PROOF OF DOUBLE SHUFFLE RELATIONS FOR p-ADIC MULTIPLE ZETA VALUES

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Abstract. We give a review of the proof of double shuffle relations for p-adic multiple zeta values in [BF]. Our techniques are a development of a higher dimensional version of Deligne's tangential basepoint [D1] and a detection of local behavior of two (and one) variable p-adic multiple polylogarithms around special divisors.

0. Introduction

In this paper we will prove a set of formulas, known as double shuffle relations, relating the p-adic multiple zeta values defined by the author in [F1]. These formulas are analogues of formulas for the usual (complex) multiple zeta values. These have a very simple proof which unfortunately does not translate directly to the p-adic world.

Recall that the (complex) multiple zeta value $\zeta(\mathbf{k})$, where $\mathbf{k}$ stands for the multi-index $\mathbf{k} = (k_1, \ldots, k_m)$, is defined by the formula

$$\zeta(\mathbf{k}) = \sum_{\substack{0<n_1<\cdots<n_m \\ n_i \in \mathbf{N}}} \frac{1}{n_1^{k_1} \cdots n_m^{k_m}} \ , \tag{0.1}$$

The series is easily seen to be convergent assuming that $k_m > 1$.

Multiple zeta values satisfy two types of so called shuffle product formulas, expressing a product of multiple zeta values as a linear combination of other such values. The first type of formulas are called series shuffle product formulas (sometimes called by harmonic product formulas). The simplest example is the relation

$$\zeta(k_1) \cdot \zeta(k_2) = \zeta(k_1, k_2) + \zeta(k_2, k_1) + \zeta(k_1 + k_2) \ , \tag{0.2}$$

which is easily obtained from the expression (0.1) by noting that the left hand side is a summation over an infinite square of pairs (n_1, n_2) of the summand in (0.1), and that summing over the lower triangle (respectively the upper triangle, respectively the diagonal) gives the three terms on the right hand side. Every series shuffle product formula has this type of proof.

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The second type of shuffle product formulas, known as iterated integral shuffle product formulas, is somewhat harder to establish and follows from the description of multiple zeta values in terms of multiple polylogarithms. More precisely. The one variable multiple polylogarithm is defined by the formula

$$\mathrm{Li}_{\mathbf{k}}(z) = \sum_{\substack{0<n_1<\cdots<n_m \\ n_i\in\mathbf{N}}} \frac{z^{n_m}}{n_1^{k_1}\cdots n_m^{k_m}}\ , \tag{0.3}$$

near $z = 0$. It can then be extended as a multi-valued function to $\mathbf{P}^1(\mathbf{C}) - \{0, 1, \infty\}$. We clearly have the relation $\lim_{z\to 1} \mathrm{Li}_{\mathbf{k}}(z) = \zeta(\mathbf{k})$.

Multiple polylogarithms can be written using the theory of iterated integrals due to Chen [Ch]. In other words, they satisfy a system of unipotent differential equations. This gives an integral expression for multiple polylogarithms. By substituting $z = 1$ and splitting the domain of integration in the right way we obtain the iterated integral shuffle product formulas, a simple example of which is the formula

(0.4)
$$\zeta(k_1)\cdot\zeta(k_2) = \sum_{i=0}^{k_1-1}\binom{k_2-1+i}{i}\zeta(k_1-i, k_2+i) + \sum_{j=0}^{k_2-1}\binom{k_1-1+j}{j}\zeta(k_2-j, k_1+j).$$

In [F1] the author defined the p-adic version of multiple zeta values and studied some of their properties. The defining formula (0.1) can not be directly used p-adically because the defining series does not converge. Instead, one must use an indirect approach based on the theory of Coleman integration [Co, Be]. Coleman's theory defines p-adic analytic continuation for solutions of unipotent differential equations "along Frobenius". Coleman used his theory initially to define p-adic polylogarithms. In [F1] Coleman integration was used to define one variable p-adic multiple polylogarithms. Taking the limit at 1 in the right way one obtains the definition of p-adic multiple zeta values. It is by no means trivial that the limit even exists or is independent of choices, and this is the main result of [F1].

Given their definition, it is not surprising that for p-adic multiple zeta values it is the iterated integral shuffle product formulas that are easier to obtain p-adically. In [F1] the series shuffle product formulas were not obtained. The purpose of this work is to prove (Theorem 3.1) these formulas, and as a consequence the double shuffle relations (Corollary 3.2) for p-adic multiple zeta values.

To prove the main theorem it is necessary to use the theory of Coleman integration in several variables developed by the first named author in [Be]. The reason for this is quite simple - If one tries to replace multiple zeta values by multiple polylogarithms in the proof of (0.2) sketched above one easily establishes the formula

$$\mathrm{Li}_{k_1}(z)\,\mathrm{Li}_{k_2}(w) = \mathrm{Li}_{k_1,k_2}(z,w) + \mathrm{Li}_{k_2,k_1}(w,z) + \mathrm{Li}_{k_1+k_2}(zw)\ , \tag{0.5}$$

which is a two variable formula. It seems impossible to obtain a one variable version of the same formula. The proof of the main theorem thus consists roughly speaking of showing that (0.5) extends to Coleman functions of several variables and then taking the limit at $(1,1)$.

