^2(1-x)dx$$
$$= 4\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{4}{3} - 1 = \frac{1}{3}$$

問題 2　配点 20

(1) $$S = \int_0^{\pi} d\theta \int_0^{a\theta} rdr = \int_0^{\pi} \left[\frac{r^2}{2}\right]_0^{a\theta} d\theta = \int_0^{\pi} \frac{a^2}{2}\theta^2 d\theta = \frac{a^2}{6}\pi^3$$

(2) $$S = \int_0^{2\pi} d\theta \int_0^{2+\cos 4\theta} rdr = \frac{1}{2}\int_0^{2\pi} (2+\cos 4\theta)^2 d\theta = \frac{1}{2}\int_0^{2\pi} 4 + 4\cos 4\theta + \cos^2 4\theta d\theta$$
$$= \frac{1}{2}\left(8\pi + [\sin 4\theta]_0^{2\pi} + \pi\right) = \frac{9}{2}\pi$$

問題 3　配点 20

(1) $$u = x - 2y,\ v = 2x + y,\ x = \frac{u+2v}{5},\ y = \frac{v-2u}{5}$$
$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{pmatrix} = \frac{1}{5}$$
$$\iint_D (x-2y)\cos(2x+y)\ dxdy = \frac{1}{5}\int_0^2 udu \int_0^{\pi/2} \cos vdv = \frac{1}{5}\left[\frac{u^2}{2}\right]_0^2 [\sin v]_0^{\pi/2} = \frac{2}{5}$$

(2) $$\iint_D x^2 e^{x^2+y^2} dxdy = \int_0^{2\pi} d\theta \int_0^1 (r\cos\theta)^2 e^{r^2} rdr = \int_0^{2\pi} \cos^2\theta d\theta \int_0^1 r^3 e^{r^2} dr$$
$$= \pi\int_0^1 \frac{r^2}{2} 2re^{r^2} dr = \pi\left(\left[\frac{r^2}{2}e^{r^2}\right]_0^1 - \int_0^1 re^{r^2} dr\right) = \pi\left(\frac{e}{2} - \left[\frac{1}{2}e^{r^2}\right]_0^1\right) = \frac{\pi}{2}$$

問題 4　配点 15

$$\frac{dx}{d\theta} = -\sin\theta \quad \frac{dy}{d\theta} = \cos\theta$$

$$\int_C ydx + 2xdy = \int_0^{2\pi} \sin\theta\frac{dx}{d\theta} + 2\cos\theta\frac{dy}{d\theta}d\theta = \int_0^{2\pi} -\sin^2\theta + 2\cos^2\theta d\theta = -\pi + 2\pi = \pi$$

問題 5　配点 20

(1) $$V = \iint_{x^2+y^2\leq 4} dxdy \int_0^{4-x^2-y^2} dz = \int_0^{2\pi} d\theta \int_0^2 rdr \int_0^{4-r^2} dz$$

$$= 2\pi \int_0^2 r(4-r^2)dr = 2\pi \left[2r^2 - \frac{r^4}{4}\right]_0^2 = 8\pi$$

(2) $$S = \iint_{x^2+y^2\leq 4} \sqrt{z_x^2 + z_y^2 + 1} + 1\, dxdy$$ 　$+1$ は底の面積

$$= \iint_{x^2+y^2\leq 4} \sqrt{4x^2+4y^2+1} + 1\, dxdy = 2\pi \int_0^2 (\sqrt{4r^2+1}+1)rdr$$

$$= 2\pi \left[\frac{1}{12}(4r^2+1)^{3/2} + \frac{r^2}{2}\right]_0^2 = 2\pi(\frac{1}{12}(17^{3/2}-1)+2)$$

$$= \frac{\pi}{6}(17^{3/2}-1) + 4\pi = \frac{\pi}{6}(17^{3/2}+23)$$

この形では $r = \sqrt{4-z}$ なので，円錐形と書いたのは間違いである．よって，高さ 4・半径 2 の円錐形に関する式で計算した人も満点とした．また，底の面積を忘れても減点しない．

問題 6　配点 20

(1) $$V = \iint_{3\leq x^2+y^2\leq 4} dxdy \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} dz = \int_0^{2\pi} d\theta \int_{\sqrt{3}}^2 rdr \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dz$$

$$= 2\pi \int_{\sqrt{3}}^2 (2\sqrt{4-r^2})rdr = 2\pi \left[-\frac{2}{3}(4-r^2)^{3/2}\right]_{\sqrt{3}}^2 = \frac{4\pi}{3}$$

(2) 図形が z 軸の回りに弧と直線を回したとにできるので，その両方を足す.

$$-1 \leq z \leq 1,\ f(z) = \sqrt{4-z^2},\ f'(z) = \frac{z}{\sqrt{4-z^2}},\ g(z) = \sqrt{3}$$

$$S = 2\pi \int_{-1}^1 f(z)\sqrt{1+f'(z)^2} + \sqrt{3}\ dz = 2\pi \int_{-1}^1 \sqrt{4-z^2}\sqrt{1+\frac{z^2}{4-z^2}}\ dz + 4\sqrt{3}\pi$$

$$= 2\pi \int_{-1}^1 \sqrt{4-z^2+z^2}\ dz + 4\sqrt{3}\pi = 8\pi + 4\sqrt{3}\pi = 4(2+\sqrt{3})\pi$$

問題 7　配点 15

教科書 119 ページを参照.

合格基準　以下の式に基いて 50 点以上を得たものを合格とする.

$$\frac{\text{中間試験成績} + \text{期末試験成績}}{2} + \text{レポート提出回数} \times 3 \geq 50$$

どちらかの試験を受けていなければ欠席となる.