練習問題 1.3.3, 2.1.1 の解答

Jacques Garrigue, 2008 年 6 月 3 日

問題 1.3.3　つぎの極限値を求めよ．

(1) $\displaystyle\lim_{x\to 0}(1+ax)^{1/x}$　　(2) $\displaystyle\lim_{x\to 0}\frac{e^x-e^{-x}}{x}$　　(3) $\displaystyle\lim_{x\to 1}x^{\frac{1}{1-x}}$

解答　定理 1.3.2 と例題 1.3.2 で得られた結果を元に，変数代入で解く．

(1)　$t=ax$ とおき，$\lim_{t\to 0}(1+t)^{1/t}=e$ を利用する．

$$\lim_{x\to 0}(1+ax)^{1/x}=\lim_{t\to 0}(1+t)^{a/t}=\lim_{t\to 0}\left((1+t)^{1/t}\right)^a=e^a$$

最後に，x^a の e での連続性を使う．

(2)　$t=-x$ とおき，$\lim_{x\to 0}\frac{e^x-1}{x}=1$ を利用する．

$$\lim_{x\to 0}\frac{e^x-e^{-x}}{x}=\lim_{x\to 0}\frac{e^x-1}{x}+\frac{e^{-x}-1}{-x}=\lim_{x\to 0}\frac{e^x-1}{x}+\lim_{t\to 0}\frac{e^t-1}{t}=2$$

(3)　$t=1-x$ とおき，問 (1) の $\lim_{t\to 0}(1+(-1)x)^{1/x}=e^{-1}$ を利用する．

$$\lim_{x\to 1}x^{\frac{1}{1-x}}=\lim_{t\to 0}(1-t)^{1/t}=e^{-1}=\frac{1}{e}$$

問題 2.1.1

(1)
$$\begin{aligned}\frac{d}{dx}(x^2+1)^5(x^3-2)^3 &= (\frac{d}{dx}(x^2+1)^5)(x^3-2)^3+(x^2+1)^5\frac{d}{dx}(x^3-2)^3\\ &=(\frac{d}{dx}x^2+1)\cdot 5(x^2+1)^4(x^3-2)^3+(x^2+1)^5(\frac{d}{dx}x^3-2)\cdot 3(x^3-2)^2\\ &=2x\cdot 5(x^2+1)^4(x^3-2)^3+(x^2+1)^5\cdot 3x^2\cdot 3(x^3-2)^2\\ &=x(x^2+1)^4(x^3-2)^2\left\{10(x^3-2)+9(x^2+1)x\right\}\\ &=x(x^2+1)^4(x^3-2)^2(19x^3+9x-20)\end{aligned}$$

(2)
$$\frac{d}{dx}\log(\log x)=(\frac{d}{dx}\log x)\frac{1}{\log x}=\frac{1}{x\log x}$$

(3)
$$\frac{d}{dx}2^x=\frac{d}{dx}e^{x\log 2}=2^x\log 2$$

(4)
$$\begin{aligned}\frac{d}{dx}x^3(x^2+1)^{3/2} &= 3x^2(x^2+1)^{3/2}+x^3\frac{d}{dx}(x^2+1)^{3/2}\\ &=3x^2(x^2+1)^{3/2}+x^3(\frac{d}{dx}x^2+1)\frac{3}{2}(x^2+1)^{1/2}\\ &=3x^2(x^2+1)^{3/2}+x^3\cdot 2x\frac{3}{2}\sqrt{x^2+1}\\ &=x^2\sqrt{x^2+1}(3x^2+3+3x^2)=3x^2\sqrt{x^2+1}(2x^2+1)\end{aligned}$$

(5)
$$\frac{d}{dx}e^{(x^x)}=(\frac{d}{dx}x^x)e^{(x^x)}=(\frac{d}{dx}e^{x\log x})e^{(x^x)}=(\frac{d}{dx}x\log x)x^xe^{x^x}=(1+\log x)x^xe^{x^x}$$

(6)
$$\begin{aligned}\frac{d}{dx}(\sin x)^{\cos x} &= \frac{d}{dx}e^{\cos x\log(\sin x)}=(\frac{d}{dx}\cos x\log(\sin x))e^{\cos x\log(\sin x)}\\ &=(-\sin x\log(\sin x)+\frac{\cos^2 x}{\sin x})(\sin x)^{\cos x}\end{aligned}$$

(7)
$$\frac{d}{dx}\mathrm{Sin}^{-1}(x^3+1)=(\frac{d}{dx}x^3+1)\frac{1}{\sqrt{1-(x^3+1)^2}}=\frac{3x^2}{\sqrt{-x^3(x^3+2)}}=\frac{-3x}{\sqrt{-x(x^3+2)}}$$
x が負なので