Since taking the limit turned out to be rather involved in [F1], we opted for an alternative approach, which was motivated by a letter of Deligne to the author [D2]. Deligne observes that taking the limit at 1 for the multiple polylogarithm can be interpreted as doing analytic continuation from tangent vectors at 0 and 1, using the theory of the tangential basepoint at infinity introduced in [D1]. To analytically continue (0.5) and obtain the series shuffle product formula we analyze a more general notion of tangential basepoint sketched in loc. cit. and examine among other things its relation with Coleman integration.

To give a precise meaning of the limit value to $(1,1)$, we work over the moduli space $\mathcal{M}_{0,5}$ of curves of $(0,5)$-type and the normal bundles for the divisors at infinity $\overline{\mathcal{M}_{0,5}} - \mathcal{M}_{0,5}$ ($\overline{\mathcal{M}_{0,5}}$: a compactification of $\mathcal{M}_{0,5}$). Two variable p-adic multiple polylogarithms are introduced. They are Coleman functions over $\mathcal{M}_{0,5}$. Their analytic continuation to the normal bundle will be discussed. In particular, we will relate the behavior of the analytic continuation of two variable multiple polylogarithm to a normal bundle with one variable multiple polylogarithm and then we get p-adic multiple zeta values as "special values" of two variable multiple polylogarithms.

1. Coleman's p-adic integration and tangential basepoints

We recall some definitions and properties of Coleman functions and tangential base points as developed in [Be],[B2] and [BF]. We fix a branch of p-adic logarithm $\log : \mathbf{Q}_p^\times \to \mathbf{Q}_p$ with value $a \in \mathbf{Q}_p$ at p for the rest of this paper.

Let X be a smooth variety over K, a finite extension of $\mathbf{Q}_p$. Let $\mathcal{NC}(X)$ denote the category of unipotent flat vector bundles on X, i.e. a vector bundle together with a flat connection on it such that it is an iterative extension of trivial vector bundles together with a trivial flat connection. This category is a neutral tannakian category (for the basics see[DM]) and any point $x \in X(K)$ defines a fiber functor ω_x from

$\mathcal{NC}(X)$ to the category Vec_K of finite dimensional K-vector spaces (cf.[D1]). In [V], Vologodsky has constructed a canonical system (after fixing a branch of p-adic logarithm) of isomorphism $a^X_{x,y} : \omega_x \longrightarrow \omega_y$ for any pair of points in $X(K)$. The properties of these isomorphism are summarized in [B2]§2. Following [B2], an abstract Coleman function is a triple (M, s, y) where $M \in \mathcal{NC}(X)$, $s \in \mathrm{Hom}_{\mathcal{O}_X}(M, \mathcal{O}_X)$ and y is a collection of $y_x \in M_x$ for all $x \in X(L)$ for any finite extension L of K, where M_x is the fiber of M over x and is an L-vector space defined by the fiber functor $\omega_x : \mathcal{N}C(X_L) \to Vec_L$.

This data must satisfy:

- For any two points $x_1, x_2 \in X(L) = X_L(L)$ we have $a^{X_L}_{x_1,x_2}(y_{x_1}) = y_{x_2}$.
- For any field homomorphism $\sigma : L \longrightarrow L'$ that fixes K and $x \in X(L)$ we have: $\sigma(y_x) = y_{\sigma(x)}$.

There is a natural notion of morphism between the abstract Coleman functions. The connected component of an abstract Coleman function is called a Coleman function. A Coleman function is also interpreted as a function on $X(\overline{K})$ by assigning to x the value $s(y_x)$. This is indeed a locally analytic function. We will use both approaches for Coleman functions, i.e. the interpretation as a triple (M, s, y) as above and the interpretation as a locally analytic function, in this paper. The set of Coleman functions on X is a ring which we denote by $\mathrm{Col}^a(X)$. Here $a \in \mathbf{Q}_p$ is the value of the chosen branch of the p-adic logarithm at p.

