ON MOTIVIC ITERATED INTEGRALS IN GENERIC POSITIONS (JOINT WORK WITH AMIR JAFARI)

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1. Introduction

This is a survey on the joint paper [FJ] with A. Jafari. We will explain a combinatorial framework for motivic study of iterated integrals on the affine line which is an extension of [GGL1] and [GGL2]. We will show that under a certain genericity condition these combinatorial objects yield to elements in the motivic Hopf algebra constructed in [BK]. It will be shown that the Hodge realization of these elements coincides with the Hodge structure induced from the fundamental torsor of path of punctured affine line.

Using cubical algebraic cycles, Bloch and Kříž [BK] constructed a category of mixed Tate motives over *any* field F (MTM(F) for short). They also constructed its Hodge realization and étale realization functor. Under the K(π,1)-conjecture (cf. §2), the desired formula

$$Ext^i_{MTM(F)}(\mathbf{Q}(0),\mathbf{Q}(n)) \cong gr^{\gamma}_i K_{2n-i}(F)_{\mathbf{Q}} \tag{1.1}$$

holds in this category. Here RHS is a graded quotient of the algebraic K-theory for F with respect to γ-filtration. Our final aim is to construct a motivic fundamental group of the affine line minus finite set in MTM(F), which is equivalent to construct motivic iterated integrals. Our main results in this paper is a construction of motivic iterated integrals in generic positions.

In contrast there is another category of mixed Tate motives constructed for a field with characteristic 0 satisfying the Beilinson-Soulé vanishing conjecture, that is, $gr^{\gamma}_r K_{2r-p}(F)_{\mathbf{Q}} = 0$ for $p \leqslant 0$ with $(r,p) \neq (0,0)$. This category is a heart of t-structure given in [L1] of the subtriangulated category generated by Tate motives inside the triangulated category of mixed motives over F constructed by Voevodsky [V] and Levine [L3]. This category admits Hodge and étale realization [L3, H] and satisfies the desired formula above. Deligne and Goncharov [DG] constructed a motivic fundamental group of the affine line minus finite set in this category. This gives motivic iterated integrals in this

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category. As far as we know, the interrelationship between this category and that of Bloch-Křiž's mixed Tate motives are not known well although they are conjecturally equivalent. Deligne-Goncharov's construction of motivic fundamental group minus finite sets does not imply the existence of motivic iterated integrals in the category $\mathrm{MTM}(F)$ of Bloch-Křiž's mixed Tate motives.

For $a, b, s_1, \ldots, s_n \in \mathbf{C}$ and a topological path $\gamma(t)$ $(0 \leqslant t \leqslant 1)$ with $\gamma(0) = a$ and $\gamma(1) = b$. and $\gamma(t) \cap \{s_i\} = \emptyset$. The iterated integral is defined to be

(1.2)

$$It^{\gamma} \int_a^b \frac{dt}{t - s_1} \circ \cdots \circ \frac{dt}{t - s_n} := \int_{0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant 1} \frac{d\gamma(t_1)}{\gamma(t_1) - s_1} \wedge \cdots \wedge \frac{d\gamma(t_n)}{\gamma(t_n) - s_n}.$$

For elements $a, b, s_1, \ldots, s_n \in \mathbb{A}^1_F$, we want to construct a motivic analogue of the above iterated integral which will be an element of $\chi_F(n)$ and we denote it by $\mathbb{I}(a; s_1, \ldots, s_n; b)$. The special case $\mathbb{I}(0; \underbrace{1, 0, \ldots, 0}_{n}; z)$ which (up to a sign) represents $\mathrm{Li}_n(z) := \sum_{k=1}^{\infty} z^k/k^n$, was constructed in [BK]. Unfortunately we have to assume a genericity assumption on the sequence $a, s_1, \ldots, s_n, b$ in order to have an admissible cycle. This assumption is as follows:

Definition 1.1. The sequence $a, s_1, \ldots, s_n, b$ is *in generic position* if either $a = b$ or the non-zero terms do not repeat.

This generic condition is more enlarged than the condition given in [GGL2], where it is assumed that all s_i's are non-zero and distinct. With allowing zero to repeat we recover the expression given in [BK] for the polylogarithm, and moreover we get expressions for (several variable) multiple polylogarithm $Li_{n_1, \ldots, n_m}(z_1, \ldots, z_m)$ for the values z_i such that $z_i \ldots z_j \neq 1$ for $1 \leqslant i \leqslant j \leqslant n$.

