

Arithmetic Galois Theory and Related Moduli Spaces

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Survey of Drinfeld's work

on GT and its associated quantum groups

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**О КВАЗИТРЕУГОЛЬНЫХ КВАЗИХОПФОВЫХ АЛГЕБРАХ И ОДНОЙ
ГРУППЕ, ТЕСНО СВЯЗАННОЙ С $\text{Gal}(\mathbb{Q}/\mathbb{Q})$**

Доказывается анонсированная ранее теорема о структуре квазитреугольных квазихопфовых алгебр в рамках теории возмущений по постоянной Планка. При этом используется проунипотентный вариант одной группы, введенной Гротендицом и содержащей $\text{Gal}(\mathbb{Q}/\mathbb{Q})$.

§ 1. Введение

Настоящая работа посвящена главным образом доказательству анонсированной в [1] теоремы о структуре квазитреугольных квазихопфовых алгебр в рамках теории возмущений по постоянной Планка h . В качестве технического средства используется проунипотентный вариант одной группы, введенной Гротендицом [2] и представляющей огромный интерес ввиду ее тесной связи с $\text{Gal}(\mathbb{Q}/\mathbb{Q})$.

Напомним основные определения из [1]. Квазихопфова алгебра отличается от алгебры Хопфа тем, что аксиома коассоциативности заменена более слабым условием. Точнее, согласно определению из [1], квазихопфова алгебра над коммутативным кольцом k — это набор $(A, \Delta, \varepsilon, \Phi)$, где A — ассоциативная k -алгебра с единицей, Δ — гомоморфизм $A \rightarrow A \otimes A$, ε — гомоморфизм $A \rightarrow k$ (предполагается, что $\Delta(1)=1$, $\varepsilon(1)=1$), а Φ — обратимый элемент $A \otimes A \otimes A$, причем

$$(\text{id} \otimes \Delta)(\Delta(a)) = \Phi \cdot (\Delta \otimes \text{id})(\Delta(a)) \cdot \Phi^{-1}, \quad a \in A, \quad (1.1)$$

$$(\text{id} \otimes \text{id} \otimes \Delta)(\Phi) \cdot (\Delta \otimes \text{id} \otimes \text{id})(\Phi) = (1 \otimes \Phi) \cdot (\text{id} \otimes \Delta \otimes \text{id})(\Phi) \cdot (\Phi \otimes 1), \quad (1.2)$$

$$(\varepsilon \otimes \text{id}) \circ \Delta = \text{id} = (\text{id} \otimes \varepsilon) \circ \Delta, \quad (1.3)$$

$$(\text{id} \otimes \varepsilon \otimes \text{id})(\Phi) = 1, \quad (1.4)$$

и выполнена аксиома, которая в хопфовом случае, т.е. при $\Phi=1$, сводится к существованию и биективности антипода. В рассматриваемой в настоящей работе ситуации, когда $(A, \Delta, \varepsilon, \Phi)$ — деформация алгебры Хопфа, зависящая от «бесконечно малого» параметра h , эта аксиома выполнена автоматически, согласно теореме 1.6 из [1]. Как и в хопфовом случае, Δ называется коумножением, а ε — коединицей.

§1. Monoidal Category

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Def $(\mathcal{C}, \otimes, I, a, l, r)$

