ON WEIGHTS OF l-ADIC POLYLOGARITHMS AND p-ADIC POLYLOGARITHMS

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Abstract. In this note, we report a l-adic behavior of Wojtkowiak's l-adic polylogarithms and a p-adic behavior of Coleman's p-adic polylogarithms with respect to weights.

1. l-adic polylogarithms

In this section, we employ notations in [NW]. Namely, let l be a prime and F be a subfield of $\mathbf{C}$. Put $\sigma \in Gal(\overline{F}/F)$ and $z \in \mathbf{P}^1(F)\backslash\{0,1,\infty\}$.

The distribution $\{\kappa_{l^n}^{z,(a)}(\sigma) \in \mathbf{Z}_l^\times\}_{n=0,1,2,\ldots}^{a\in\mathbf{Z}/l^n}$ (loc. cit. Definition 2) which determines a measure $\mu_F^{z,\sigma}$ of $\mathbf{Z}_l$ satisfies the following:

$$\kappa_{l^n}^{z,(a)}(\sigma) = \kappa_{l^{n+1}}^{z^l,(la)}(\sigma). \tag{1}$$

Let us note that actually the definition of $\kappa_{l^n}^{z,(a)}(\sigma)$ in LHS depends on a choice of a topological path in $\pi_1^{\mathrm{top}}(\mathbf{P}^1(\mathbf{C})\backslash\{0,1,\infty\};\overrightarrow{01},z)$ and we take its corresponding image in $\pi_1^{\mathrm{top}}(\mathbf{P}^1(\mathbf{C})\backslash\{0,1,\infty\};\overrightarrow{01},z^l)$ by the endomorphism of $\mathbf{P}^1(\mathbf{C})\backslash\{0,1,\infty\}$ sending $x \mapsto x^l$ when we consider the definition of $\kappa_{l^{n+1}}^{z^l,(la)}(\sigma)$ in RHS.

For $m \geqslant 1$, the l-adic generalized Soulé element $\tilde{\chi}_m^z : Gal(\overline{F}/F) \to \mathbf{Z}_l$ is defined by the following congruence property (loc. cit. Definition 3):

$$\tilde{\chi}_m^z(\sigma) \equiv \sum_{a=0}^{l^n-1} a^{m-1}\kappa_{l^n}^{z,(a)}(\sigma) \mod l^n\mathbf{Z}_l \qquad (n \geqslant 1),$$

which gives its integral expression

$$\tilde{\chi}_m^z(\sigma) = \int_{\mathbf{Z}_l} x^{m-1} d\mu_F^{z,\sigma}.$$

Particularly when $\sigma \in \mathrm{Gal}\left(\overline{F}/F(z^{\frac{1}{l^\infty}})\right)$, we have $\ell_m^z(\sigma) = (-1)^{m-1}\frac{\tilde{\chi}_m^z(\sigma)}{(m-1)!}$ (loc. cit. Corollary) where $\ell_m^z(\sigma)$ stands for the l-adic polylogarithm studied by Wojtkowiak. Put

$$\chi_m^{(l),\sigma}(z) := \tilde{\chi}_m^{z^l}(\sigma) - l^{m-1}\tilde{\chi}_m^z(\sigma) \in \mathbf{Z}_l.$$

Proposition 1. *Fix $\sigma \in Gal(\overline{F}/F)$ and $z \in \mathbf{P}^1(F)\backslash\{0,1,\infty\}$. Let $m, m', M \in \mathbf{N}$. If $m \equiv m' \mod (l-1)l^M$, then*

$$\chi_m^{(l),\sigma}(z) \equiv \chi_{m'}^{(l),\sigma}(z) \mod l^{M+1}\mathbf{Z}_l.$$

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Proof . By (1), we compute

$$\begin{aligned}\tilde{\chi}^z_m(\sigma) &= \lim_{n\to\infty}\sum_{0\leqslant a\leqslant l^n-1} a^{m-1}\kappa^{z,(a)}_{l^n}(\sigma) = \lim_{n\to\infty}\frac{1}{l^{m-1}}\sum_{0\leqslant a\leqslant l^n-1}(la)^{m-1}\kappa^{z^l,(la)}_{l^{n+1}}(\sigma)\\ &= \lim_{n\to\infty}\frac{1}{l^{m-1}}\sum_{\substack{0\leqslant a\leqslant l^{n+1}-1\\ l\mid a}} a^{m-1}\kappa^{z^l,(a)}_{l^{n+1}}(\sigma) = \frac{1}{l^{m-1}}\int_{l\mathbf{Z}_l} x^{m-1}d\mu^{z^l,\sigma}_F.\end{aligned}$$