In fact the method will work for certain sequences which are not generic, examples of such sequences are $(0; s_1, \ldots, s_n; a)$ where at most two of s_i's is equal to 1 and the rest are zero. This will imply the existence of motivic (one variable) double polylogarithm $Li_{n,m}(a)$ and also the multiple zeta value $\zeta(n, m)$ ([FJ]).

Our construction of motivic iterated integrals in generic position is based on the combinatorial framework of trees. We will consider a sum of trivalent trees with prescribed decorations and define motivic iterated integrals to be its corresponding cycle in Definition 4.3.

Theorem 1.2. *If F is a subfield of $\mathbf{C}$, the Hodge realization of the motivic iterated integral $(-1)^n \mathbb{I}(a; s_1, \ldots, s_n; b)$ agree with the framed mixed Hodge Tate structure corresponding to the iterated integral* (1.2).

In [FJ] it is shown that motivic iterated integrals satisfy the usual properties of the iterated integrals, namely:

Theorem 1.3. *Under the above genericity assumption for a, b, s_i the elements $\mathbb{I}(a; s_1, \ldots, s_n; b)$ satisfy the following properties:*

(1) *Triviality:*

$$\mathbb{I}(a; b) = 1,$$
$$\mathbb{I}(a; s_1, \ldots, s_n; a) = 0.$$

(2) *Shuffle relation:*

$$\mathbb{I}(a; s_1, \ldots, s_n; b) \cdot \mathbb{I}(a; s_{n+1}, \ldots, s_{n+m}; b) = \sum_{\sigma \in Sh(n,m)} \mathbb{I}(a; s_{\sigma(1)}, \ldots, s_{\sigma(n+m)}; b).$$

Here the shuffle product is defined to be the following:

$$Sh(n,m) := \Big\{ \tau : \{1, \ldots, n+m\} \to \{1, \ldots, n+m\} \;\Big|\; \tau \textit{ is bijective}, \\ \tau(1) < \cdots < \tau(n), \tau(n+1) < \cdots \tau(n+m) \Big\}.$$

(3) *Path composition:*

$$\mathbb{I}(a; s_1, \ldots, s_n; b) = \sum_{k=0}^{n} \mathbb{I}(a; s_1, \ldots, s_k, c) \cdot \mathbb{I}(c; s_{k+1}, \ldots, s_n; b).$$

(4) *Antipode relation:*

$$\mathbb{I}(a; s_1, \ldots, s_n; b) = (-1)^n \mathbb{I}(b; s_n, \ldots, s_1; a).$$

(5) *Coproduct formula:*

$$\Delta \mathbb{I}(a; s_1, \ldots, s_n; b) = \sum \mathbb{I}(a; s_{i_1}, s_{i_2}, \ldots, s_{i_k}; b) \otimes \prod_{j=0}^{k} \mathbb{I}(s_{i_j}; s_{i_j+1}, \ldots, s_{i_{j+1}-1}; s_{i_{j+1}})$$

where the sum is over all indices $0 = i_0 < i_1 < \cdots < i_k < i_{k+1} = n+1$ and $s_0 := a$ and $s_{n+1} := b$ and $k = 0, 1, \ldots$.

2. Review on Bloch and Kříž' mixed Tate motives

A differential graded algebra (DGA) with Adams grading is a bigraded $\mathbb{Q}$-vector space, $\mathcal{A} = \oplus \mathcal{A}^n(r)$ where $n \in \mathbb{Z}$ and $r \geqslant 0$, such that $\mathcal{A}^n(r) = 0$ for $n > 2r$ and $\mathcal{A}(0) = \mathbb{Q}$, together with a product $\mathcal{A}^n(r) \otimes \mathcal{A}^m(s) \longrightarrow \mathcal{A}^{n+m}(r+s)$ that makes $\mathcal{A}$ into a graded commutative algebra with identity (the signs are contributed from the differential grading and not the Adams grading), and a differential $d : \mathcal{A}^n(r) \longrightarrow \mathcal{A}^{n+1}(r)$ that satisfies the Leibniz rule.