(8)
$$\begin{aligned}\frac{d}{dx}\mathrm{Tan}^{-1}\frac{1-x^2}{1+x^2} &= (\frac{d}{dx}\frac{1-x^2}{1+x^2})(1+(\frac{1-x^2}{1+x^2})^2)^{-1}=\frac{-2x(1+x^2)-(1-x^2)2x}{(1+x^2)^2}(1+(\frac{1-x^2}{1+x^2})^2)^{-1}\\ &=\frac{-4x}{(1+x^2)^2}(1+(\frac{1-x^2}{1+x^2})^2)^{-1}=\frac{-4x}{(1+x^2)^2}\frac{(1+x^2)^2}{(1+x^2)^2+(1-x^2)^2}=\frac{-2x}{1+x^4}\end{aligned}$$

(9) $\dfrac{d}{dx}\sqrt{1+2\log x} = (\dfrac{d}{dx}1+2\log x)\dfrac{1}{2\sqrt{1+2\log x}} = \dfrac{1}{x\sqrt{1+2\log x}}$

(10) $\dfrac{d}{dx}\exp(\mathrm{Tan}^{-1}x) = \dfrac{\exp(\mathrm{Tan}^{-1}x)}{1+x^2}$

(11) $\dfrac{d}{dx}x\sqrt{a^2-x^2}+a^2\mathrm{Sin}^{-1}\dfrac{x}{a} = \sqrt{a^2-x^2}+\dfrac{x(\frac{d}{dx}a^2-x^2)}{2\sqrt{a^2-x^2}}+\dfrac{a^2}{a}\dfrac{1}{\sqrt{1-(x/a)^2}}$

$= \sqrt{a^2-x^2}-\dfrac{x^2}{\sqrt{a^2-x^2}}+\dfrac{a^2}{\sqrt{a^2-x^2}} = 2\sqrt{a^2-x^2}$

(12) $\dfrac{d}{dx}\mathrm{Sin}^{-1}\dfrac{x}{\sqrt{1+x^2}} = (\dfrac{d}{dx}\dfrac{x}{\sqrt{1+x^2}})\dfrac{1}{\sqrt{1-(x/\sqrt{1+x^2})^2}} = \dfrac{\sqrt{1+x^2}-x\frac{2x}{2\sqrt{1+x^2}}}{1+x^2}\sqrt{1+x^2} = \dfrac{1}{1+x^2}$

(13) $\dfrac{d}{dx}2\mathrm{Cos}^{-1}\sqrt{\dfrac{x+1}{2}} = -2(\dfrac{d}{dx}\sqrt{\dfrac{x+1}{2}})\dfrac{1}{\sqrt{1-\frac{x+1}{2}}} = -\sqrt{2}\dfrac{1}{2\sqrt{x+1}}\sqrt{\dfrac{2}{1-x}} = \dfrac{-1}{\sqrt{1-x^2}}$

対数微分法　積や指数の多い関数を微分するときに便利な方法

$$\frac{d}{dx}\log f(x) = \frac{f'(x)}{f(x)} \qquad \text{なので} \qquad f'(x) = f(x)\frac{d}{dx}\log f(x)$$

(14) $\dfrac{d}{dx}\log\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)}} = \dfrac{1}{2}\dfrac{d}{dx}(\log(x-1)+\log(x-2)-\log(x-3)-\log(x-4))$

$\dfrac{d}{dx}\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)}} = \sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)}}\dfrac{1}{2}\left(\dfrac{1}{x-1}+\dfrac{1}{x-2}-\dfrac{1}{x-3}-\dfrac{1}{x-4}\right)$

(15) $\dfrac{d}{dx}\log\sqrt[3]{\dfrac{x^2+1}{(x-1)^2}} = \dfrac{1}{3}\left(\dfrac{2x}{x^2+1}-\dfrac{2}{x-1}\right)$

$\dfrac{d}{dx}\sqrt[3]{\dfrac{x^2+1}{(x-1)^2}} = \dfrac{2}{3}\sqrt[3]{\dfrac{x^2+1}{(x-1)^2}}\left(\dfrac{x}{x^2+1}-\dfrac{1}{x-1}\right)$

(16) $\dfrac{d}{dx}\sinh x = \dfrac{d}{dx}\dfrac{e^x-e^{-x}}{2} = \dfrac{e^x+e^{-x}}{2} = \cosh x$　　定義はページ 19 参照

(17) $\dfrac{d}{dx}\cosh x = \dfrac{d}{dx}\dfrac{e^x+e^{-x}}{2} = \dfrac{e^x-e^{-x}}{2} = \sinh x$

(18) $\dfrac{d}{dx}\tanh x = \dfrac{d}{dx}\dfrac{\sinh x}{\cosh x} = \dfrac{\cosh^2 x-\sinh^2 x}{\cosh^2 x} = \dfrac{e^{2x}+2+e^{-2x}-e^{2x}+2-e^{-2x}}{4\cosh^2 x} = \dfrac{1}{\cosh^2 x}$

問題 2.1.4　略解参照