Let X be a smooth $\mathcal{O}_K$-scheme and $D = \sum_{i\in I} D_i$ be a divisor with relative normal crossings over $\mathcal{O}_K$, with D_i's smooth and irreducible over $\mathcal{O}_K$. Let J be a nonempty subset of I. In [BF] a tangential morphism $Res_{D,J} : \mathcal{NC}((X - D)_K) \longrightarrow \mathcal{NC}(\mathcal{N}^{00}_J)$ was constructed. Here $\mathcal{N}^{00}_J$ is the normal bundle of $D_J = \cap_{j\in J} D_j$ minus the normal bundles of $D_{J-\{j\}}$ for all $j \in J$ (the normal bundle $\mathcal{N}_\emptyset$ is considered as the zero section of of $\mathcal{N}_{D_J}$) , and then restricted to $D_J - \cup_{j\notin J}(D_j \cap D_J)$. The construction is given as follows (cf. loc. cit. §3): For each $j \in J$ consider the valuation v_j on $K(X)$ associated with the divisor D_j. Let $\mathcal{O}_X(D^{-1})$ be the localization of $\mathcal{O}_X$ at D. There exists a multi-filtration F_J on $\mathcal{O}_X(D^{-1})$, indexed by tuples $\chi = (\chi_j \in \mathbf{Z})_{j\in J}$, such that F^χ_J is the $\mathcal{O}_X$-module generated by $\{f \in \mathcal{O}_X(D^{-1}),\ v_j(f) \geqslant \chi_j$ for all $j \in J\}$. It is easy to see that $Spec(Gr_J \mathcal{O}_X(D^{-1}))$ is precisely $\mathcal{N}^{00}_J$. Suppose we have a connection $\nabla : M \to M \otimes_{\mathcal{O}_X} \Omega^1_X(\log D)$ with logarithmic singularities along D. We give $\Omega^1_X(D^{-1}) = \Omega^1_X(\log D) \otimes \mathcal{O}_X(D^{-1})$ the induced filtrations from the filtration on $\mathcal{O}_X(D^{-1})$. It is easy to see that the differential d preserves the filtration. Now $M(D^{-1}) = M \otimes \mathcal{O}_X(D^{-1})$ and $M \otimes \Omega^1_X(D^{-1})$ have the induced filtrations. It follows that the extended connection $\nabla : M(D^{-1}) \to M \otimes \Omega^1_X(D^{-1})$ respects the filtration. The connection $Res_{D,J}(M)$ is the graded quotient of this connection.

Let κ be the residue field of $\mathcal{O}_K$. It was shown in [BF] that if the Frobenius endomorphism of $(X,D)_\kappa$ locally lifts to an algebraic endomorphism of (X,D) then this morphism respect the action of the Frobenius endomorphism. Indeed by [S], [CLS] the categories $\mathcal{NC}(X-D)$ and $\mathcal{NC}(\mathcal{N}^{00}_J)$ are isomorphic to the categories of the unipotent isocrystals $\mathcal{NC}^\dagger((X-D)_\kappa)\otimes K$ and $\mathcal{NC}^\dagger((\mathcal{N}^{00}_J)_\kappa)\otimes K$ on the reductions $(X-D)_\kappa$ and $(\mathcal{N}^{00}_J)_\kappa$ and therefore admit a natural action of the Frobenius endomorphism. Choose a point $\tilde{t} \in (\mathcal{N}^{00}_J)_\kappa(\overline{\kappa})$ which is the reduction of a point $t \in \mathcal{N}^{00}_J(L)$ for some extension L of K. The point $\tilde{t}$ defines a fiber functor $\omega_{\tilde{t}}$ from $\mathcal{NC}^\dagger((X-D)_\kappa)$ to Vec_L, which is Frobenius invariant if we take a high power of the Frobenius. Then following [Be] for any point $\tilde{x} \in (X-D)_\kappa(\overline{\kappa})$, which is the reduction of $x \in \mathcal{N}^{00}_J(L)$, we get a canonical Frobenius invariant isomorphism $\tilde{a}_{\tilde{x},\tilde{t}} : \omega_{\tilde{x}} \longrightarrow \omega_{\tilde{t}}$. The above categorical equivalence gives an isomorphism $a_{x,t} : \omega_x \longrightarrow \omega_t$. Now for any $x' \in (X-D)(L)$ and $t' \in \mathcal{N}^{00}_J(L)$ we define :

$$a_{x',t'} = a_{x',x} \circ a_{x,t} \circ a_{t,t'}.$$

This is independent of the choice of x and t. Using this we get a way (developped in [BF] §4) to extend a certain type of Coleman functions (which were called Coleman functions of 'algebraic origin' in loc. cit.) (M,s,y) on $X-D$ to a Coleman function (M',s',y') on $\mathcal{N}^{00}_J$ as follows. Let $M \in \mathcal{NC}(X-D)$ and y be a compatible system over $X-D$ as before. The morphism $s : M \to \mathcal{O}_X$ induces a morphism $s_D : M(D^{-1}) \to \mathcal{O}_X(D^{-1})$ which we assume to be compatible with the filtration F_J. Then the Coleman function (M',s',y') is defined: $M' = Res_{D,J}(M)$ as described above and the morphism s' is $Gr(s_D) : Res_{D,J}M \to \mathcal{O}_{\mathcal{N}^{00}_J}$. The section y' will be a collection of y'_t ($t \in \mathcal{N}^{00}_J(L)$) with $y'_t = a_{x,t}(y_x)$ for some $x \in (X-D)(L)$.

2. The analytic continuation

We introduce two (and one) variable p-adic multiple polylogarithms and discuss their analytic continuation to the normal bundle of the divisor o the Deligne-Mumford compactification of the moduli space $\mathcal{M}_{0,5}$ of genus 0 curves with 5 distinct marked points.