We now recall the bar construction. For our future need we develop the theory in a more general setting. Let A be a DGA with Adams grading and let M be a right DG module over A with a compatible Adams grading. Let $A^+ = \oplus_{r>0} A(r)$. Define

$$B(M,A) = M \otimes \left(\dot{\bigotimes} A^+[1] \right).$$

The elements of $B(M,A)$ are denoted by a bar notation as $m[a_1|\ldots|a_r]$ which has a degree equal to $\deg(m) + \deg(a_1) + \ldots \deg(a_r) - r$. The graded vector space $B(M,A)$ is a differential graded with total Adams grading. The differential d is given as the sum of two differentials, the external and internal differentials:

$$\begin{aligned} d_{ext}(m[a_1|\ldots|a_r]) =& dm[a_1|\ldots|a_r] \\ &+ \sum_{i=1}^{r} Jm[Ja_1|\ldots|Ja_{i-1}|da_i|a_{i+1}|\ldots|a_r], \\ d_{int}(m[a_1|\ldots|a_r]) =& Jm \cdot a_1[a_2|\ldots|a_r] \\ &+ \sum_{i=1}^{r-1} Jm \cdot [Ja_1|\ldots|Ja_{i-1}|Ja_i \cdot a_{i+1}|\ldots|a_r]. \end{aligned}$$

the operation J is given by $J(a) = (-1)^{\deg(a)-1}a$ on the homogeneous elements. Note that $\deg(a)-1$ is the degree if a in the shifted complex $A[1]$. If M is the trivial DGA, we have a coproduct (where the empty tensor is 1 by convention):

$$\Delta([a_1|\ldots|a_r]) = \sum_{s=0}^{r} [a_1|\ldots|a_s] \otimes [a_{s+1}|\ldots|a_r].$$

Up to this point all the constructions work for a DGA which is not necessarily commutative. The product is defined only when both M and A are commutative DGA's by the shuffle:

$$m[a_1|\ldots|a_r] \cdot m'[a_{r+1}|\ldots|a_{r+s}] := \sum sgn_{\mathbf{a}}(\sigma) m \cdot m'[a_{\sigma(1)}|\ldots|a_{\sigma(r+s)}]$$

where σ runs over the (r,s)-shuffles (an (r,s)-shuffle is a permutation σ on $1,\ldots,r+s$ such that $\sigma^{-1}(1) < \cdots < \sigma^{-1}(r)$ and $\sigma^{-1}(r+1) < \cdots < \sigma^{-1}(r+s)$) and the sign is obtained by giving a_i's weights $\deg(a_i)-1$. These data make $B(A) := B(k,A)$ into a commutative differential Hopf algebra with an Adams grading. Taking H^0 with respect to the differential gives a commutative graded Hopf algebra $H^0B(A)$.

Following [BK], we now introduce the cubical cycle complex. Let F be a field and denote $\square_F := \mathbb{P}^1_F \backslash \{1\}$. Then the permutation group on

n letters Σ_n acts on $\square_F^n$ and also we have an action of $(\mathbb{Z}/2)^n$ given by

$$\epsilon \cdot (x_1, \ldots, x_n) = (x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n})$$

where $\epsilon \in \{1, -1\}^n$. Therefor we have an action of $G_n := (\mathbb{Z}/2)^n \rtimes \Sigma_n$ on $\square_F^n$. Let $\mathrm{Alt}_n \in \mathbb{Q}[G_n]$ be the element

$$\frac{1}{|G_n|} \sum_{g \in G_n} sgn(g) g$$

where for $g = (\epsilon, \sigma) \in G_n$, sign is defined by $(\prod_i \epsilon_i) sgn(\sigma)$. We also define a face of $\square_F^n$ as a subset defined by setting certain coordinates equal zero or infinity. We are now prepared to define the DGA with an Adams grading $\mathcal{N} = \mathcal{N}_F$:

$$\mathcal{N}^n(r) := \mathrm{Alt}_{2r-n} Z(\square_F^{2r-n}, r).$$

Notice that although $r \geqslant 0$, n can be negative. Here $Z(\square_F^n, r)$ denotes the $\mathbb{Q}$-span of the admissible codimension r subvarieties (i.e. closed and integral subschemes that intersect all the faces of codimension $\geqslant 1$ properly, i.e. in codimension r) of $\square_F^n$. The product structure is given by

$$Z_1 \cdot Z_2 := \mathrm{Alt}(Z_1 \times Z_2).$$

The differential $d : \mathcal{N}^n(r) \longrightarrow \mathcal{N}^{n+1}(r)$ is given by

