問題 4.3.2-3，4.3.5，4.3.7，4.4.2-3 の解答

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問題 4.3.2

(1) $z = x^3y + xy^2 \qquad \dfrac{\partial z}{\partial x} = 3x^2y + y^2 \qquad \dfrac{\partial z}{\partial y} = x^3 + 2xy$

$$\frac{\partial^2 z}{\partial x^2} = 6xy \qquad \frac{\partial z^2}{\partial x \partial y} = \frac{\partial z^2}{\partial y \partial x} = 3x^2 + 2y \qquad \frac{\partial^2 z}{\partial y^2} = 2x$$

(2) $z = x\sin(xy) \qquad \dfrac{\partial z}{\partial x} = \sin xy + xy\cos xy \qquad \dfrac{\partial z}{\partial y} = x^2\cos xy$

$$\frac{\partial^2 z}{\partial x^2} = 2y\cos xy - xy^2\sin xy \qquad \frac{\partial z^2}{\partial x \partial y} = \frac{\partial z^2}{\partial y \partial x} = 2x\cos xy - x^2y\sin xy$$

$$\frac{\partial^2 z}{\partial y^2} = -x^3\sin xy$$

(3) $z = e^{x^2y} \qquad \dfrac{\partial z}{\partial x} = 2xye^{x^2y} \qquad \dfrac{\partial z}{\partial y} = x^2e^{x^2y}$

$$\frac{\partial^2 z}{\partial x^2} = 2y(1 + 2x^2y)e^{x^2y} \qquad \frac{\partial z^2}{\partial x \partial y} = \frac{\partial z^2}{\partial y \partial x} = 2x(1 + x^2y)e^{x^2y} \qquad \frac{\partial^2 z}{\partial y^2} = x^4e^{x^2y}$$

(4) $z = \tan^{-1}(xy) \qquad \dfrac{\partial z}{\partial x} = \dfrac{y}{1 + x^2y^2} \qquad \dfrac{\partial z}{\partial y} = \dfrac{x}{1 + x^2y^2}$

$$\frac{\partial^2 z}{\partial x^2} = \frac{-2xy^3}{(1 + x^2y^2)^2} \qquad \frac{\partial z^2}{\partial x \partial y} = \frac{\partial z^2}{\partial y \partial x} = \frac{1 - x^2y^2}{(1 + x^2y^2)^2} \qquad \frac{\partial^2 z}{\partial y^2} = \frac{-2x^3y}{(1 + x^2y^2)^2}$$

問題 4.3.3

(1) $z = \log(x^2 + y^2) \qquad \dfrac{\partial z}{\partial x} = \dfrac{2x}{x^2 + y^2} \qquad \dfrac{\partial z}{\partial y} = \dfrac{2y}{x^2 + y^2}$

$$\frac{\partial^2 z}{\partial x^2} = \frac{2(x^2 + y^2) - 4x^2}{(x^2 + y^2)^2} \qquad \frac{\partial^2 z}{\partial y^2} = \frac{2(x^2 + y^2) - 4y^2}{(x^2 + y^2)^2}$$

$\Delta z = 0$

(2) $z = \dfrac{x}{x^2 + y^2} \qquad \dfrac{\partial z}{\partial x} = -\dfrac{x^2 - y^2}{(x^2 + y^2)^2} \qquad \dfrac{\partial z}{\partial y} = -\dfrac{2xy}{(x^2 + y^2)^2}$

$$\frac{\partial^2 z}{\partial x^2} = \frac{-2x(x^2 + y^2) + 2(x^2 - y^2)(2x)}{(x^2 + y^2)^3} = \frac{2x^3 - 6xy^2}{(x^2 + y^2)^3}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{-2x(x^2 + y^2) + 2(2xy)(2y)}{(x^2 + y^2)^3} = \frac{-2x^3 + 6xy^2}{(x^2 + y^2)^3}$$