The moduli space $\mathcal{M}_{0,5} = \{(P_i)_{i=1}^5 \in (\mathbf{P}^1)^5 | P_i \neq P_j \ (i \neq j)\}/PGL(2)$ is identified with $\{(x,y) \in \mathbf{A}^2\} \backslash \{x=0\} \cup \{y=0\} \cup \{x=1\} \cup \{y=1\} \cup \{xy=1\}$. This identification is given by sending (x,y) to 5 marked points in $\mathbf{P}^1$ given by $(0,x,1,\frac{1}{y},\infty)$. The symmetric group S_5

acts on $\mathcal{M}_{0,5}$ by $\sigma(P_i)=P_{\sigma^{-1}(i)}$ $(1\leqslant i\leqslant 5)$ for $\sigma\in S_5$. Especially for $c=(1,3,5,2,4)\in S_5$ its action is described by $x\mapsto\frac{1-y}{1-xy}$, $y\mapsto x$.

The Deligne-Mumford compactification of $\mathcal{M}_{0,5}$ is denoted by $\overline{\mathcal{M}_{0,5}}$. This space classifies stable curves of $(0,5)$-type and the above S_5-action extends to the action on $\overline{\mathcal{M}_{0,5}}$. This space is the blow-up of $(\mathbf{P}^1)^2(\supset\mathcal{M}_{0,5})$ at $(x,y)=(1,1)$, $(0,\infty)$ and $(\infty,0)$. The complement $\overline{\mathcal{M}_{0,5}}-\mathcal{M}_{0,5}$ is a divisor with 10 components: $\{x=0\}$, $\{y=0\}$, $\{x=1\}$, $\{y=1\}$, $\{xy=1\}$, $\{x=\infty\}$, $\{y=\infty\}$ and 3 exceptional divisors obtained by blowing up at $(x,y)=(1,1)$, $(\infty,0)$ and $(0,\infty)$. In particular for our convenience we denote $\{y=0\}$, $\{x=1\}$, the exceptional divisor at $(1,1)$, $\{y=1\}$ and $\{x=0\}$ by D_1, D_2, D_3, D_4 and D_5 (or sometimes D_0) respectively. It is because $c^i(D_0)=D_i$. These five divisors form a pentagon and we denote each vertex $D_i\cap D_{i-1}$ by P_i. Hence we have $c^i(P_0)=P_i$. The two dimensional affine space $U_1=Spec\mathbf{Q}[x,y]$ gives an open affine subset of $\overline{\mathcal{M}_{0,5}}$. The S_5-action gives other open subsets $U_i=c^{i-1}(U_1)=Spec\mathbf{Q}[z_i,w_i]$ $(1\leqslant i\leqslant 5)$ in $\overline{\mathcal{M}_{0,5}}$ where $(z_1,w_1)=(x,y)$, $(z_2,w_2)=(y,\frac{1-x}{1-xy})$, $(z_3,w_3)=(\frac{1-x}{1-xy},1-xy)$, $(z_4,w_4)=(1-xy,\frac{1-y}{1-xy})$ and $(z_5,w_5)=(\frac{1-y}{1-xy},x)$.

For $\mathbf{a}=(a_1,\cdots,a_k)\in\mathbf{Z}^k_{>0}$, $\mathbf{b}=(b_1,\cdots,b_l)\in\mathbf{Z}^l_{>0}$, and x, $y\in\mathbf{Q}_p$ with $|x|_p<1$ and $|y|_p<1$ we define **two variable p-adic multiple polylogarithm** by

$$\mathrm{Li}_{\mathbf{a},\mathbf{b}}(x,y):=\sum_{\substack{0<m_1<\cdots<m_k\\<n_1<\cdots<n_l}}\frac{x^{m_k}\qquad y^{n_l}}{m_1^{a_1}\cdots m_k^{a_k}n_1^{b_1}\cdots n_l^{b_l}}\in\mathbf{Q}_p[[x,y]],$$

and for $\mathbf{c}=(c_1,\cdots,c_h)\in\mathbf{Z}^h_{>0}$ **one variable p-adic multiple polylogarithm** by

$$\mathrm{Li}_{\mathbf{c}}(y):=\sum_{0<m_1<\cdots<m_h}\frac{y^{m_h}}{m_1^{c_1}\cdots m_h^{c_h}}\in\mathbf{Q}_p[[y]]\subset\mathbf{Q}_p[[x,y]].$$

By the differential equations [BF](5.2)$\sim$(5.4), $Li_{\mathbf{a},\mathbf{b}}(x,y)$, $Li_{\mathbf{c}}(xy)$ and $Li_{\mathbf{c}}(y)$ are all iterated integrals of $\frac{dx}{x}$, $\frac{dx}{1-x}$, $\frac{dy}{y}$, $\frac{dy}{1-y}$ and $\frac{xdy+ydx}{1-xy}$, differential forms over $\mathcal{M}_{0,5}$. Whence they are obtained from some triple (M,s,y) over $\mathcal{M}_{0,5}$. We interpret them as Coleman functions over the rigid triple $(\mathcal{M}_{0,5},\overline{\mathcal{M}_{0,5}})$. This means that they are analytically continued to $\mathcal{M}_{0,5}(\mathbf{Q}_p)$ as Coleman functions by the methods of analytically continuation along Frobenius in §1.