$$d = \sum_{i=1}^{2r-n} (-1)^{i-1} (\partial_0^i - \partial_\infty^i)$$

where for $c = 0, \infty$, ∂_c^i is obtained by the pull-back of the cycles under the inclusions $\square_F^{2r-n-1} \hookrightarrow \square_F^{2r-n}$ given by letting the i^{th} coordinate equal to c. Using the results of [B1],[B2] and [L2] it is shown in [BK] that:

$$\mathrm{H}^n(\mathcal{N}(r)) \cong \mathrm{CH}^r(F, 2r-n) \otimes \mathbb{Q} \cong gr_r^\gamma K_{2r-n}(F)_{\mathbb{Q}}. \tag{2.1}$$

Here $\mathrm{CH}^r(F, n)$ denotes the Bloch's higher Chow group of $\mathrm{Spec}(F)$.

Definition 2.1. ([BK]) The category $\mathrm{MTM}(F)$ of mixed Tate motives over F is the category of finite dimensional graded comodules over the graded commutative Hopf algebra

$$\chi_F := \mathrm{H}^0 B(\mathcal{N}_F).$$

More explicitly a mixed Tate motive over F is a graded finite dimensional $\mathbb{Q}$-vector space M with a linear map $\nu : M \longrightarrow M \otimes \chi_F$ such that it respects the grading and

$$(id \otimes \Delta)(\nu(a)) = (\nu \otimes id)(\nu(a))$$

$$(id \otimes \epsilon)(\nu(a)) = a$$

where Δ and ϵ are the coproduct and the counit of χ_F.

The K(π, 1)-conjecture says that the complex $\mathcal{N}_F$ should be quasi-isomorphic to the complex of its Sullivan 1-minimal model. This is stronger than the Beilinson-Soulé's vanishing conjecture saying that the complex should be cohomologically connected. Under the K(π,1)-conjecture, we have

$$Ext^i_{MTM(F)}(\mathbf{Q}(0), \mathbf{Q}(n)) \cong H^i(\mathcal{N}_F(n)) \tag{2.2}$$

(see [BK] and also [KM]). Here $\mathbf{Q}(n)$ is a copy of $\mathbf{Q}$ sitting in degree $-n$. The equalities of (2.1) and (2.2) give the desired formula (1.1). As far as we know the validity of Beilinson-Soulé's vanishing conjecture is known for number fields, function fields of the curve with genus 0 defined over number field and their inductive limits whereas the K(π,1)-conjecture is open for all fields.

3. Trees and cycles

This section is a complement to [GGL1]. Let S be a set. We define the DGA $\mathcal{T}_S$ (with an Adams grading) of rooted S-decorated planar trees.

A rooted S-decorated planar tree is a connected finite graph with no loops, such that each edge has exactly two vertex and no vertex has degree 2. Each vertex of degree one is decorated by an element of the given set S and one such vertex is distinguished as a root. Decorated vertices which are not a root are called the ends. External edges which are not with a root are called leaves. Furthermore such a tree has a given embedding into the plane. The embedding into the plane defines a canonical ordering on the edges which is obtained by going counter clockwise starting with the root edge. The root defines a direction on each edge which is the direction away from the root.

Definition 3.1. (Tree algebra $\mathcal{T}_S$). The free graded commutative algebra generated by the rooted S-decorated planar trees, where each tree has weight equal to its number of edges, is denoted by $\mathcal{T}_S$. There is a bi-grading where a rooted tree in $\mathcal{T}_S^n(r)$ has r ends and $2r - n$ edges. (Notice that the differential grading is different with the number of edges but is congruent to it modulo 2, so for sign purposes both give the same value).

For a rooted S decorated planar tree T, with a given edge e, the contracted tree $T/e \in \mathcal{T}_S$ is defined as follows: If e is an internal edge then T/e is obtained by contracting the edge e, keeping the root and decoration and embedding in the plane as before. If e is an external edge (i.e. an edge with a decorated vertex) then T/e is obtained in several stages. First remove e and its two vertices. Denote the connected components by $T_1, \ldots, T_k$ (in the order dictated by the embedding), add a vertex to the open edges of these trees and decorate it by the decoration of the vertex of e. For the trees that do not have a root, make this newly added vertex their root. T/e is the product of rooted S decorated trees $\epsilon(e)T_1 \cdots T_k$. Here $\epsilon(e)$ is ± 1 according to the orientation as we now explain. Each planar tree has a canonical ordering on its edges given by going counter clockwise on the edges starting from the root. The sign $\epsilon(e)$ is defined from comparing the ordering defined by $T_1 \cdots T_k$ and by T with the edge e removed. If these two ordering differ by an odd permutation the sign is minus and otherwise it is plus.