$\Delta z = 0$

(3) $z = \tan^{-1}\dfrac{y}{x} \qquad \dfrac{\partial z}{\partial x} = -\dfrac{y}{x^2 + y^2} \qquad \dfrac{\partial z}{\partial y} = \dfrac{x}{x^2 + y^2}$

$$\frac{\partial^2 z}{\partial x^2} = \frac{2xy}{(x^2 + y^2)^2} \qquad \frac{\partial^2 z}{\partial y^2} = -\frac{2xy}{(x^2 + y^2)^2}$$

$\Delta z = 0$

(4) $z = x^3 + xy + y^3 \qquad \dfrac{\partial z}{\partial x} = 3x^2 + y \qquad \dfrac{\partial z}{\partial y} = x + 3y^2$

$\dfrac{\partial^2 z}{\partial x^2} = 6x \qquad \dfrac{\partial^2 z}{\partial y^2} = 6y \qquad \Delta z = 6(x+y)$

問題 4.3.5

$n = 2$ のマクローリン展開は、

$$f(h,k) = f(0,0) + hf_x(0,0) + kf_y(0,0) + \frac{1}{2}(h^2 f_{xx}(\theta h, \theta k) + 2hkf_{xy}(\theta h, \theta k) + k^2 f_{yy}(\theta h, \theta k))$$

(1) $f(x,y) = e^{x-y}, \quad f(0,0) = 1 \quad f_x = e^{x-y}, \quad f_x(0,0) = 1 \quad f_y = -e^{x-y}, \quad f_y(0,0) = -1$

$f_{xx} = e^{x-y} \qquad f_{xy} = -e^{x-y} \qquad f_{yy} = e^{x-y}$

$$e^{h-k} = 1 + h - k + \frac{1}{2}(h^2 e^{\theta h - \theta k} - 2hke^{\theta h - \theta k} + k^2 e^{\theta h - \theta k}) = 1 + h - k + \frac{1}{2}(h-k)^2 e^{\theta(h-k)}$$

(2) $f(x,y) = \cos(x+2y), \quad f(0,0) = 1$

$f_x = -\sin(x+2y), \quad f_x(0,0) = 0 \qquad f_y = -2\sin(x+2y), \quad f_y(0,0) = 0$

$f_{xx} = -\cos(x+2y) \qquad f_{xy} = -2\cos(x+2y) \qquad f_{yy} = -4\cos(x+2y)$

$$\cos(h+2k) = 1 + \frac{1}{2}(h^2(-\cos(\theta h + 2\theta k)) + 2hk(-2\cos(\theta h + 2\theta k)) + k^2(-4\cos(\theta h + 2\theta k)))$$

$$= 1 - \frac{1}{2}(h+2k)^2 cos(\theta(h+2k))$$

問題 4.3.7 次の関数の極値を求めよ

(1) $f(x,y) = x^2 + xy + 2y^2 - 4y \qquad f_x(x,y) = 2x + y \qquad f_y(x,y) = x + 4y - 4$

$f_{xx}(x,y) = 2 \qquad f_{xy}(x,y) = 1 \qquad f_{yy}(x,y) = 4$

$D = 8 - 1 > 0,\ f_{xx} > 0$ なので極小値がありうる

$$f_x(x,y) = f_y(x,y) = 0 \ \Leftrightarrow \ \begin{cases} 2x + y = 0 \\ x + 4y - 4 = 0 \end{cases} \ \Leftrightarrow \ \begin{cases} y = -2x \\ x + 4(-2x) - 4 = 0 \end{cases}$$

$\left(-\dfrac{4}{7}, \dfrac{8}{7}\right)$ で極小値 $\dfrac{16 - 32 + 2 \times 64}{49} - \dfrac{32}{7} = -\dfrac{16}{7}$

(2) $f(x,y) = x^3 + 2xy - x - 2y \qquad f_x(x,y) = 3x^2 + 2y - 1 \qquad f_y(x,y) = 2x - 2$