For a Coleman function f over $\mathcal{M}_{0,5}$, $f^{(D_i)}$ means the analytic continuation of f to $N^{00}_{D_i}$ $(i\in\mathbf{Z}/5)$. For $\mathbf{a}=(a_1,\cdots,a_k)\in\mathbf{Z}^k_{>0}$ and $\mathbf{b}=(b_1,\cdots,b_l)\in\mathbf{Z}^l_{>0}$, $F_{\mathbf{a},\mathbf{b}}$ stands for the Coleman function $Li_{\mathbf{a},\mathbf{b}}(x,y)-$

$Li_{\mathbf{ab}}(xy)$ and for $\mathbf{c}=(c_1,\cdots,c_h)\in\mathbf{Z}^h_{>0}$, $G_{\mathbf{c}}$ stands for the Coleman function $Li_{\mathbf{c}}(xy)-Li_{\mathbf{c}}(y)$ over $\mathcal{M}_{0,5}$.

Lemma 2.1. *$F^{(D_1)}_{\mathbf{a},\mathbf{b}}=0$ and $G^{(D_1)}_{\mathbf{c}}=0$ for any index $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.*

Proof . The constant terms of $Li_{\mathbf{a},\mathbf{b}}(x,y)$, $Li_{\mathbf{c}}(xy)$ and $Li_{\mathbf{c}}(y)$ at the origin P_5 are zero because there are no constant terms in their power series expansions. We take their differentials and take their residues at $y=0$. It gives 0 by induction because each term will be a multiple polylogarithm with one lower weight than the original one. Whence $Li_{\mathbf{a},\mathbf{b}}(x,y)$, $Li_{\mathbf{c}}(xy)$ and $Li_{\mathbf{c}}(y)$ are identically zero. It gives our claim. $\square$

Lemma 2.2. *$F^{(D_2)}_{\mathbf{a},\mathbf{b}}\equiv 0$ if $\mathbf{a}$ is admissible* [1] *and $G^{(D_2)}_{\mathbf{c}}\equiv 0$ for any index $\mathbf{c}$.*

Proof . On the affine coordinate (z_2,w_2) for U_2, the divisor D_2 is defined by $w_2=0$. We have $dx=\frac{w_2(1-w_2)}{(z_2w_2-1)^2}dz_2+\frac{z_2-1}{(z_2w_2-1)^2}dw_2$ and $dy=dz_2$. By taking the residue of the differential equations [BF](5.2)∼(5.4), we get that differentials of $F^{(D_2)}_{\mathbf{ab}}$ and $G^{(D_2)}_{\mathbf{c}}$ with respect to $\bar{z}_2$ and $\bar{w}_2$ are zero by induction. Therefore they must be constant. By Lemma 2.1 their constant terms at P_1 is zero. So they are identically zero. $\square$

Lemma 2.3. *$F^{(D_3)}_{\mathbf{a},\mathbf{b}}=0$ if $\mathbf{a}$ and $\mathbf{b}$ are admissible and $G^{(D_3)}_{\mathbf{c}}=0$ if $\mathbf{c}$ is admissible.*

Proof . On the affine coordinate (z_3,w_3) for U_3, the divisor D_3 is defined by $w_3=0$. We have $dx=-w_3dz_3-z_3dw_3$ and $dy=\frac{w_3(1-w_3)}{(z_3w_3-1)^2}dz_3+\frac{z_3-1}{(z_3w_3-1)^2}dw_3$. By taking the residue of the differential equations [BF](5.2)∼(5.4), we get to know that differentials of $F^{(D_3)}_{\mathbf{a},\mathbf{b}}$ and $G^{(D_3)}_{\mathbf{c}}$ with respect to $\bar{z}_3$ and $\bar{w}_3$ are zero by induction. Therefore they must be constant. By Lemma 2.2 their constant term at P_2 is zero. So they are identically zero. $\square$

In [F1] it was shown that the limit (in a certain way) to $z=1$ of $\mathrm{Li}_{k_1,\cdots,k_m}(z)$, which is a Coleman function over $\mathbf{P}^1\backslash\{0,1,\infty\}$, exists when $k_m>1$ (loc. cit. Theorem 2.18) and p-adic multiple zeta value $\zeta_p(k_1,\cdots,k_m)$ is defined to be this limit value (loc. cit. Definition 2.17), but by using the terminologies in §1 we reformulate its definition as follows.

Definition 2.4. For $k_m>1$, the **p-adic multiple zeta value** $\zeta_p(k_1,\cdots,k_m)$ is the constant term of $\mathrm{Li}_{k_1,\cdots,k_m}(z)$ at $z=1$.

[1] An index $\mathbf{a}=(a_1,\cdots,a_k)$ $(a_i\in\mathbf{N})$ is called *admissible* if $a_k>1$.

In the case for $k_m = 1$, the constant term of $\mathrm{Li}_{k_1,\cdots,k_m}(z)$ at $z = 1$ is actually equal to the (canonical) regularization $(-1)^m I_p(BA^{k_{m-1}-1}B\cdots A^{k_1-1}B)$ of p-adic multiple zeta values by loc. cit. Theorem 2.22 (for this notation, see loc. cit. Theorem 3.30).

The following is important to prove double shuffle relations for p-adic multiple zeta values.