\mathcal{C} : category

$\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$: tensor product

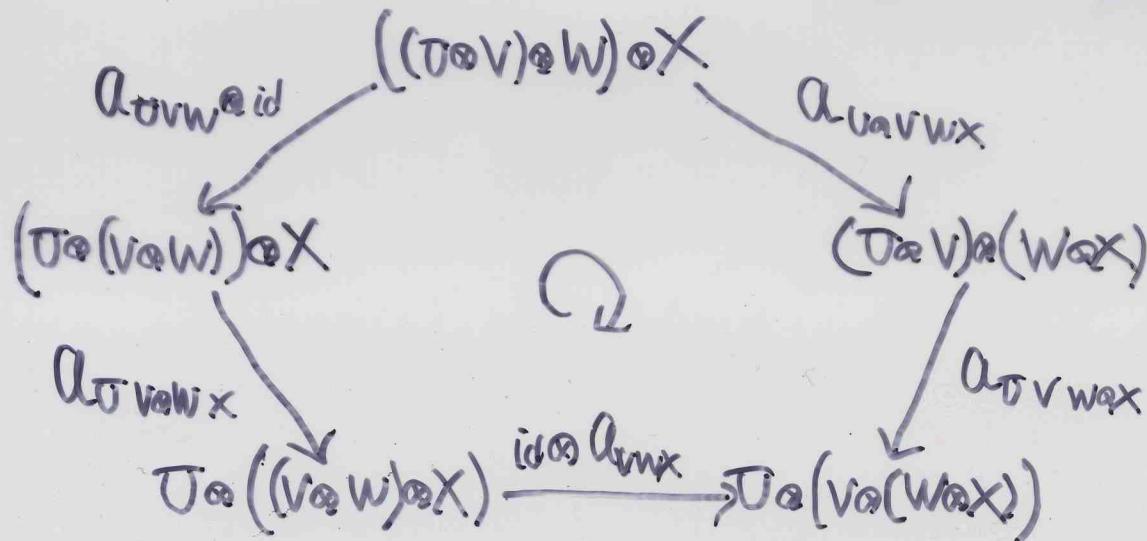
$I \in \mathcal{C}$: unit

$a_{uvw}: (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)$: associativity constraint

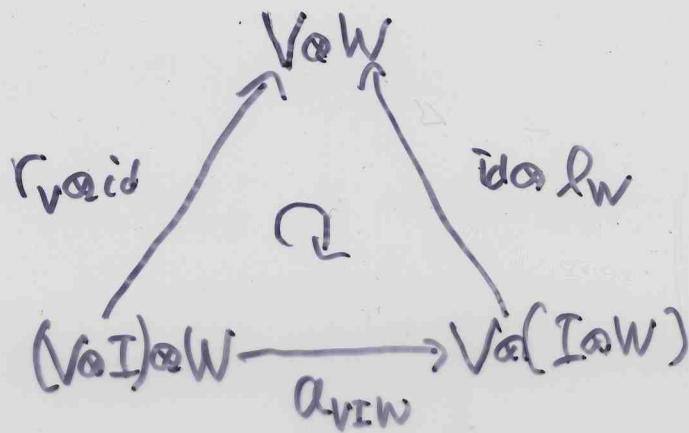
$l_v: I \otimes V \rightarrow V$: left unit constraint

$r_v: V \otimes I \rightarrow V$: right unit constraint

forms **monoidal category** if and only if it satisfies



and



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Assume k to be a unitary commutative ring.

Def (A, μ, δ)

$A : k\text{-mod}$

$\mu : A \otimes_k A \longrightarrow A : \text{product} \quad \leftarrow k\text{-mod hom}$

$\delta : k \longrightarrow A : \text{unit} \quad \leftarrow k\text{-mod hom}$

forms $k\text{-algebra}$ if and only if it satisfies

$$\begin{array}{ccc} A \otimes_k A \otimes_k A & \xrightarrow{\mu \otimes \text{id}} & A \otimes_k A \\ \downarrow \text{id} \otimes \mu & \curvearrowright & \downarrow \mu \\ A \otimes_k A & \xrightarrow{\mu} & A \end{array} \quad \text{associativity condition}$$

and

$$\begin{array}{ccccc} k \otimes_k A & \xrightarrow{\eta \otimes \text{id}} & A \otimes_k A & \xleftarrow{\text{id} \otimes \gamma} & A \otimes_k k \\ & \swarrow = & \downarrow \mu & \curvearrowright & \searrow = \\ & & A & & \end{array} \quad \text{unit condition}$$

Particularly it is commutative if and only if

it satisfies

$$\begin{array}{ccc} & A & \\ \mu \nearrow & \curvearrowright & \mu \searrow \\ A \otimes_k A & \xrightarrow{\tau} & A \otimes_k A \\ a \otimes b & \longmapsto & b \otimes a \end{array}$$

commutativity condition

Problem $(\text{left } A\text{-mod}) \ni V, W$

$$\begin{matrix} \cup & \cup \\ A & A \end{matrix}$$

$$\begin{array}{ccc} V \otimes_k W & , & k \\ \uparrow \varepsilon & & \uparrow \varepsilon \\ A \otimes_k A & \xrightarrow{\Delta} & A \otimes_k A \end{array}$$

