THE EXPLICIT FORMULA FOR A CERTAIN SOLUTION OF THE KZ-LIKE DIFFERENTIAL EQUATION

HIDEKAZU FURUSHO

Fix $m \in \mathbf{N}$.

Notation 1. Let $A^{\wedge}_{\mathbf{C}} = \mathbf{C}\langle\langle A, B_1, \cdots, B_m\rangle\rangle$ be the non-commutative formal power series ring with complex number coefficients generated by A and B_i $(i \in \mathbf{Z}/m)$. We consider the following differential equation

$$\frac{dG}{du}(u) = \left(\frac{A}{u} + \sum_{a\in\mathbf{Z}/m} \frac{B_a}{u-\zeta^{-a}}\right) \cdot G(u) \qquad , \tag{1}$$

where $\zeta = \zeta_m = exp(2\pi i/m)$ and $G(u)$ is an analytic function in complex variable u with values in $A^{\wedge}_{\mathbf{C}}$ where 'analytic' means each of whose coefficient is analytic.

The equation (1) has singularities only at $u = 0$, ∞ and ζ^i $(i \in \mathbf{Z}/m)$.

Lemma 2. *Let $G(u)$ and $H(u)$ be local solutions of* (1)*. Suppose that $H(u)$ is invertible. Then $H(u)^{-1}G(u)$ is constant.*

Proof .

$$\begin{aligned}
\frac{d}{du}H(u)^{-1}G(u) &= -H(u)^{-1}\cdot\frac{d}{du}H(u)\cdot H(u)^{-1}G(u) + H(u)^{-1}\frac{d}{du}G(u)\\
&= -H(u)^{-1}\left(\frac{A}{u} + \sum_{a\in\mathbf{Z}/m}\frac{B_a}{u-\zeta^{-a}}\right)H(u)\cdot H(u)^{-1}G(u)\\
&\qquad + H(u)^{-1}\left(\frac{A}{u} + \sum_{a\in\mathbf{Z}/m}\frac{B_a}{u-\zeta^{-a}}\right)G(u)\\
&= -H(u)^{-1}\left(\frac{A}{u} + \sum_{a\in\mathbf{Z}/m}\frac{B_a}{u-\zeta^{-a}}\right)G(u)\\
&\qquad + H(u)^{-1}\left(\frac{A}{u} + \sum_{a\in\mathbf{Z}/m}\frac{B_a}{u-\zeta^{-a}}\right)G(u) = 0 \ .
\end{aligned}$$

Therefore $H(u)^{-1}G(u)$ is constant. □

Proposition 3. *There exists a unique local solution $G_0(u)$ of* (1)*, which satisfies $G_0(u) \approx u^A$ $(u \to 0)$.*

Here u^A stands for the local section $1 + \frac{Alogu}{1!} + \frac{(Alogu)^2}{2!} + \frac{(Alogu)^3}{3!} + \cdots$ and $G_0(u) \approx u^A$ $(u \to 0)$ means that the local section $P(u) := G_0(u)\cdot u^{-A}$ has an analytic continuation in a neighborhood of 0 with value 1 at 0.

Proof . *Uniqueness*: Suppose that $H_0(u)$ is another solution satisfying above properties, which follows especially that $H_0(u)$ is invertible. By Lemma 2, there exists a unique constant series $c \in A^{\wedge}_{\mathbf{C}}$ such that $G_0(u) = H_0(u)\cdot c$. By the assumption, we get $u^A \cdot c \cdot u^{-A} \to 1$ as $u \to 0$, which means $c = 1$.

Existence: By substituting $G(u)=P(u)\cdot u^A$ into (1), we get

$$\frac{dP}{du}(u)=\Big[\frac{A}{u},P(u)\Big]+\sum_{a\in\mathbf{Z}/m}\frac{B_a}{u-\zeta^{-a}}P(u)\quad .$$

We can construct inductively a local solution $P(u)$ of the above differential equation which takes value $1\in A^{\wedge}_{\mathbf{C}}$ at $u=0$. ☐

It is clear , by definition, that the constant term of $G_0(z)$ is 1.

Definition 4 ([AK])**.** For an index set $(\mathbf{k};\mathbf{a})=(k_1,\cdots,k_n;a_1,\cdots,a_n)$ where $k_i\geqslant 1$, $a_i\in\mathbf{Z}/m$ $(1\leqslant i\leqslant m)$, we define the following series:

$$Li_{\mathbf{k};\mathbf{a}}(z):=\sum_{m_1>m_2>\cdots>m_n}\frac{z^{m_1}\zeta^{a_1(m_1-m_2)}\ldots\zeta^{a_{n-1}(m_{n-1}-m_n)}\zeta^{a_nm_n}}{m_1^{k_1}m_2^{k_2}\cdots m_n^{k_n}},$$

which is absolutely convergent for $|z|<1$.