Proposition 2.5. (1) *The analytic continuation* $\mathrm{Li}^{(D_3)}_{\mathbf{a},\mathbf{b}}(x, y)$ *is constant and equal to* $\zeta_p(\mathbf{a}, \mathbf{b})$ *if* $\mathbf{a}$ *and* $\mathbf{b}$ *are admissible.*

(2) *The analytic continuation* $\mathrm{Li}^{(D_3)}_{\mathbf{c}}(xy)$ *and* $\mathrm{Li}^{(D_3)}_{\mathbf{c}}(y)$ *are constant and take value* $\zeta_p(\mathbf{c})$ *if* $\mathbf{c}$ *is admissible.*

Proof . By Lemma 2.3 it is enough to prove this for $Li^{(D_3)}_{\mathbf{c}}(y)$. By the argument in Lemma 2.1 $Li^{(D_1)}_{\mathbf{c}}(y) = 0$. By the computation in Lemma 2.2 $Li^{(D_2)}_{\mathbf{c}}(y) = Li^{(D_2)}_{\mathbf{c}}(\bar{z}_2)$. So the constant term of $Li^{(D_2)}_{\mathbf{c}}(y)$ at P_2 is equal to the constant term of $Li_{\mathbf{c}}(\bar{z}_2)$ at $\bar{z}_2 = 1$, which is $\zeta_p(\mathbf{c})$. By the computation in Lemma 2.3 $Li^{(D_3)}_{\mathbf{c}}(y)$ must be constant if $c_h > 1$. Since this constant term must be the constant term of $Li^{(D_2)}_{\mathbf{c}}(y)$, $Li^{(D_3)}_{\mathbf{c}}(y) \equiv \zeta_p(\mathbf{c})$ for $c_h > 1$. □

By discussing on the opposite divisors D_5, D_4 and D_3, we also obtain the following.

Proposition 2.6. (1) *The analytic continuation* $\mathrm{Li}^{(D_3)}_{\mathbf{a},\mathbf{b}}(y, x)$ *is constant and equal to* $\zeta_p(\mathbf{a}, \mathbf{b})$ *if* $\mathbf{a}$ *and* $\mathbf{b}$ *are admissible.*

(2) *The analytic continuation* $\mathrm{Li}^{(D_3)}_{\mathbf{c}}(xy)$ *and* $\mathrm{Li}^{(D_3)}_{\mathbf{c}}(x)$ *are constant and equal to* $\zeta_p(\mathbf{c})$ *if* $\mathbf{c}$ *is admissible.*

3. The double shuffle relations

In this section, we prove double shuffle relations for p-adic multiple zeta values (Definition 2.4). Firstly we recall double shuffle relations for complex multiple zeta values. Let $\mathbf{a} = (a_1, \cdots, a_k)$ and $\mathbf{b} = (b_1, \cdots, b_l)$ be admissible indices (i.e. $a_k > 1$ and $b_l > 1$). The **series shuffle product formulas** (called by harmonic product formulas in [F1] and first shuffle relations in [G1]) are relations

$$\zeta(\mathbf{a}) \cdot \zeta(\mathbf{b}) = \sum_{\sigma \in Sh^{\leqslant}(k,l)} \zeta(\sigma(\mathbf{a}, \mathbf{b})) \tag{3.1}$$

which is obtained by expanding the summation on the left hand side into the summation which give multiple zeta values. Here

$$Sh^{\leqslant}(k,l) := \bigcup_N \Big\{\sigma : \{1,\cdots,k+l\} \to \{1,\cdots,N\} \Big| \ \sigma \text{ is onto},$$
$$\sigma(1) < \cdots < \sigma(k), \sigma(k+1) < \cdots < \sigma(k+l)\Big\}$$

and $\sigma(\mathbf{a},\mathbf{b}) = (c_1,\cdots,c_N)$ where N is the cardinality of the image of σ and

$$c_i = \begin{cases} a_s + b_{t-k} & \text{if } \sigma^{-1}(i) = \{s,t\} \text{ with } s < t, \\ a_s & \text{if } \sigma^{-1}(i) = \{s\} \quad \text{with } s \leqslant k. \\ b_{s-k} & \text{if } \sigma^{-1}(i) = \{s\} \quad \text{with } s > k. \end{cases}$$

One of the easiest example of (3.1) is (0.2).

On the other hand, multiple zeta values admit an iterated integral expression (cf. [G1], [IKZ] see also [F0])

$$\zeta(\mathbf{a}) = \int_0^1 \underbrace{\frac{du}{u} \circ \cdots \circ \frac{du}{u} \circ \frac{du}{1-u}}_{a_k} \circ \frac{du}{u} \circ \cdots\cdots \circ \frac{du}{1-u}$$
$$\circ \underbrace{\frac{du}{u} \circ \cdots \circ \frac{du}{u} \circ \frac{du}{1-u}}_{a_1}.$$