For a rooted S decorated planar tree T in $\mathcal{T}_S$ the differential is given by

$$dT = \sum_{i=1}^{e(T)} (-1)^{i-1}(T/e_i). \tag{3.1}$$

Here the ordering is given by the embedding by going counter clockwise starting from the root and $e(T)$ is the number of edges of T. This will be extended by Leibniz rule to all of $\mathcal{T}_S$. The differential satisfies $d^2 = 0$, from which it follows that $\mathcal{T}_S$ becomes a commutative DGA.

Definition 3.2. Let S and S' be two sets. The DGA of double decorated rooted trees $\mathcal{T}_{S,S'}$ is the free graded algebra generated by rooted S-decorated planar trees together with a decoration of edges with values in S'. We require three conditions:

(1) The two decoration of the leaves are distinct (we do not make this assumption for the root).
(2) For any edge decorated with s' and final vertex v, either v is a decorated vertex or there is an edge starting from v with the same edge decoration.
(3) Any path between two labeled vertex with the same label s, passes through the vertex of an edge decorated with s.

The previous differential (3.1) with a slight modification provides a differential structure on this algebra [FJ]. Bearing in mind the integral $\int_a^c \frac{dt}{t-b}$, for an oriented edge e with origin vertex labeled by a and the final vertex labeled by b (they can be elements of F or variables) and the edge labeled by c define the function

$$f(e) = \frac{a-b}{c-b}.$$

We will define the following map which generalizes a construction of Gangl, Goncharov and Levin in [GGL1]:

Definition 3.3. (Forest cycling map). Let S be a non-empty subset of $\mathbb{A}^1_F$. Decorate the internal vertices of elements of $\mathcal{T}_{S,S'}$ by independent variables. Give each edge the direction that points away from the root. The forest cycling map to an element T of $\mathcal{T}^n_S(r)$ associates the following cycle of codimension r inside $\square_F^{2r-n}$:

$$\rho(T) = \mathrm{Alt}_{2r-n}(f(e_1), \dots, f(e_{2r-n})).$$

Here $e_1, \dots, e_{2r-n}$ are the edges of T with the ordering induced from its embedding.

In [FJ] it is shown that the map $\rho : \mathcal{T}_{S,S'} \longrightarrow \mathcal{N}$ is a well-defined morphism of DGA's.

4. Iterated Integrals and trees

Let S be a non-empty subset of $\mathbb{A}^1$. For $a_i \in S$, for $i = 0, \dots, n+1$ define $\tilde{t}(a_0; a_1, \dots, a_n) \in \mathcal{T}^1_S(n)$ as the sum of all rooted planar 3-valent trees with n leaves decorated by $a_1, \dots, a_n$ (in this order) and its root decorated by a_0. We define

$$t(a_0; a_1, \dots, a_n; a_{n+1}) := \tilde{t}(a_0; a_1, \dots, a_n) - \tilde{t}(a_{n+1}; a_1, \dots, a_n).$$

In this definition $t(a; b) = 0$ by our convention.

Definition 4.1. (Admissible decomposition). An admissible decomposition of $(a_0; a_1, \dots, a_n; a_{n+1})$ is an ordered decomposition $D = P_1 \cup \dots \cup P_k$ of the regular polygon with vertices (with clockwise order) $a_0, \dots, a_{n+1}$ into subpolygons P_i by diagonals. The ordering should satisfy the following admissibility condition. If for $i < j$, P_i and P_j have a common edge and their union has vertices $a_{i_1}, \dots, a_{i_m}$ in clockwise ordering starting with the vertex with the smallest index, then the edge $a_{i_1}a_{i_m}$ should belong to P_i.
For a subpolygon P with vertices $a_{i_1}, \dots, a_{i_m}$ ordered as above we let

$$t(P) := t(a_{i_1}; \dots; a_{i_m}),$$

and for a decomposition $D = P_1 \cup \dots \cup P_k$ we let:

$$t(D) := [t(P_1)| \dots |t(P_k)] \in B(\mathcal{T})^0.$$

Lemma 4.2 ([FJ]). *The element $T(a_0; \dots; a_{n+1}) := \sum_D t(D) \in B(\mathcal{T})^0$ (where the sum is over all the admissible decomposition of $\mathbf{a}$ as above) has differential zero so defines an element in $H^0B(\mathcal{T})$ denoted again by $T(\mathbf{a})$.*