Daf (C, Δ, ε)

$C : k\text{-mod}$

$\Delta : C \rightarrow C \otimes_k C$: coproduct

$\varepsilon : C \rightarrow k$: counit

← $k\text{-mod hom}$

← $k\text{-mod hom}$

forms $k\text{-coalgebra}$ if and only if it satisfies

$$\begin{array}{ccc}
 C & \xrightarrow{\Delta} & C \otimes_k C \\
 \downarrow \Delta & \curvearrowright & \downarrow \text{id} \otimes \Delta \\
 C \otimes_k C & \xrightarrow{\Delta \otimes \text{id}} & C \otimes_k C \otimes_k C
 \end{array}
 \quad \text{coassociativity condition}$$

$(\iff (\text{id} \otimes \Delta)(\Delta(a)) = (\Delta \otimes \text{id})(\Delta(a)) \text{ for } a \in C)$

and

$$\begin{array}{ccccc}
 k \otimes_k C & \xleftarrow{\varepsilon \otimes \text{id}} & C \otimes_k C & \xrightarrow{\text{id} \otimes \varepsilon} & C \otimes_k k \\
 & \curvearrowright & \uparrow \Delta & \curvearrowright & \\
 & & C & &
 \end{array}
 \quad \text{counit condition}$$

$(\iff (\varepsilon \otimes \text{id})(\Delta(a)) = a, (\text{id} \otimes \varepsilon)(\Delta(a)) = a \text{ for } a \in C)$

Particularly it is cocommutative if and only if it satisfies

$$\begin{array}{ccc}
 C & \xrightarrow{\Delta} & C \otimes_k C \\
 & \Delta \swarrow \searrow & \\
 C \otimes_k C & \xleftarrow{\text{id}} & C \otimes_k C
 \end{array}
 \quad \text{cocommutativity condition}$$

$(\iff \Delta^{\text{op}}(a) = \Delta(a) \text{ for } a \in C)$

Def $(A, \mu, \eta, \Delta, \varepsilon)$

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$A: k\text{-mod}$

$\mu: A \otimes_k A \rightarrow A : \text{product}$

$\leftarrow k\text{-mod hom}$

$\eta: k \rightarrow A : \text{unit}$

$\leftarrow k\text{-mod hom}$

$\Delta: A \rightarrow A \otimes_k A : \text{coproduct}$

$\leftarrow k\text{-mod hom}$

$\varepsilon: A \rightarrow k : \text{counit}$

$\leftarrow k\text{-mod hom}$

forms **bialgebra** if and only if it satisfies

$(A, \mu, \eta) : k\text{-alg}$

$(A, \Delta, \varepsilon) : k\text{-coalg}$

$\Delta : k\text{-alg hom}$

$\varepsilon : k\text{-alg hom}$

Particularly it forms **Hopf algebra** if and only if
it equips with a bijective **antipode**.

Proposition $A: \text{bialg} \Rightarrow (A\text{-mod})$ forms monoidal category

Problem. How about \Leftarrow ?

Def $(A, \Delta, \varepsilon, \Phi, l, r)$

$A: k\text{-alg}$

$\Delta: A \rightarrow A \otimes_k A : \text{coproduct} \quad \leftarrow k\text{-alg hom}$

$\varepsilon: A \longrightarrow k : \text{counit} \quad \leftarrow k\text{-alg hom}$

$\Phi \in (A \otimes_k A \otimes_k A)^\times : \text{Drinfel'd associator}$

$l \in A^\times : \text{left unit}$

$r \in A^\times : \text{right unit}$

forms **quasi-bialgebra** if and only if it satisfies

$$(\text{id} \otimes \Delta)(\Delta(a)) = \Phi((\Delta \otimes \text{id})(\Delta(a))) \Phi^{-1} \quad \text{quasi-coassociativity}$$

$$(\varepsilon \otimes \text{id})(\Delta(a)) = l^{-1}a l, \quad (\text{id} \otimes \varepsilon)(\Delta(a)) = r^{-1}a r \quad \text{quasi-counit}$$

$$(\text{id} \otimes \text{id} \otimes \Delta)(\Phi) \cdot (\Delta \otimes \text{id} \otimes \text{id})(\Phi) = \Phi_{234} (\text{id} \otimes \Delta \otimes \text{id})(\Phi) \Phi_{123}$$

$$(\text{id} \otimes \varepsilon \otimes \text{id})(\Phi) = r \circ l^{-1}$$

Particularly it forms **quasi-Hopf algebra** if and only if it equips with a bijective 'antipode'.