$f_{xx}(x,y) = 6x \qquad f_{xy}(x,y) = 2 \qquad f_{yy}(x,y) = 0$

$D = -4 < 0$ なので極値がありえない

(3) $f(x,y) = x^3 + y^3 + x^2 + 2xy + y^2 \qquad f_x(x,y) = 3x^2 + 2x + 2y \qquad f_y(x,y) = 3y^2 + 2x + 2y$

$f_{xx}(x,y) = 6x + 2 \qquad f_{xy}(x,y) = 2 \qquad f_{yy}(x,y) = 6y + 2$

$$\begin{cases} 3x^2 + 2x + 2y = 0 \\ 3y^2 + 2x + 2y = 0 \end{cases} \ \Leftrightarrow \ \begin{cases} x = y \\ 3x^2 + 4x = 0 \end{cases} \vee \begin{cases} x = -y \\ 3x^2 = 0 \end{cases}$$

$(0,0)$ では $D = 0$ なので極値かどうか直接に判定できない

$\forall x, f(x,-x) = 0$ なので,$f(0,0) < f(\epsilon, -\epsilon)$ がなりたたず,極値ではない

$\left(-\dfrac{4}{3}, -\dfrac{4}{3}\right)$ では $D = (2 - 24/3)^2 - 4 = 32 > 0,\ f_{xx}\left(-\dfrac{4}{3}, -\dfrac{4}{3}\right) = -6 < 0$ なので極大値

$f(-4/3, -4/3) = -2(4/3)^3 + 4(4/3)^2 = -\dfrac{128}{27} + 3\dfrac{64}{27} = \dfrac{64}{27}$

(4) $f(x,y)=x^4+y^2+2x^2-4xy+1$ $\qquad f_x(x,y)=4x^3+4x-4y$ $\qquad f_y(x,y)=2y-4x$
$f_{xx}(x,y)=12x^2+6$ $\qquad f_{xy}(x,y)=-4$ $\qquad f_{yy}(x,y)=2$

$$\begin{cases} 4x^3+4x-4y=0 \\ 2y-4x=0 \end{cases} \Leftrightarrow \begin{cases} y=2x \\ 4x^3-4x=0 \end{cases} \Leftrightarrow\ x=y=0 \vee x^2=1, y=2x$$

$(0,0)$ では $D=12-16<0$ なので極値がない
$(1,2)$ では $D=36-16>0, f_{xx}(1,2)>0$ なので極小値 $f(1,2)=1+4+2-8+1=0$
$(-1,-2)$ では同様に極小値 $f(-1,-2)=0$

(5) $f(x,y)=x^2-xy+y^2+2x-y+7$ $\qquad f_x(x,y)=2x-y+2$ $\qquad f_y(x,y)=-x+2y-1$
$f_{xx}(x,y)=2$ $\qquad f_{xy}(x,y)=-1$ $\qquad f_{yy}(x,y)=2$ $\qquad D>0$ と $f_{xx}>0$ なので極小値

$$\begin{cases} 2x-y+2=0 \\ 2y-x-1=0 \end{cases} \Leftrightarrow \begin{cases} y=2x+2 \\ 3x+3=0 \end{cases} \Leftrightarrow\ x=-1,\ y=0$$

$(-1,0)$ で極小値 $f(-1,0)=6$

問題 4.4.2 P 付近の陰関数の存在を示し，その導関数を求めよ

(1) $f(x,y)=x^3+3xy+y^5-x+1$ $\ P(2,-1)$ $\qquad f\in C^\infty$
$f(2,-1)=8-6-1-2+1=0$ $\qquad f_y(x,y)=3x+5y^4$ $\qquad f_y(2,-1)=11\neq 0$
よって陰関数が存在する

$$\varphi'(x)=-\frac{f_x(x,\varphi(x))}{f_y(x,\varphi(x))}=-\frac{3x^2+3\varphi(x)-1}{3x+5(\varphi(x))^4} \qquad \varphi'(2)=-\frac{12-3-1}{6+5}=-\frac{8}{11}$$

