問題 5.4.1,4 の解答

Jacques Garrigue, 2009 年 1 月 26 日

問題 5.4.1

つぎの図形の体積を求めよ．

(1) $V : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \le 1 \quad x = at,\ y = bu,\ z = cv \quad W : t^2 + u^2 + v^2 \le 1$

$$\frac{\partial(x,y,z)}{\partial(t,u,v)} = \det\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

$$\iiint_V dxdydx = \iiint_W \left|\frac{\partial(x,y,z)}{\partial(t,u,v)}\right| dtdudv = \iiint_W abc\, dtdudv = \frac{4\pi}{3}abc$$

(2) $V : x^{2/3} + y^{2/3} + z^{2/3} \le a^{2/3} \quad (x,y,z \ge 0)$

$x = t^3,\ y = u^3,\ z = v^3 \quad V' : t^2 + u^2 + v^2 < a^{2/3} \quad (t,u,v \ge 0)$

$$\frac{\partial(x,y,z)}{\partial(t,u,v)} = \det\begin{pmatrix} 3t^2 & 0 & 0 \\ 0 & 3u^2 & 0 \\ 0 & 0 & 3v^2 \end{pmatrix}$$

$t = r\sin\theta\cos\varphi,\ u = r\sin\theta\sin\varphi,\ v = r\cos\theta$

$W : 0 \le r \le a^{1/3},\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \varphi \le \frac{\pi}{2} \qquad \frac{\partial(t,u,v)}{\partial(r,\theta,\varphi)} = r^2\sin\theta$

$$\iiint_V dxdydz = \iiint_{V'} 27t^2u^2v^2 dtdudv = \iiint_W 27r^8\sin^5\theta\cos^2\theta\cos^2\varphi\sin^2\varphi drd\theta d\varphi$$

$$= 27\int_0^{a^{1/3}} r^8 dr \int_0^{\pi/2} \sin^5\theta\cos^2\theta\, d\theta \int_0^{\pi/2} \cos^2\varphi\sin^2\varphi\, d\varphi$$

$$= 3(a^{1/3})^9 \frac{1}{6}\int_0^{\pi/2} \sin^7\theta\, d\theta\ \frac{1}{3}\int_0^{\pi/2} \cos^4\varphi\, d\varphi$$ 部分積分

$$= \frac{a^3}{6}\int_0^{\pi/2} \cos^7\theta\, d\theta\ \int_0^{\pi/2} \cos^4\varphi\, d\varphi$$ $\sin\theta = \cos(\frac{\pi}{2} - \theta)$ より

$$= \frac{a^3}{\cancel{6}}\frac{\cancel{6}}{7}\frac{\cancel{4}}{5}\frac{\cancel{2}}{\cancel{3}}\int_0^{\pi/2} \cos\theta\, d\theta\ \frac{\cancel{3}}{\cancel{4}}\frac{1}{\cancel{2}}\int_0^{\pi/2} d\varphi = \frac{1}{35}a^3\frac{\pi}{2} = \frac{1}{70}a^3\pi$$

答は教科書と違うが，教科書は多分 $|x|^{2/3} + |y|^{2/3} + |z|^{2/3}$を解いている

(3) $V : x^2 + y^2 \le z \le 2x$

$$\iiint_V dxdydz = \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2\cos\theta} rdr \int_{r^2}^{2r\cos\theta} dz = \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2\cos\theta} (2r\cos\theta - r^2)rdr$$

$$= \int_{-\pi/2}^{\pi/2} \left[\frac{2}{3}r^3\cos\theta - \frac{1}{4}r^4\right]_{r=0}^{r=2\cos\theta} d\theta = \int_{-\pi/2}^{\pi/2} \frac{16}{3}\cos^4\theta - 4\cos^4\theta\, d\theta$$

$$= \frac{4}{3}\int_{-\pi/2}^{\pi/2} \cos^4\theta d\theta = \frac{4}{3}\frac{3}{4}\frac{1}{2}\int_{-\pi/2}^{\pi/2} d\theta = \frac{\pi}{2}$$

(4) $V : 0 \le x + y \le 1,\ 0 \le y + z \le 1,\ 0 \le z + x \le 1 \quad t = x + y,\ u = y + z,\ v = z + x$

$W: 0 \le t \le 1,\ 0 \le u \le 1, 0 \le v \le 1 \quad x = \dfrac{t-u+z}{2},\ y = \dfrac{t+u-z}{2},\ z = \dfrac{-t+u+z}{2}$

$$\frac{\partial(x,y,z)}{\partial(t,u,v)} = \det\begin{pmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{pmatrix} = \frac{1}{8}((1+1)-(-1-1)-(1-1)) = \frac{1}{2}$$

$$\iiint_V dxdydz = \iiint_W \frac{dtdudv}{2} = \frac{1}{2}\int_0^1 dt \int_0^1 du \int_0^1 dv = \frac{1}{2}$$

(5) 曲面 $(x^2+y^2+z^2)^2 = z$ で囲まれる図形

$x = r\sin\theta\cos\varphi,\ y = r\sin\theta\sin\varphi,\ z = r\cos\theta \quad dxdydz = r^2\sin\theta\, drd\theta d\varphi$