Ih $(A, \Delta, \varepsilon, \Phi, l, r): \text{q.b.alg} \iff (A\text{-mod}): \text{monoidal cat}$

$$\Rightarrow \underset{\text{cov}}{\alpha}(\varepsilon_{Q(u)} \otimes w) = \tilde{\Phi}(u \otimes u \otimes w)$$

$$\begin{aligned} \tilde{\Phi} &= Q_{AAA}((r \otimes \text{id})) \\ l &= l_A((\text{id} \otimes l)) \end{aligned} \quad \Leftarrow \quad \begin{aligned} l \circ (l \otimes u) &= lu \\ r \circ (r \otimes u) &= ru \end{aligned}$$

$$r = r_A((\text{id}))$$

§2. Quasi-tensored category

L6

Def (Drinfel'd) $(\mathcal{C}, \otimes, I, a, l, r, c)$:

$(\mathcal{C}, \otimes, I, a, l, r)$: monoidal category

$C_{VW}: V \otimes W \longrightarrow W \otimes V$: commutativity constraint
(braiding, R-matrix)

forms **quasi-tensored category** if and only if it satisfies
(braided tensor category)

$$\begin{array}{ccccc}
 & U \otimes (V \otimes W) & \xrightarrow{\text{Co.vw}} & (V \otimes W) \otimes U & \\
 \alpha_{U,V} \swarrow & & & & \downarrow \alpha_{W,U} \\
 (U \otimes V) \otimes W & & \curvearrowleft & & V \otimes (W \otimes U) \\
 \text{Co.vid} \searrow & & & & \downarrow \text{id} \circ \text{Co.w} \\
 & (V \otimes U) \otimes W & \xrightarrow{\alpha_{V,U}} & V \otimes (U \otimes W) &
 \end{array}$$

and

$$\begin{array}{ccccc}
 & (U \otimes V) \otimes W & \xrightarrow{\text{Co.v,w}} & W \otimes (U \otimes V) & \\
 \alpha^{-1}_{U,V} \swarrow & & & & \downarrow \alpha^{-1}_{W,U} \\
 U \otimes (V \otimes W) & & \curvearrowleft & & (W \otimes U) \otimes V \\
 \text{id} \circ C_{VW} \searrow & & & & \downarrow \text{Co.w id} \\
 & U \otimes (W \otimes V) & \xrightarrow{\alpha^{-1}_{U,WV}} & (U \otimes W) \otimes V &
 \end{array}$$

Particularly it forms **Tensor category** (symmetric monoidal cat)
if and only if $C_{WV} \circ C_{VW} = \text{id}$.

Def $(A, \Delta, \varepsilon, \Phi, \ell, r, R)$:

$(A, \Delta, \varepsilon, \Phi, \ell, r)$: q.b. alg

$R \in (A \otimes A)^*$: R-matrix

forms quasi-triangular quasi-bialgebra

if and only if it satisfies

$$\Delta^{\text{op}}(a) = R \Delta(a) R^{-1} \quad \text{quasi-cocommutativity}$$

$$(\text{id} \otimes \Delta)(R) = \Phi_{231}^{-1} R_{13} \Phi_{213} R_{12} \Phi_{123}^{-1}$$

$$(\Delta \otimes \text{id})(R) = \Phi_{312} R_{13} \Phi_{132}^{-1} R_{23} \Phi_{123}$$

Particularly it forms quasi-triangular quasi-Hopf algebra

if and only if it equips with a bijective 'antipode'.

