問題5.2.1,2,4の解答

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問題5.2.1

(1) $D : a^2 \le x^2 + y^2 \le 4a^2 \qquad E : a \le r \le 2a$

$$\iint_D \frac{dxdy}{x^2+y^2}^m = \iint_E \frac{rdrd\theta}{r^{2m}} \int_a^{2a} \frac{dr}{r^{2m-1}} \int_0^{2\pi} d\theta = 2\pi \int_a^{2a} r^{1-2m} dr$$

$$= \frac{\pi}{1-m}\left[r^{2-2m}\right]_a^{2a} = \frac{\pi}{1-m} a^{2-2m}(4^{1-m}-1) \qquad (m \neq 1)$$

$$= 2\pi\left[\log r\right]_a^{2a} = 2\pi \log 2 \qquad (m = 1)$$

(2) $D : x^2 + y^2 \le 1 \qquad E : r \le 1$

$$\iint_D \sqrt{1-x^2-y^2}dxdy = \iint_E \sqrt{1-r^2} rdrd\theta = 2\pi \int_0^1 r\sqrt{1-r^2}dr = -\frac{2\pi}{3}\left[(1-r^2)^{3/2}\right]_0^1$$

$$= \frac{2\pi}{3}$$

(3) $D : x^2 + y^2 \le x \qquad E : -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ r \le \cos\theta$

$$\iint_D xdxdy = \iint_E r^2 \cos\theta drd\theta = \int_{-\pi/2}^{\pi/2} d\theta \int_0^{\cos\theta} r^2 \cos\theta dr = \frac{1}{3}\int_{-\pi/2}^{\pi/2} \left[r^3\right]_{r=0}^{r=\cos\theta} \cos\theta d\theta$$

$$= \frac{1}{3}\int_{-\pi/2}^{\pi/2} \cos^4\theta d\theta = \frac{1}{12}\left[\cos^3\theta \sin\theta\right]_{-\pi/2}^{\pi/2} + \frac{1}{4}\int_{-\pi/2}^{\pi/2} \cos^2\theta d\theta$$

$$= \frac{1}{8}\left[\cos\theta\sin\theta\right]_{-\pi/2}^{\pi/2} + \frac{1}{8}\int_{-\pi/2}^{\pi/2} d\theta = \frac{\pi}{8}$$

問題5.2.2

(1) $D : 0 \le x+y \le 2,\ 0 \le x-y \le 2 \quad x = u+v,\ y = u-v \quad E : 0 \le u \le 1,\ 0 \le v \le 1$

$$\iint_D (x-y)e^{x+y}dxdy = \iint_E ve^u \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = \iint_E ve^u \left|\det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right| dudv$$

$$= \int_0^1 du \int_0^1 2ve^{2u}2dv = 2\int_0^2 \left[v^2 e^{2u}\right]_{v=0}^{v=1} du = 2\int_0^1 e^{2u}du = \left[e^{2u}\right]_0^1 = e^2 - 1$$

(2) $D : (x/a)^2 + (y/b)^2 \le 1 \qquad x = ar\cos\theta,\ y = br\sin\theta \qquad E : 0 \le r \le 1,\ 0 \le \theta \le 2\pi$

$$\iint_D x^2 dxdy = \iint_E a^2 r^2 \cos^2\theta \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| drd\theta$$

$$= \iint_E a^2 r^2 \cos^2\theta \left|\det\begin{pmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{pmatrix}\right| drd\theta$$

$$= \int_0^{2\pi} \cos^2\theta d\theta \int_0^1 a^3 br^3 dr = \pi\frac{1}{4}\left[a^3 br^4\right]_0^1 = \frac{\pi}{4}a^3 b$$

(3) $D : x^2 + 2xy + 2y^2 \le 1 \quad y = r\sin\theta,\ x = r(\cos\theta - \sin\theta) \quad E : 0 \le r \le 1,\ 0 \le \theta \le 2\pi$

$$\iint_D (x+y)^4 dxdy = \iint_E r^4\cos^4\theta \left|\det\begin{pmatrix} \cos\theta - \sin\theta & -r\sin\theta - r\cos\theta \\ \sin\theta & r\cos\theta \end{pmatrix}\right| drd\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 r^5 \cos^4\theta dr = \frac{1}{6}\int_0^{2\pi} \cos^4\theta d\theta = \frac{1}{6}\frac{3}{8}\int_0^{1\pi} d\theta = \frac{\pi}{8}$$

問題 5.2.4

$D = \{(r\cos\theta, r\sin\theta) \mid \alpha \le \delta \le \beta,\ 0 \le r \le f(\theta)\}$

$S(D) = \iint_D dxdy = \iint_E rdrd\theta = \int_\alpha^\beta d\theta \int_0^{f(\theta)} rdr = \int_\alpha^\beta \left[\frac{1}{2}r^2\right]_0^{f(\theta)} d\theta = \frac{1}{2}\int_\alpha^\beta f(\theta)^2 d\theta$

(1) $S(D) = \dfrac{1}{2}\displaystyle\int_0^\pi \sin^2\theta d\theta = \dfrac{\pi}{4}$

(2) $(x^2+y^2)^2 = x^2 - y^2 \ \Leftrightarrow\ r^4 = r^2(\cos^2\theta - \sin^2\theta) \ \Leftrightarrow\ r^2 = \cos^2\theta - \sin^2\theta$

$$S(D) = 2\frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2\theta - \sin^2\theta d\theta = 2\int_{-\pi/4}^{\pi/4}\cos^2\theta d\theta - \frac{\pi}{2}$$
$$= [\sin\theta\cos\theta]_{-\pi/4}^{\pi/4} + \int_{-\pi/4}^{\pi/4} d\theta - \frac{\pi}{2} = 1$$

(3) $S(D) = \dfrac{1}{2}\displaystyle\int_0^{2\pi} a^2(1+\cos\theta)^2 d\theta = \dfrac{a^2}{2}\displaystyle\int_0^{2\pi} 1 + 2\cos\theta + \cos^2\theta d\theta$

$$= \frac{a^2}{2}\left\{2\pi + 2\left[\sin\theta\right]_0^{2\pi} + \pi\right\} = \frac{3\pi}{2}a^2$$