問題 5.1.1-2 の解答

Jacques Garrigue, 2008 年 12 月 22 日

問題 5.1.1

(1) $\displaystyle \int_0^2 dx \int_{x^2}^{2x} xe^y dy = \int_0^2 [xe^y]_{y=x^2}^{y=2x}\, dx = \int_0^2 xe^{2x} - xe^{x^2} dx$

$\displaystyle = \frac{1}{2}\left[xe^{2x}\right]_0^2 - \frac{1}{2}\int_0^2 e^{2x}dx - \frac{1}{2}\left[e^{x^2}\right]_0^2 = e^4 - \frac{1}{4}e^4 + \frac{1}{4} - \frac{1}{2}e^4 + \frac{1}{2} = \frac{e^4+3}{4}$

(2) $\displaystyle \int_0^1 dy \int_0^{\pi/2} y \sin xy\, dx = \int_0^1 [-\cos xy]_{x=0}^{x=\pi/2} dy = \int_0^1 1 - \cos y\frac{\pi}{2}\, dy = 1 - [\frac{2}{\pi}\sin y\frac{\pi}{2}]_0^1 = 1 - \frac{2}{\pi}$

問題 5.1.2

(1) $D: 0 \le x \le \frac{\pi}{2},\ 0 \le y \le \frac{\pi}{2}$

$\displaystyle \iint_D \sin(2x+y)dxdy = \int_0^{\pi/2} dx \int_0^{\pi/2} \sin(2x+y)dy = \int_0^{\pi/2} [-\cos(2x+y)]_{y=0}^{y=\pi/2} dx$

$\displaystyle = \int_0 -\pi/2\cos 2x - \cos(2x + \frac{\pi}{2})dx = \frac{1}{2}\left[\sin 2x - \sin(2x+\frac{\pi}{2})\right]_0^{\pi/2} = -\frac{1}{2} + \frac{1}{2} = 1$

(2) $D: 1 \le x \le 2,\ 2 \le y \le 3$

$\displaystyle \iint_D (x^2y + y^2)dxdy = \int_2^3 \left[\frac{1}{3}x^3y + xy^2\right]_{x=1}^{x=2} dy = \int_2^3 \frac{7}{3}y + y^2 dy = \left[\frac{7}{6}y^2 + \frac{1}{3}y^3\right]_2^3$

$\displaystyle = \frac{7(9-4) + 2(27-8)}{6} = \frac{73}{6}$

(3) $D: x^2 + y^2 \le 1,\ x \ge 0$

$\displaystyle \iint_D xdxdy = \int_0^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} xdy = \int_0^1 2x\sqrt{1-x^2}\, dx = -\frac{2}{3}[(1-x^2)^{3/2}]_0^1 = \frac{2}{3}$

(4) $D: x^2 + y^2 \le a^2$

$\displaystyle \iint_D \sqrt{a^2-y^2}\, dxdy = \int_{-a}^a dy \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \sqrt{a^2-y^2}\, dx = \int_{-a}^a 2(a^2-y^2)dy$

$\displaystyle = 4a^3 - \frac{2}{3}[y^3]_{-a}^a = \frac{8}{3}a^3$

(5) $D: 0 \le y \le x \le 1$

$\displaystyle \iint_D xy^2dxdy = \int_0^1 dx \int_0^x xy^2 dy = \frac{1}{3}\int_0^1 [xy^3]_{y=0}^{y=x} dx = \frac{1}{3}\int_0^1 x^4 dx = \frac{1}{15}[x^5]_0^1 = \frac{1}{15}$

(6) $D: x \le y \le 2x,\ x + y \le 3 \quad \Rightarrow \quad 0 \le x \le \frac{3}{2}$

$\displaystyle \iint_D 2x - ydxdy = \int_0^1 dx \int_x^{2x} 2x - ydy + \int_1^{3/2} dx \int_x^{3-x} 2x - ydy$

$\displaystyle = \int_0^1 \left[2xy - \frac{1}{2}y^2\right]_{y=x}^{y=2x} dx + \int_1^{3/2} \left[2xy - \frac{1}{2}y^2\right]_{y=x}^{y=3-x} dx$

$\displaystyle = \int_0^1 2x^2 - \frac{3}{2}x^2dx + \int_1^{3/2} 2x(3-2x) - \frac{1}{2}(9-6x)dx$

$\displaystyle = \frac{1}{6}\left[x^3\right]_0^1 + \int_1^{3/2} 9x - 4x^2 - \frac{9}{2}dx = \frac{1}{6} + \left[\frac{9}{2}x^2 - \frac{4}{3}x^3 - \frac{9}{2}x\right]_1^{3/2}$

$\displaystyle = \frac{1}{6} + \frac{9}{2}\frac{5}{4} - \frac{4}{3}\frac{19}{8} - \frac{9}{2}\frac{1}{2} = \frac{4 + 135 - 76 - 54}{24} = \frac{3}{8}$

(7) $D: 0 \leq x \leq 1,\ 0 \leq y \leq 1-x,\ 0 \leq z \leq 1-x-y$

$$\iiint_D z dx dy dz = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} z dz = \frac{1}{2}\int_0^1 dx \int_0^{1-x} [z^2]_{z=0}^{z=1-x-y} dy$$

$$= \frac{1}{2}\int_0^1 dx \int_0^{1-x} (1-x-y)^2 dy = -\frac{1}{6}\int_0^1 \left[(1-x-y)^3\right]_{y=0}^{y=1-x} dx$$

$$= \frac{1}{6}\int_0^1 (1-x)^3 dx = \frac{1}{6}\int_0^1 x^3 dx = \frac{1}{24}$$

(8) $D: x \geq 0,\ y \geq 0,\ z \geq 0,\ x+2y+3z \leq 6$

$$\iiint_D y dx dy dz = \int_0^3 dy \int_0^{6-2y} dx \int_0^{2-x/3-2y/3} y dz = \int_0^3 dy \int_0^{6-2y} y(2-\frac{x}{3}-\frac{2y}{3}) dx$$

$$= \frac{1}{6}\int_0^3 y\left[x(12-4y)-x^2\right]_{x=0}^{x=6-2y} dy = \frac{1}{6}\int_0^3 y(6-2y)(12-4y-6+2y) dy$$

$$= \frac{1}{3}\int_0^3 y(3-y)(6-2y) dy = \frac{2}{3}\int_0^3 y^2(3-y) dy = \frac{2}{3}\left[y^3-\frac{1}{4}y^4\right]_0^3 = \frac{1}{6}27 = \frac{9}{2}$$