Proposition 5 ([AK])**.** *The above series $Li_{\mathbf{k};\mathbf{a}}(z)$ admits the following iterated integral expression:*

$$Li_{\mathbf{k};\mathbf{a}}(z)=\int_0^z\underbrace{\frac{du}{u}\circ\cdots\circ\frac{du}{u}\circ\frac{du}{\zeta^{-a_1}-u}}_{k_1}\circ\frac{du}{u}\circ\cdots\cdots\circ\frac{du}{\zeta^{-a_{n-1}}-u}$$
$$\circ\underbrace{\frac{du}{u}\circ\cdots\circ\frac{du}{u}\circ\frac{du}{\zeta^{-a_n}-u}}_{k_n}$$

For the precise meaning of the above expression, see [F] §3.2.

Notation 6. (1) Let $\mathbf{A}.=\underset{w\geqslant 0}{\oplus}\mathbf{A}_w=\mathbf{Q}\langle A,B_1,\cdots,B_m\rangle(\subset A^{\wedge}_{\mathbf{C}})$ be the non-commutative graded polynomial ring over $\mathbf{Q}$ with variables A and B_a with $degA=degB_a=1$ $(a\in\mathbf{Z}/m)$. Here $\mathbf{A}_w$ is the homogeneous degree w part of $\mathbf{A}.$. We call an element of $\mathbf{A}.$ which is monic and monomial by a word. But exceptionally we shall not call 1 a word. For each word W, the weight and depth of W are as follows.

$wt(W):=$ 'the sum of exponents of A and B_a $(a\in\mathbf{Z}/m)$ in W'
$dp(W):=$ 'the sum of exponents of B_a $(a\in\mathbf{Z}/m)$ in W'

(2) Put $M'=\mathbf{A}\,.\cdot B_1+\mathbf{A}\,.\cdot B_2+\cdots+\mathbf{A}\,.\cdot B_m$ which is the $\mathbf{Q}$-linear subspace of $\mathbf{A}.$. Note that there is a natural surjection from $\mathbf{A}.$ to $\mathbf{A}.\big/\mathbf{A}\,.\,A$. By identifying the latter space with $\mathbf{Q}\cdot 1+M'$, we obtain the $\mathbf{Q}$-linear map $f':\mathbf{A}.\twoheadrightarrow\mathbf{A}.\big/\mathbf{A}\,.\,A\overset{\sim}{\to}\mathbf{Q}\cdot 1+M'\hookrightarrow\mathbf{A}.$. Abbreviately we denote $f':A^{\wedge}_{\mathbf{C}}\to A^{\wedge}_{\mathbf{C}}$ to be the $\mathbf{C}$-linear map induced from $f':\mathbf{A}.\to\mathbf{A}.$.

(3) For each word $W=A^{k_1-1}B_{a_1}\cdots A^{k_n-1}B_{a_n}$ in M' $(n\geqslant 1,\,k_i\geqslant 1)$, we define

$$Li_W(z):=Li_{k_1,\cdots,k_n;a_1,\cdots,a_n}(z)\qquad\text{where}\qquad |z|<1.$$

By extending linearly, we get the $\mathbf{Q}$-linear map $Li(z):M'\to\Gamma(D,\mathcal{O}_D^{\mathrm{an}})$ which sends each word W in M' to $Li_W(z)$, where $D:=\{z\in\mathbf{C}\mid |z|<1\}$.

Theorem 7 (Explicit Formulae)**.** *Let $G_0(z)$ be the fundamental solution of* (1) *in Proposition 3. Expand $G_0(z)=1+\sum_{W:words}J(W)(z)\,W$. Then each coefficients $J(W)(z)$ can be expressed as follows.*

(a) *When* W *is in* M', $\quad J(W)(z) = (-1)^{dp(W)} Li_W(z)$.
(b) *When* W *is written as* $VA^r (r \geqslant 0, V \in M')$,

$$J(W)(z) = \sum_{\substack{s+t=r \\ 0 \leqslant s, 0 \leqslant t}} (-1)^{dp(W)+s} Li_{f'(V \circ A^s)}(z) \frac{\{log(z)\}^t}{t!} \quad .$$

(c) *When* W *is written as* $A^r (r \geqslant 0)$, $\quad J(W)(z) = \frac{\{log(z)\}^r}{r!}$.

For the defeinition of the shuffle product '$\circ$', see [F] Definition 3.2.2. We give the proof of this explicit formulae below, whose techniques are esentially borrowed from [LM].