Here for differential 1-forms $\omega_1, \omega_2, \ldots, \omega_n$ ($n \geqslant 1$) on $\mathbf{C}$ an iterated integral $\int_0^1 \omega_1 \circ \omega_2 \circ \cdots \circ \omega_n$ is defined inductively as $\int_0^1 \omega_1(t_1) \int_0^{t_1} \omega_2 \circ \cdots \circ \omega_n$. There are the well-known shuffle product formulas (for example see loc. cit.) of iterated integration

$$\int_0^1 \omega_1 \circ \cdots \circ \omega_k \cdot \int_0^1 \omega_{k+1} \circ \cdots \circ \omega_{k+l} = \sum_{\tau \in Sh(k,l)} \int_0^1 \omega_{\tau(1)} \circ \cdots \circ \omega_{\tau(k+l)},$$

where $Sh(k,l)$ is the set of shuffles defined by

$$Sh(k,l) := \Big\{\tau : \{1,\cdots,k+l\} \to \{1,\cdots,k+l\} \Big| \ \tau \text{ is bijective},$$
$$\tau(1) < \cdots < \tau(k), \tau(k+1) < \cdots < \tau(k+l)\Big\}.$$

They induce the **iterated integral shuffle produce formulas** (called by shuffle product formulas simply in [F1] and second shuffle relations in [G1]) for multiple zeta values

$$\zeta(\mathbf{a}) \cdot \zeta(\mathbf{b}) = \sum_{\tau \in Sh(N_{\mathbf{a}}, N_{\mathbf{b}})} \zeta(I_{\tau(W_{\mathbf{a}}, W_{\mathbf{b}})}) \tag{3.2}$$

where $N_{\mathbf{a}} = a_1 + \cdots + a_k$, $N_{\mathbf{b}} = b_1 + \cdots + b_l$. For $\mathbf{c} = (c_1, \cdots, c_h)$ with $h, c_1, \ldots, c_h \geqslant 1$ the symbol $W_{\mathbf{c}}$ means a word $A^{c_h-1}BA^{c_{h-1}-1}B\cdots A^{c_1-1}B$ and conversely for given such W we denote its corresponding index by I_W. For words, $W = X_1\cdots X_k$ and $W' = X_{k+1}\cdots X_{k+l}$ with $X_i \in \{A, B\}$, and $\tau \in Sh(k, l)$ the symbol $\tau(W, W')$ stands for the word $Z_1\cdots Z_{k+l}$ with $Z_i = X_{\tau^{-1}(i)}$. One of the easiest example of (3.2) is (0.4).

The **double shuffle relations** for multiple zeta values are linear relations which are obtained by combining two shuffle relations (3.3), i.e. series shuffle product formulas (3.1) and iterated integral shuffle produce formulas (3.2)

$$\sum_{\sigma\in Sh^{\leqslant}(k,l)} \zeta(\sigma(\mathbf{a},\mathbf{b})) = \sum_{\tau\in Sh(N_{\mathbf{a}},N_{\mathbf{b}})} \zeta(I_{\tau(W_{\mathbf{a}},W_{\mathbf{b}})}). \tag{3.3}$$

The following is the easiest example of the double shuffle relations obtained from (0.2) and (0.4):

$$\begin{aligned}&\zeta(k_1,k_2)+\zeta(k_2,k_1)+\zeta(k_1+k_2)\\ =&\sum_{i=0}^{k_1-1}\binom{k_2-1+i}{i}\zeta(k_1-i,k_2+i)+\sum_{j=0}^{k_2-1}\binom{k_1-1+j}{j}\zeta(k_2-j,k_1+j)\end{aligned}$$

for $k_1, k_2 > 1$.

Theorem 3.1. *p-adic multiple zeta values in convergent case (i.e. for admissible indices) satisfy the series shuffle product formulas, i.e.*

$$\zeta_p(\mathbf{a})\cdot\zeta_p(\mathbf{b}) = \sum_{\sigma\in Sh^{\leqslant}(k,l)} \zeta_p(\sigma(\mathbf{a},\mathbf{b})) \tag{3.4}$$

for admissible indices $\mathbf{a}$ *and* $\mathbf{b}$.

Proof . Put $\mathbf{a} = (a_1, \cdots, a_k)$ and $\mathbf{b} = (b_1, \cdots, b_l)$. By the power series expansion of $\mathrm{Li}_{\mathbf{a},\mathbf{b}}(x, y)$ and $\mathrm{Li}_{\mathbf{a}}(x)$, we obtain the following formula

$$\mathrm{Li}_{\mathbf{a}}(x)\cdot\mathrm{Li}_{\mathbf{b}}(y) = \sum_{\sigma\in Sh^{\leqslant}(k,l)} \mathrm{Li}^{\sigma}_{\mathbf{a},\mathbf{b}}(x,y). \tag{3.5}$$

Here

$$\mathrm{Li}^{\sigma}_{\mathbf{a},\mathbf{b}}(x,y) := \sum_{(m_1,\cdots,m_k,n_1,\cdots,n_l)\in Z^{\sigma}_{++}} \frac{x^{m_k} \qquad y^{n_l}}{m_1^{a_1}\cdots m_k^{a_k} n_1^{b_1}\cdots n_l^{b_l}}$$

with

$$Z^{\sigma}_{++} = \left\{(c_1,\cdots,c_{k+l})\in\mathbf{Z}^{k+l}_{>0} \;\middle|\; c_i<c_j \text{ if } \sigma(i)<\sigma(j), c_i=c_j \text{ if } \sigma(i)=\sigma(j)\right\}.$$