Then we have

$$\chi^{(l),\sigma}_m(z) = \tilde{\chi}^{z^l}_m(\sigma) - l^{m-1}\tilde{\chi}^z_m(\sigma) = \int_{\mathbf{Z}_l^\times} x^{m-1}d\mu^{z^l,\sigma}_F.$$

The equality $x^{m-1}\equiv x^{m'-1} \mod l^{M+1}\mathbf{Z}_l$ for $x\in\mathbf{Z}_l^\times$ by $m\equiv m' \mod (l-1)l^M$ gives an expression $x^{m-1}-x^{m'-1} = l^{M+1}h(x)$ for $h(x)\in \mathrm{Cont}(\mathbf{Z}_l^\times,\mathbf{Z}_l)$. By $\int_{\mathbf{Z}_l^\times} h(x)d\mu^{z^l,\sigma}_F\in\mathbf{Z}_l$,

$$\int_{\mathbf{Z}_l^\times} x^{m-1}d\mu^{z^l,\sigma}_F \equiv \int_{\mathbf{Z}_l^\times} x^{m'-1}d\mu^{z^l,\sigma}_F \mod l^{M+1}\mathbf{Z}_l,$$

from which we get the claim. □

Especially by restricting into $\sigma\in Gal\left(\overline{F}/F(z^{\frac{1}{l^\infty}})\right)$, we see that a modification

$$\ell^{(l),\sigma}_m(z) := \Big\{(m-1)!l^m\ell^z_m(\sigma)\Big\} - \frac{1}{l^{m-1}}\Big\{(m-1)!l^m\ell^{z^l}_m(\sigma)\Big\} \tag{2}$$

of the l-adic polylogarithm $\ell^z_m(\sigma)$ admits a nice l-adic behavior with respect to the weight m.

2. p-adic polylogarithms

Let p be a prime. Put $m\in\mathbf{N}$. The p-adic polylogarithm studied by Coleman [C] is the following series

$$Li_m(z) = \sum_{n=1}^\infty \frac{z^n}{n^m},$$

which converges on $\{z\in\mathbf{C}_p : |z|_p<1\}$. Here $|\cdot|_p$ means the standard multiplicative valuation of $\mathbf{C}_p$. By using his p-adic iterated integration theory (loc. cit.), this function is analytically continued to $\mathbf{P}^1(\mathbf{C}_p)\backslash\{1,\infty\}$ as (not a rigid analytic function but) a Coleman function of $\mathbf{P}^1(\mathbf{C}_p)\backslash\{0,1,\infty\}$ (for a brief expository, see [F]). Consider its modification

$$\ell^{(p)}_m(z) := Li_m(z) - \frac{1}{p^m}Li_m(z^p) \tag{3}$$

which has an expansion $\ell^{(p)}_m(z) = \sum_{(n,p)=1}\frac{z^n}{n^m}$ on $|z|_p<1$. It was shown in [C] Proposition 6.2 that actually this function extends to a rigid analytic function of $B := \mathbf{P}^1(\mathbf{C}_p) - \left\{|z-1|_p < p^{\frac{-1}{p-1}}\right\}$.

Proposition 2. *Let $m, m', M\in\mathbf{N}$. If $m\equiv m' \mod (p-1)p^M$, then*

$$\ell^{(p)}_m(z)\equiv \ell^{(p)}_{m'}(z) \mod p^{M+1}\mathbf{Z}_p[[z]].$$

Proof . It is clear by the congruence $n^m\equiv n^{m'} \mod p^{M+1}\mathbf{Z}$ for $(n,p)=1$. □

Since the ring of rigid analytic functions on B is embedded into $\mathbf{C}_p[[z]]$ by taking expansions at $z=0$, we see that the modification $\ell_m^{(p)}(z)=Li_m(z)-\frac{1}{p^m}Li_m(z^p)$ of the p-adic polylogarithm $Li_m(z)$ admits a nice p-adic behavior with respect to the weight m as a rigid analytic function.

But it looks a bit strange because the adjustment factor is $\frac{1}{p^m}$ in (3) while it is $\frac{1}{l^{m-1}}$ in (2). The author would like to clarify this and expects a theoretical argument to explain both Proposition 1 and Proposition 2 at once.

References

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