(2) $f(x,y)=\cos x+2y\cos xy+2x\cos y-\pi$ $\ P(\pi/2,0)$
$f(\pi/2,0)=0+0+2\pi/2-\pi=0$ $\qquad f_y(x,y)=2\cos xy-2xy\sin xy-2xy\sin y$
$f_y(\pi/2,0)=2\neq 0$ よって陰関数が存在する

$$\varphi'(x)=-\frac{-\sin x+2xy\sin xy+2\cos y}{2\cos xy-2xy\sin xy-2xy\sin y} \qquad \varphi'(\pi/2)=-\frac{-1+2}{2}=-\frac{1}{2}$$

問題 4.4.3 P における接線と法線の方程式を与えよ

(1) $f(x,y)=3x^2-xy^3+2xy+y-x=0$ $\ P(1,2)$
$f(1,2)=3-8+4+2-1=0$ $\qquad f_x(1,2)=6-8+4-1=1$ $\qquad f_y(1,2)=-12+2+1=-9$

接線の方程式：

$$y-2=-\frac{f_x(1,2)}{f_y(1,2)}(x-1)\ \Leftrightarrow\ (x-1)f_x(1,2)+(y-2)f_y(1,2)=0\ \Leftrightarrow\ x-9y+17=0$$

法線の方程式： 接線と直角で，接線の方向ベクトルが $\left(1,-\dfrac{f_x(1,2)}{f_y(1,2)}\right)$ または
$(f_y(1,2),-f_x(1,2))$ なので，

$$(x-1)f_y(1,2)-(y-2)f_x(1,2)=0\ \Leftrightarrow\ -9x-y+11=0$$

(2) $f(x,y)=xe^{2y}-e^{xy}+\sin\pi xy+y$ $\ P(0,1)$
$f(0,1)=0-1+0+1=0$ $\qquad f_x(0,1)=e^2-1+\pi$ $\qquad f_y(0,1)=1$

接線の方程式： $xf_x(0,1)+(y-1)f_y(0,1)=0 \;\Leftrightarrow\; (e^2-1+\pi)x+y-1=0$
法線の方程式： $xf_y(0,1)-(y-1)f_x(0,1)=0 \;\Leftrightarrow\; x-(e^2-1+\pi)(y-1)=0$

問題 4.4.5 陰関数 $y=\varphi(x)$ の極値を求めよ

(1) $x^2+xy+2y^2=1$ $\qquad \dfrac{dy}{dx}=-\dfrac{f_x}{f_y}=-\dfrac{2x+y}{x+2y}$

$$\frac{dy}{dx}=0 \;\Leftrightarrow\; f_x=0 \;\Leftrightarrow\; 2x+y=0 \;\Leftrightarrow\; y=-2x \;(x+2y\neq 0)$$

$$x^2-2x^2+8x^2=1 \;\Leftrightarrow\; 7x^2=1 \;\Leftrightarrow\; x=\pm\frac{1}{\sqrt{7}}$$

$$\frac{d^2y}{dx^2}=-\frac{f_{xx}f_y^2-2f_{xy}f_xf_y+f_{yy}f_x^2}{f_y^3}=-\frac{f_{xx}}{f_y}=-\frac{2}{x+2y}=\frac{2}{3x}$$

$(1/\sqrt{7},-2/\sqrt{7})$ では $\dfrac{d^2y}{dx}=\dfrac{2\sqrt{7}}{3}>0$ なので極小値

$(-1/\sqrt{7},2/\sqrt{7})$ では $\dfrac{d^2y}{dx}=-\dfrac{2\sqrt{7}}{3}<0$ なので極大値