$r^4 = r\cos\theta \ \Leftrightarrow\ r = (\cos\theta)^{1/3}$

$V: 0 \le r \le (\cos\theta)^{1/3}$

$$\iiint_V r^2\sin\theta\, drd\theta d\varphi = \int_0^{2\pi} d\varphi \int_0^{\pi/2} d\theta \int_0^{(\cos\theta)^{1/3}} r^2\sin\theta\, dr$$

$$= 2\pi\int_0^{\pi/2}\left[\frac{r^3}{3}\right]_0^{(\cos\theta)^{1/3}} \sin\theta\, d\theta$$

$$= \frac{2\pi}{3}\int_0^{\pi/2}\cos\theta\sin\theta\, d\theta$$

$$= \frac{2\pi}{3}\left[-\frac{1}{2}\cos\theta\right]_0^{\pi/2} = \frac{\pi}{3}$$

問題 5.4.4

つぎの図形の曲面積を求めよ．

(1) 円柱 $y^2+z^2=a^2$ と円柱 $x^2+y^2=a^2$ の内部にある部分．

$K: y^2 = \min(a^2-z^2, a^2-x^2)$

対称性を使うと総面積は $x,y,z \ge 0,\ x^2 \le z^2$ の 16 倍になる．

$x,y,z \ge 0,\ x \le z \Rightarrow y = f(x,z) = \sqrt{a^2-z^2}$

曲面積の定理を使う．

$$\frac{S(K)}{16} = \iint_{0\le x\le z\le a} \sqrt{f_x(x,z)^2 + f_z(x,z)^2 + 1}\ dxdz$$

$$= \int_0^a dz \int_0^z \sqrt{0 + \left(\frac{-z}{\sqrt{a^2-z^2}}\right)^2 + 1}\ dx$$

$$= \int_0^a z\sqrt{\frac{a^2}{a^2-z^2}}\ dz = \int_0^a \frac{az}{\sqrt{a^2-z^2}}\ dz$$

$$= a\left[-\sqrt{a^2-z^2}\right]_0^a = a^2$$

$S(K) = 16a^2$

教科書の答 $8a^2$ は不可能である．この形が半径 a の球を含むので，面積は $4\pi a^2$ 以上でなければならない．

(2) 円柱 $y^2+z^2=a^2$ と球 $x^2+y^2+z^2=2a^2$ の内部にある部分.

$K: z^2=\min(a^2-y^2, 2a^2-x^2-y^2)$

対称性を使うと総面積は $x,y,z\geq 0$ の 8 倍になる.

$x,y,z\geq 0,\ x^2\leq a^2 \Rightarrow z=f(x,y)=\sqrt{a^2-y^2}$

$x,y,z\geq 0,\ x^2\geq a^2 \Rightarrow z=g(x,y)=\sqrt{2a^2-x^2-y^2}$

曲面積の定理を使う.

$$\frac{S(K)}{8}=\int_0^a dx\int_0^a \sqrt{f_x(x,y)^2+f_y(x,y)^2+1}\ dy + \int_a^{\sqrt{2}a} dx\int_0^{\sqrt{2a^2-x^2}} \sqrt{g_x(x,y)^2+g_y(x,y)^2+1}\ dy$$

$$=\int_0^a dx\int_0^a \sqrt{\frac{y^2}{a^2-y^2}+1}\ dy + \int_a^{\sqrt{2}a} dx\int_0^{\sqrt{2a^2-x^2}} \sqrt{\frac{x^2}{2a^2-x^2-y^2}+\frac{y^2}{2a^2-x^2-y^2}+1}\ dy$$

$$=\int_0^a dx\int_0^a \sqrt{\frac{a^2}{a^2-y^2}}\ dy+\int_a^{\sqrt{2}a} dx\int_0^{\sqrt{2a^2-x^2}}\sqrt{\frac{2a^2}{2a^2-x^2-y^2}}\ dy$$

$$=\int_0^a a\left[\mathrm{Sin}^{-1}\frac{y}{a}\right]_0^a dx+\int_a^{\sqrt{2}a}\sqrt{2}a\left[\mathrm{Sin}^{-1}\frac{y}{\sqrt{2a^2-x^2}}\right]_0^{\sqrt{2a^2-x^2}} dx$$

$$=\int_0^a a\frac{\pi}{2}dx+\int_a^{\sqrt{2}a}\sqrt{2}a\frac{\pi}{2}dx=\frac{\pi}{2}a^2+\frac{\pi}{2}(2-\sqrt{2})a^2=\frac{3-\sqrt{2}}{2}\pi a^2$$

$S(K)=4(3-\sqrt{2})\pi a^2$

(3) 曲面 $z=x^2+y^2$ の $z=a$ 以下の部分.