At first, by applying f' to $G_0(z)$ term by term, we get

Lemma 8. $f'(G_0(z)) = 1 + \sum_{W \in M' : words} J(W)(z)\ W$.

On the other hand,

Lemma 9. $f'(G_0(z)) = 1 + \sum_{W \in M' : words} (-1)^{dp(W)} Li_W(z)\ W$.

Proof . By Proposition 5, the differential equation (1) and $\lim_{z \to 0} G_0(z) \cdot z^{-A} = 1$, we can show this formula inductively. $\square$

By combining Lemma 8 with Lemma 9, we get Theorem 7.(a).

Notation 10. Let $A^{\wedge}_{\mathbf{C}}[[\alpha]] := A^{\wedge}_{\mathbf{C}} \widehat{\otimes} \mathbf{C}[[\alpha]]$ be the one variable formal power series ring with the non-commutative algebra $A^{\wedge}_{\mathbf{C}}$-coefficient. Let $g'_1 : A^{\wedge}_{\mathbf{C}} \to A^{\wedge}_{\mathbf{C}}[[\alpha]]$ be the algebra homorphism which sends A, B to $A - \alpha$, B respectively and let $g'_2 : A^{\wedge}_{\mathbf{C}}[[\alpha]] \to A^{\wedge}_{\mathbf{C}}$ be the well-defined $\mathbf{C}$-linear map which sends $W \otimes \alpha^q$ to WA^q for each word W and $q \geqslant 0$.

Secondly, consider the $\mathbf{C}$-linear map $g'_2 \circ g'_1 : A^{\wedge}_{\mathbf{C}} \to A^{\wedge}_{\mathbf{C}}$.

Lemma 11. $g'_2 \circ g'_1(VA) = 0$ *for* $V \in A^{\wedge}_{\mathbf{C}}$.

Proof . It is clear by definition. $\square$

Lemma 12. $G_0(z) = g'_2 \circ g'_1\Big(f'\big(G_0(z)\big)\Big) \cdot z^A$.

Proof . By Lemma 11, we get

$$g'_2 \circ g'_1\Big(f'\big(G_0(z)\big)\Big) = g'_2 \circ g'_1\big(G_0(z)\big) \quad . \tag{2}$$

Both $G_0(A-\alpha, B)(u)$ and $u^{-\alpha} G_0(A,B)(u)$ are local solutions of the differential equation $\frac{dG}{du}(u) = (\frac{A-\alpha}{u} + \sum_{a \in \mathbf{Z}/m} \frac{B_a}{u - \zeta^{-a}}) G(u)$ and satisfies the same asymptotic behavior $G(u) \approx u^{A-\alpha}$ as $u \to 0$. Therefore by the uniqueness of solution of the above differential equation (which can be proved analogously to the proof of Proposition 3) we get $G_0(A-\alpha, B)(u) = G_0(A,B)(u) \cdot u^{-\alpha}$, from which it follows that

$$g'_2 \circ g'_1\big(G_0(z)\big) = G_0(z) \cdot z^{-A} \quad . \tag{3}$$

By (2) and (3), we get $G_0(z) = g'_2 \circ g'_1\Big(f'\big(G_0(z)\big)\Big) \cdot z^A$. $\square$

Therefore we see that $P(z) = g_2' \circ g_1'\Big(f'\big(G_0(z)\big)\Big)$ (for $P(z)$, see Proposition 3).

Proof of Theorem 7. By combining Lemma 9 with Lemma 12, we get the explicit formula in Theorem 7. □

Examples 13. When $m = 1$, we get a low degree part of $G_0(A,B)(z)$ as following.

$$\begin{aligned}
G_0(A,B)(z) = {} & 1 + (logz)A + log(1-z)B + \frac{(logz)^2}{2}A^2 - Li_2(z)AB \\
& + \{Li_2(z) + (logz)log(1-z)\}\, BA + \frac{\{log(1-z)\}^2}{2}B^2 + \frac{(logz)^3}{6}A^3 \\
& - Li_3(z)A^2B + \{2Li_3(z) + (logz)Li_2(z)\}\, ABA + Li_{1,2}(z)AB^2 \\
& - \left[Li_3(z) - (logz)Li_2(z) - \frac{(logz)^2 log(1-z)}{2}\right] BA^2 + Li_{2,1}(z)BAB \\
& - \left[Li_{1,2}(z) + Li_{2,1}(z) - \frac{logz\{log(1-z)\}^2}{2}\right] B^2A \\
& + \frac{\{log(1-z)\}^3}{6}B^3 + \cdots
\end{aligned}$$

References

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Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502 Japan

E-mail address: `furusho@kurims.kyoto-u.ac.jp`