Proof. Combinatorial arguments with the formula

$$dt(a_0;a_1,\dots,a_n;a_{n+1}) = \\ -\sum_{0\leqslant i<j\leqslant n} t(a_0;a_1,\dots,a_i,a_{j+1},\dots;a_{n+1})t(a_i;a_{i+1},\dots,a_j;a_{j+1})$$

proved in [FJ] give the claim. □

By quite combinatorial arguments the proof of Theorem 1.3 for $T(\mathbf{a})$ is shown in [FJ].

Let $\mathcal{T}'_S$ be the sub-DGA of $\mathcal{T}_S$ generated by admissible trees, i.e. those trees that the non-zero labels do not repeat. There is a map of DGA's :

$$dec : \mathcal{T}'_S \longrightarrow \mathcal{T}_{S,\{0,1\}}$$

given by decorating the leaves with zero label by 1 and the rest of edges by 0.

Definition 4.3. For generic sequence $(a_0;a_1,\dots,a_n;a_{n+1})$, i.e. a sequence with non repeating non-zero terms, define the motivic analogue of the iterated integral

$$(-1)^n\int_{a_0}^{a_{n+1}}\frac{dt}{t-a_1}\circ\cdots\circ\frac{dt}{t-a_n}$$

by

$$\rho(dec(T(a_0;a_1,\dots,a_n;a_{n+1}))).$$

This is an element of $\chi_F(n) = \mathrm{H}^0B(\mathcal{N})(n)$ and will be denoted by $\mathbb{I}(a_0;\dots;a_{n+1})$. Here $\rho : \widetilde{\mathcal{T}}_{S,\{0,1\}} \longrightarrow \mathcal{N}$ and it induces a morphism $\rho : \mathrm{H}^0B(\widetilde{\mathcal{T}}_{S,\{0,1\}}) \longrightarrow \mathrm{H}^0B(\mathcal{N})$.

In §7, it will be shown that the Hodge realization of this element agrees with the framed mixed Hodge Tate structure of the above iterated integral.

5. Bloch-Křiž' recipe

In this section we review the construction of Bloch-Křiž for associating a framed MHTS to an element of $\mathrm{H}^0B(\mathcal{N})$. In fact for our application it is only enough to recall §8 of [BK] where they give the Hodge realization for $\mathrm{H}^0B(\mathcal{N}')$ for a particular sub-DGA $\mathcal{N}'$ of $\mathcal{N}$.
Let $\omega(n,r)$ be a collection of real oriented subvarieties of $(\mathbb{P}^1)^n$ of codimension $2r$. Assume that for $S_1 \in \omega(n,r)$ and $S_2 \in \omega(m,s)$, $S_1\times S_2 \in \omega(n+m,r+s)$, and for $\sigma \in G_n = (\mathbb{Z}/2)^n \rtimes \Sigma_n$, $\sigma(S_1) \in \omega(n,r)$. Moreover assume that the intersection of an element of $\omega(n,r)$ with a

hyperplane $t_i = 0$ or ∞ belongs to $\omega(n-1, r)$. Define a DGA with an Adams grading by:

$$\mathcal{D}^n(r) := \mathrm{Alt} \varinjlim \mathrm{H}_{2r-2n}(S \cup \mathcal{J}^{2r-n}, \mathcal{J}^{2r-n})$$

where $\mathcal{J}^n$ is the union of all the codimension 1 hyper planes of $(\mathbb{P}^1)^n$ obtained by letting one coordinate equal to 1. The limit is taken over all $S \in \omega(2r-n, r)$. This has a natural structure of DGA (refer to [BK] §8).

Let us quickly recall some basic notions from mixed Hodge-Tate structures.