In q.b.alg A forms q.t.q.b.alg \Leftrightarrow $(A\text{-mod})$ forms q-tensor cat

$$\Rightarrow (\text{End}(A)) = \text{End}(R(\alpha_w))$$

$$R = \text{End}(C_{AA}(\alpha_1)) \Leftarrow$$

§3. Grothendieck-Teichmüller group

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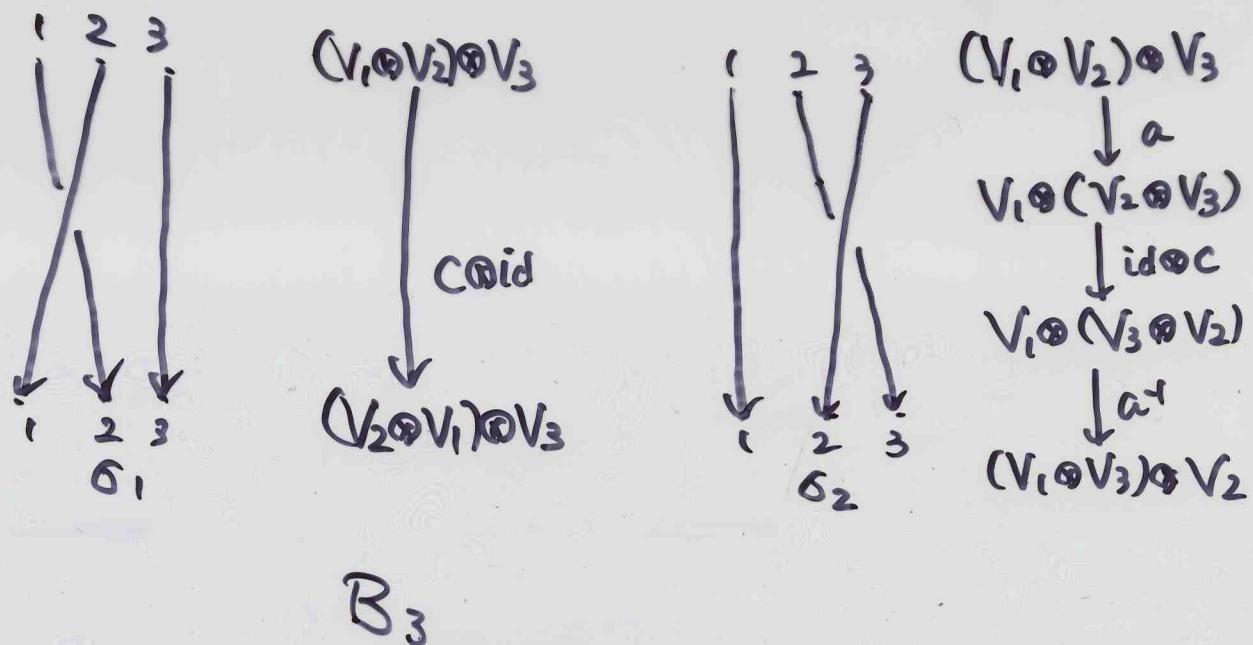
Let $(\mathcal{C}, \otimes, I, a, c, \ell, r)$ be a q-tensor cat.

Consider a deformation of a and c .

$$c : V_1 \otimes V_2 \rightarrow V_2 \otimes V_1$$

$$\rightsquigarrow c' = c^{2m+1} (m \in \mathbb{Z})$$

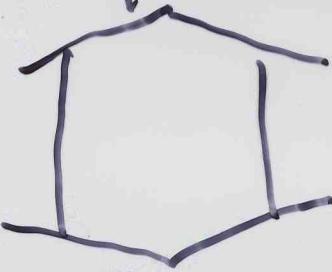
$$a : (V_1 \otimes V_2) \otimes V_3 \rightarrow V_1 \otimes (V_2 \otimes V_3) \rightsquigarrow a' = af \ (f \in P_3)$$



We get a new data $\mathcal{C}' = (\mathcal{C}, \otimes, I, a', c', \ell, r)$.

Problem When \mathcal{C}' forms a quasi-tensor category?

Th \mathcal{C}' forms quasi-tensored category.