Then for each $\sigma \in Sh^{\leqslant}(k,l)$, $\mathrm{Li}^{\sigma}_{\mathbf{a},\mathbf{b}}(x,y)$ can be written $\mathrm{Li}_{\mathbf{a}',\mathbf{b}'}(x,y)$, $\mathrm{Li}_{\mathbf{a}',\mathbf{b}'}(y,x)$ or $\mathrm{Li}_{\mathbf{a}',\mathbf{b}'}(xy)$ for some indices $\mathbf{a}'$ and $\mathbf{b}'$. We note that, if $\mathbf{a}$ and $\mathbf{b}$ are admissible, then these $\mathbf{a}'$ and $\mathbf{b}'$ are also admissible. By Proposition 2.5 and Proposition 2.6, we know that analytic continuations $\mathrm{Li}^{(D_3)}_{\mathbf{a},\mathbf{b}}(x,y)$, $\mathrm{Li}^{(D_3)}_{\mathbf{b},\mathbf{a}}(y,x)$, $\mathrm{Li}^{(D_3)}_{\mathbf{a},\mathbf{b}}(xy)$, $\mathrm{Li}^{(D_3)}_{\mathbf{a}}(x)$ and $\mathrm{Li}^{(D_3)}_{\mathbf{b}}(y)$ are all constant and take values $\zeta_p(\mathbf{a},\mathbf{b})$, $\zeta_p(\mathbf{b},\mathbf{a})$, $\zeta_p(\mathbf{a},\mathbf{b})$, $\zeta_p(\mathbf{a})$ and $\zeta_p(\mathbf{b})$ respectively when $\mathbf{a}$ and $\mathbf{b}$ are admissible. Therefore by taking an analytic continuation along Frobenius of both hands sides of (3.5) into $N^{00}_{D_3}(\mathbf{Q}_p)$, we obtain the series shuffle product formulas (3.4) for p-adic multiple zeta value in convergent case. □

By this theorem we say for example

$$\zeta_p(k_1)\cdot\zeta_p(k_2)=\zeta_p(k_1,k_2)+\zeta_p(k_2,k_1)+\zeta_p(k_1+k_2)$$

for $k_1,k_2>1$ which is a p-adic analogue of (0.2).

Corollary 3.2. *p-adic multiple zeta values in convergent case satisfy double shuffle relations. Namely*

$$\sum_{\sigma\in Sh^{\leqslant}(k,l)}\zeta_p(\sigma(\mathbf{a},\mathbf{b}))=\sum_{\tau\in Sh(N_{\mathbf{a}},N_{\mathbf{b}})}\zeta_p(I_{\tau(W_{\mathbf{a}},W_{\mathbf{b}})}).$$

holds for $a_k>1$ and $b_l>1$.

Proof . It was shown in [F1] Corollary 3.46 that p-adic multiple zeta values satisfy iterated integral shuffle product formulas

$$\zeta_p(\mathbf{a})\cdot\zeta_p(\mathbf{b})=\sum_{\tau\in Sh(N_{\mathbf{a}},N_{\mathbf{b}})}\zeta_p(I_{\tau(W_{\mathbf{a}},W_{\mathbf{b}})}). \tag{3.6}$$

By combining it with Theorem 3.1, we obtain double shuffle relations for p-adic multiple zeta values. □

Therefore we say for example

$$\begin{aligned}&\zeta_p(k_1,k_2)+\zeta_p(k_2,k_1)+\zeta_p(k_1+k_2)\\&=\sum_{i=0}^{k_1-1}\binom{k_2-1+i}{i}\zeta_p(k_1-i,k_2+i)+\sum_{j=0}^{k_2-1}\binom{k_1-1+j}{j}\zeta_p(k_2-j,k_1+j)\end{aligned}$$

for $k_1,k_2>1$ which is a p-adic analogue of (0.4).

Remark 3.3. In complex case there are two regularizations of multiple zeta values in divergent case, integral regularization and power series regularization (see [IKZ], [G1]§2.9 and §2.10). The first ones satisfy iterated integral shuffle product formulas, the second ones satisfy series shuffle product formulas and these two regularizations are related by regularization relations. Actually these provide new type of relations among multiple zeta values. In the case of p-adic multiple zeta values, p-adic analogue of integral regularization appear on coefficients of p-adic Drinfel'd associator (see [F1]) and they satisfy iterated integral shuffle product formulas like (3.6). On the other hand, it is not clear at all to say that p-adic analogue of power series regularization satisfy series shuffle product formulas and regularization relation. It is because that in the complex case the definition of this regularization and the proof of their series shuffle product formulas and regularization relation essentially based on the asymptotic behaviors of power series summations of multiple zeta values (see [G1] Proposition 2.19) however in the p-adic case our p-adic multiple zeta values do not have power series sum expression like (0.1). Recently the validity of these type of relations among p-adic multiple zeta values were achieved in [FJ] by using several variable p-adic multiple polylogarithm and a stratification of the stable compactification of the moduli $\mathcal{M}_{0,N+3}$ $(N \geqslant 3)$. By combining results of [F2] we solved in [FJ] the problem [D3] posed by Deligne in 2002 Arizona Winter School.

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