Definition 5.1. A MHTS H is a finite dimensional $2\mathbb{Z}$-graded $\mathbb{Q}$-vector space $H_{dR} = \oplus_n H_{2n}$ together with a $\mathbb{Q}$-subspace H_B of $H_{dR} \otimes \mathbb{C}$ such that for all m:

$$\mathrm{Im}\left(H_B \cap (\bigoplus_{n \leqslant m} H_{2n} \otimes \mathbb{C}) \xrightarrow{Proj} H_{2m} \otimes \mathbb{C} \right) = (2\pi i)^m H_{2m}.$$

A morphism f is a graded morphism f_{dR} from H_{dR} to H'_{dR} such that $f_{dR} \otimes 1 : H_{dR} \otimes \mathbb{C} \longrightarrow H'_{dR} \otimes \mathbb{C}$ sends H_B to H'_B, this induced map is denoted by f_B.

The notion of framed MHTS was introduced by Beilinson, Goncharov, Schechtman and Varchenko in [BGSV]. This will give a concrete way of thinking about the Hopf algebra of the coordinate ring of the tannakian Galois group.

Definition 5.2. A framing on a MHTS H is the choice of a frame vector $v \in H_{2r}$ and a coframe vector $\hat{v} \in \mathrm{Hom}(H_{2s}, \mathbb{Q})$. Two framed MHTS $(H, v, \hat{v})$ and $(H', v', \hat{v}')$ such that for an integer n there is a morphism from n-Tate twisting $H(n) \longrightarrow H'$ that respects the frames, are called equivalent. This relation generates an equivalence relation. The equivalence classes of framed MHTS is denoted by χ_{MHTS}. This is graded where $[(H, v, \hat{v})]$ where $v \in H_{2r}$ and $\hat{v} \in \mathrm{Hom}(H_{2s}, \mathbb{Q})$ has degree $r - s$. It is easily shown that the negative weights vanish. The following definitions of product and coproduct make χ_{MHTS} into a Hopf algebra:

$$[(H, v, \hat{v})] \cdot [(H', v', \hat{v}')] = [(H \otimes H', v \otimes v', \hat{v} \otimes \hat{v}')]$$

$$\Delta = \bigoplus_m \Delta_m : \quad \Delta_m[(H, v, \hat{v})] = \sum_i [(H, v, \hat{b}_i)] \otimes [(H, b_i, \hat{v}')]$$

where b_i is a basis for H_{2m} and $\hat{b}_i$ is the dual basis for $\mathrm{Hom}(H_{2m}, \mathbb{Q})$. It can be shown that the category of finite dimensional graded comodules over χ_{MHTS} is equivalent to the category of MHTS's.

Bloch-Kříž [BK] prove that:

Theorem 5.3. *Under the admissibility condition (cf. loc.cit) for $(\mathcal{N}', \mathcal{D})$ the graded $\mathbb{Q}$-vector space $H^0B(\mathcal{N}') = \oplus_r H^0B(\mathcal{N}')(r)$ where $H^0B(\mathcal{N}')(r)$ means its degree $2r$-part, together with the subspace defined by the image of the map λ defined as composition*

$$H^0B(\mathcal{D}, \mathcal{N}') \xrightarrow{B(\lambda_0, id)} H^0B(\mathbb{C}[x], \mathcal{N}') \xrightarrow{x=1} H^0B(\mathcal{N}')_{\mathbb{C}}$$

with the integration map λ_0 (cf. loc.cit.) is a pro-MHTS. We denote its sub-MHTS on $\oplus_{0 \leqslant r \leqslant n} H^0B(\mathcal{N}')(r)$ by $H(\mathcal{N}', n)$.

Definition 5.4. The realization map $\mathrm{Real}'_{MHTS} : \mathrm{H}^0B(\mathcal{N}')(n) \to \chi_{MHTS}(n)$ is given by sending $a \in \mathrm{H}^0B(\mathcal{N}')(n)$ to the framed MHTS $H := H(\mathcal{N}', n)$ whose frame is given by $a \in \mathrm{H}^0B(\mathcal{N}')(n) = H_{2n}$ and whose coframe is induced from the augmentation isomorphism $\mathrm{H}^0B(\mathcal{N}')(0) \to \mathbb{Q}$.

It is proven in [BK] that:

Theorem 5.5. *The realization map $Real'_{MHTS}$ is morphism of graded Hopf algebras. It is equal to the composition of $H^0B(\mathcal{N}') \longrightarrow H^0B(\mathcal{N})$ with the general Hodge realization $Real_{MHTS} : H^0B(\mathcal{N}) \longrightarrow \chi_{MHTS}$ constructed in § 7 of* [BK].