- $f \in F_2' := [F_2, F_2]$ for $F_2 = \langle \sigma_1^2, \sigma_2^2 \rangle \subset P_3$
- $f(X, Y) f(Y, X) = 1$ 2-cycle relation
- $f(X, Y) X^m f(Z, X) Z^m f(Y, Z) Y^m = 1$
for $X Y Z = 1$ 3-cycle relation
- $f(x_{12}, x_{23}) f(x_{34}, x_{45}) f(x_{51}, x_{12}) f(x_{23}, x_{34}) f(x_{45}, x_{51}) = 1$
In $P_5^* = P_5 / \text{sphere center}$ 5-cycle relation

Def (Drinfeld)

$$GT := \left\{ (\lambda, f) \in \mathbb{Z} \times F_2 \mid \begin{array}{l} (\lambda, f) \text{ satisfy} \\ \text{the above} \end{array} \right\} \quad \begin{array}{l} \text{(primitive)} \\ \text{discrete GT} \end{array}$$

multiplication $(\lambda, f) = (\lambda_1, f_1) \circ (\lambda_2, f_2)$

$$\begin{cases} \lambda = \lambda_1 \lambda_2 \\ f = f_1 (f_2(x, y) X^{\lambda_2} f_2(x, y)^*, Y^{\lambda_2}) f_2(x, y) \end{cases}$$

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GT-action Let $(\lambda, f) = (\lambda_{\text{rel}}, f) \in \text{GT}$ and

$(A, \Delta, \varepsilon, \Phi, R)$: qtqb-alg with $\ell=1, r=1$.

$\mathcal{C} = (A, \Delta, \varepsilon, \Phi, R)$ -mod

$\xrightarrow{(\lambda, f)}$ $\mathcal{C}' = (A, \Delta, \varepsilon, \Phi', R')$ -mod

$$\Phi' = f(\Phi R^{21} R^{12} \Phi^{-1}, R^{32} R^{23}) \Phi$$

$$R' = (R^{12} \cdot R^{21})^m R$$

Tragedy Actually $\text{GT} = \{\pm 1, 1\} \cong \mathbb{Z}/2$!! 😕

Def k : a field with $\text{char} = 0$.

$$\text{GT}(k) := \left\{ (\lambda, f) \in k^\times \times \underline{F}_2(k) \mid \begin{array}{l} \lambda \in k^\times, f \in \underline{F}_2'(k) \\ \text{satisfy 2-, 3-, 5-} \\ \text{cycle relation} \end{array} \right\}$$

: (pro-algebraic) Grothendieck-Teichmüller group.

In (Drinfeld : '91) $\underline{\text{GT}}(k)$ IS TOO BIG !!

Def k : a field with $\text{char} = 0$

$(A, \Delta, \varepsilon, \Phi, R)$

$A: k[[R]]\text{-alg}$

$\Delta: A \rightarrow A \hat{\otimes}_{k[[R]]} A$: coproduct ← conti

$\varepsilon: A \rightarrow k[[R]]$: counit ← conti

$\Phi \in (A \hat{\otimes}_{k[[R]]} A \hat{\otimes}_{k[[R]]} A)^*$: Drinfeld associator

$R \in (A \hat{\otimes}_{k[[R]]} A)^*$: R-matrix

forms quasi-triangular quasi-Hopf Quantized Universal enveloping algebra if and only if

$(A, \Delta, \varepsilon, \Phi, R)$: qtb alg

$(A, \Delta, \varepsilon, \Phi, R)_{R=0}$: UEA

A : top free

Rmk The existence of the 'antipode' automatically follows.