$K: z=x^2+y^2\leq a,\ x=r\cos\theta,\ y=r\sin\theta$ とおく.

$$S(K)=\iint_{x^2+y^2\leq a}\sqrt{4x^2+4y^2+1}\ dxdy=\iint_{r\leq\sqrt{a}}\sqrt{4r^2\cos^2\theta+4r^2\sin^2\theta+1}\ rdrd\theta$$

$$=2\pi\int_0^{\sqrt{a}}\sqrt{1+4r^2}\ rdr=2\pi\left[\frac{1}{8}\frac{2}{3}(1+4r^2)^{3/2}\right]_0^{\sqrt{a}}=\frac{\pi}{6}((1+4a)^{3/2}-1)$$

(4) 曲面 $z=xy$ の円柱 $x^2+y^2=a^2$ の内部.

$K: z=xy,\ x^2+y^2\leq a,\ x=r\cos\theta,\ y=r\sin\theta$ とおく.

$$S(K)=\iint_{x^2+y^2\leq a^2}\sqrt{x^2+y^2+1}\ dxdy=\iint_{r\leq a}\sqrt{r^2\cos^2\theta+r^2\sin^2\theta+1}\ rdrd\theta$$

$$=2\pi\int_0^a r\sqrt{1+r^2}dr=2\pi\left[\frac{1}{3}(1+r^2)^{3/2}\right]_0^a=\frac{2\pi}{3}((1+a^2)^{3/2}-1)$$

(5) 球面 $x^2+y^2+z^2=4$ の $x^2+y^2=2z+1$ より上の部分.

$z^2+2z+1=4 \iff z^2+2z-3=0 \iff (z+3)(z-1)=0$

$z=-3$ が錐形よりしたなので，球面の $z=1$ より上の部分になる.

$z=2\sin\theta$ とおく.

$$S=2\pi\int_1^2\sqrt{4-z^2}\sqrt{1+\left(\frac{z}{\sqrt{4-z^2}}\right)^2}=2\pi\int_{\pi/6}^{\pi/2}2\cos\theta\sqrt{1+\tan^2\theta}\ 2\cos\theta\, d\theta$$

$$= 8\pi \int_{\pi/6}^{\pi/2} \cos\theta \sqrt{\cos^2\theta + \sin^2\theta}\, d\theta = 8\pi \int_{\pi/6}^{\pi/2} \cos\theta\, d\theta = 8\pi [\sin\theta]_{\pi/6}^{\pi/2} = 4\pi$$

おまけ　球 $x^2 + y^2 + z^2 = a^2$ の面積

回転体の表面積の定理を使い，$x = a\sin\theta$ とおく．

$$S = 2\pi \int_{-a}^{a} \sqrt{a^2 - x^2} \sqrt{1 + \left(\frac{x}{\sqrt{a^2 - x^2}} \right)^2}\, dx = 2\pi \int_{-\pi/2}^{\pi/2} a\cos\theta \sqrt{1 + \tan^2\theta} a\cos\theta\, d\theta$$

$$= 2\pi a^2 \int_{-\pi/2}^{\pi/2} \cos\theta \sqrt{\cos^2\theta + \sin^2\theta}\, d\theta = 2\pi a^2 \int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta = 4\pi a^2$$

問題 5.5.6

つぎの広義重積分の値を求めよ．

両方とも正の関数なので，重積分可能性が領域の列の取り方によらない．

(1)　$D = \{(x, y) \mid x, y \geq 1\}$　　$D_a = \{(x, y) \mid 1 \leq x, y \leq a\}$

$$\iint_{D_a} \frac{xy}{(x^2 + y^2)^3} dxdy = \int_1^a dx \int_1^a \frac{xy}{(x^2 + y^2)^3} dy$$

$$= \int_1^a \left[-\frac{1}{4} \frac{x}{(x^2 + y^2)^2} \right]_{y=1}^{y=a} dx = \int_1^a \frac{x}{4(x^2 + 1)^2} - \frac{x}{4(x^2 + a^2)} dx$$

$$= \frac{1}{8} \left[\frac{1}{x^2 + a^2} - \frac{1}{x^2 + 1} \right]_1^a = \frac{1}{8} \left(\frac{1}{2a^2} - \frac{2}{1 + a^2} + \frac{1}{2} \right) = S_a$$

$$\iint_D \frac{xy}{(x^2 + y^2)^3} dxdy = \lim_{a\to\infty} S_a = \frac{1}{16}$$

(2)　$D = \{(x, y) \mid x, y \geq 0\}$　　$D_a = \{(r\cos\theta, r\sin\theta) \mid 0 \leq \theta \leq \frac{\pi}{2},\ 0 \leq r \leq a\}$

$$\iint_D x^2 e^{-(x^2+y^2)} dxdy = \int_0^{\pi/2} \cos^2\theta d\theta \int_0^\infty r^2 e^{-r^2} r dr$$

$$= \frac{\pi}{4} \left(\left[-\frac{r^2}{2} e^{-r^2} \right]_0^\infty + \int_0^\infty r e^{-r^2} dr \right) = \frac{\pi}{4} \left(0 + \left[-\frac{1}{2} e^{-r^2} \right]_0^\infty \right) = \frac{\pi}{8}$$