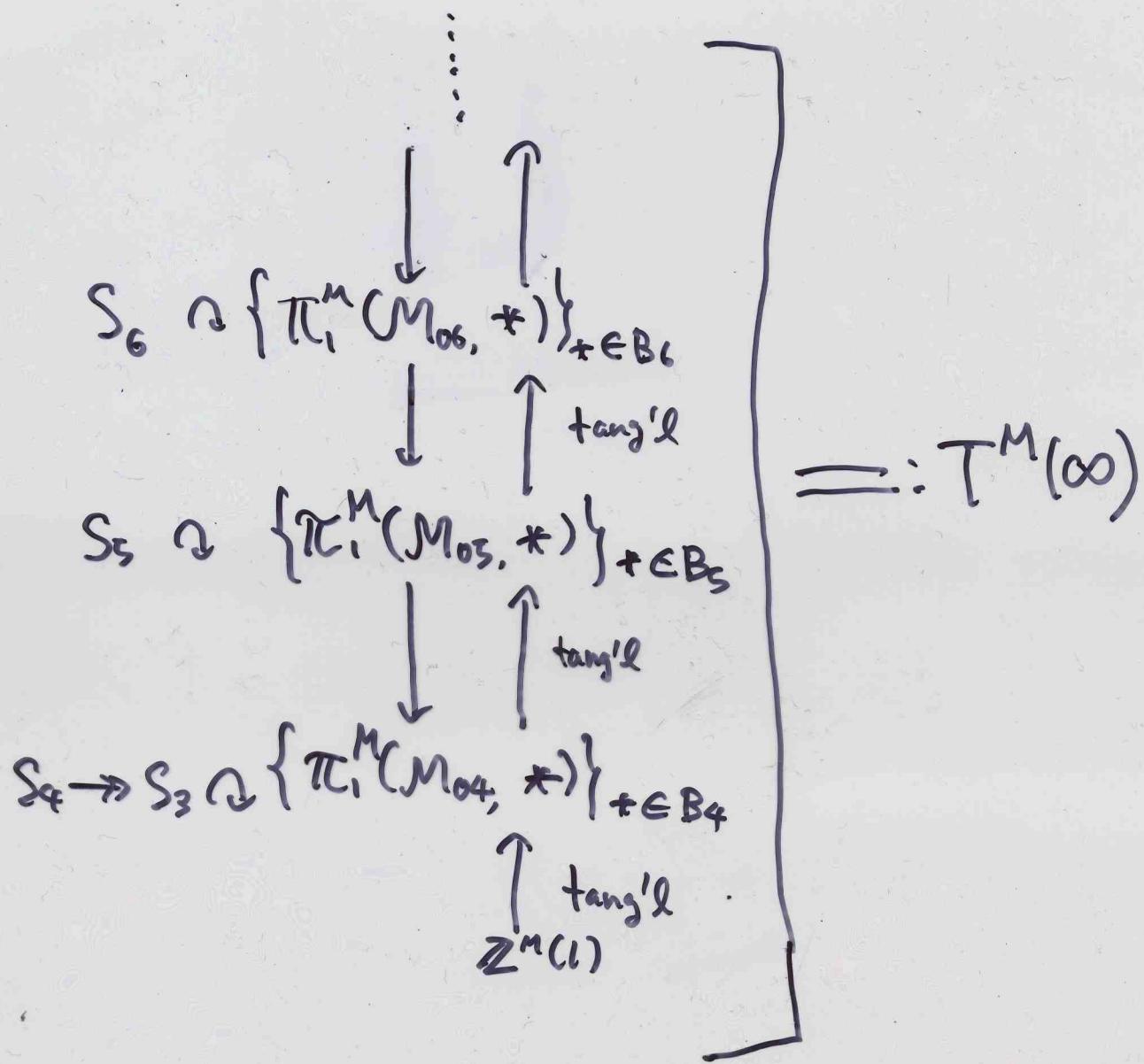
Thus it is really quasi-Hopf algebra

$\text{GT}(k) \cancel{\hookrightarrow} (\text{e}, \otimes, I, a, c)$, $\text{GT}(k) \hookrightarrow \{\text{qtbHQUE alg}/k\}$

§4. Motivic Teichmüller-Lego Phylosophy

L2

(cf Esquisse d'un programme , Grothendieck 1984)



$$\begin{array}{ccc} \text{Prop} & \underline{\text{Aut}}^{\otimes}(MTM(Z), \omega_{B_6}) & \longrightarrow \underline{\text{Aut}}(\omega_{B_6}(T^M(\infty))) \\ & \parallel & \parallel \\ & \text{Gal}^M(Z) & \text{GT}_{\otimes} \end{array}$$

$$\begin{array}{ccc} \text{Problem} & \text{Gal}^M(Z) & \xrightarrow{\stackrel{?}{\sim}} \underline{\text{GT}} \end{array}$$