6. Hodge realization of the motivic iterated integrals

In this section we assume that F is a subfield of $\mathbb{C}$ and S is a finite subset of F. We will calculate the Hodge realization of $\mathbb{I}(\mathbf{a}) \in \mathrm{H}^0B(\mathcal{N}')$, where $\mathbf{a} = (a_0; a_1, \ldots, a_n; a_{n+1})$ is a *generic* sequence in S. We will do this parallel to the special case $\mathbb{I}_n(a) := \mathbb{I}(0; \underbrace{1, 0, \ldots, 0}_{n}; a)$ for $a \in F$ which is done in [BK]. Here $\mathcal{N}' = \mathcal{N}_{\mathbf{a}}$ is the sub-DGA of $\mathcal{N}$ generated by the cycles $\rho \circ dec \circ t(\mathbf{a}') \in \mathcal{N}^1(n)$ for all sub-sequences $\mathbf{a}'$ of $\mathbf{a}$. Notice that by definition $\mathcal{N}'$ is a minimal DGA, i.e. it is connected and $d(\mathcal{N}') \subseteq \mathcal{N}'^+ \cdot \mathcal{N}'^+$.

Once and for all choose a path γ with interior in $\mathbb{A}^1(\mathbb{C}) - S$ from the tangential base point a_0 to the tangential base point a_{n+1}. For $1 \leqslant i \leqslant n$ we define intermediate cycles $\eta_i(\mathbf{a})$ inside $\square_{\mathbb{C}}^{2n-i}$ with (real) dimension $2n-i$. It is defined as follows. Consider the sum of all rooted planar forests with i connected components and n leaves, decorated by $a_1, \ldots, a_n$. Its roots are decorated by variables $\gamma(s_1), \ldots, \gamma(s_i)$ for $0 \leqslant s_1 \leqslant \cdots \leqslant s_i \leqslant 1$. Apply the morphism $\rho \circ dec$ to this which will give an oriented (real) cycle (with boundary) of dimension $2n - i$

inside $\square_{\mathbb{C}}^{2n-i}$. Let Γ be a small disk around zero in $\square_{\mathbb{C}}$ with its canonical orientation. We define:

$$\tau_i(\mathbf{a}) = (\delta\eta_i(\mathbf{a})) \cdot \Gamma \cdot (\delta\Gamma)^{i-1} + (-1)^i \eta_i(\mathbf{a}) \cdot (\delta\Gamma)^i$$

where $\cdot$ denotes the usual alternating product and δ denotes the topological boundary defined for a cycle $f(s_1, \dots, s_n)$ by:

$$(\delta f)(s_1, \dots s_{n-1}) = -f(0, s_1, s_2, \dots, s_{n-1}) + f(s_1, s_1, s_2, \dots, s_{n-1}) - \cdots$$
$$\cdots + (-1)^n f(s_1, \dots, s_{n-1}, s_{n-1}) + (-1)^{n+1} f(s_1, \dots, s_{n-1}, 1).$$

Finally denote

$$\xi_\gamma(\mathbf{a}) = \sum_{k=1}^{n} \tau_k(\mathbf{a}).$$

Let $\omega(*,*)$ be the subvarieties which support the cycles $\xi(\mathbf{a}')\rho(\mathbf{a}'')$ and $\rho(\mathbf{a}')$ for all sub sequences of $\mathbf{a}'$ of $\mathbf{a}$, here $\mathbf{a}''$ is the complement of $\mathbf{a}'$ inside $\mathbf{a}$. Denote by $\mathcal{D} = \mathcal{D}_{\mathbf{a}}$ the corresponding DGA obtained from $\omega(*,*)$. Hence $\xi_\gamma(\mathbf{a}) \in \mathcal{D}^0(n)$. As in lemma 4.2 for an admissible decomposition $D = P_1 \cup \cdots \cup P_k$ let $\xi_\gamma(D) := \xi_\gamma(P_1)[\rho(P_2)| \dots |\rho(P_k)] \in B(\mathcal{D}, \mathcal{N}')$.

Lemma 6.1. *Define $Z_\gamma(\mathbf{a}) := 1 \cdot [\mathbb{I}(\mathbf{a})] + \sum_D \xi_\gamma(D)$ where the sum is over all admissible decompositions of $\mathbf{a}$. Then $d(Z_\gamma(\mathbf{a})) = 0$. Therefore $Z_\gamma(\mathbf{a})$ defines an element of $H^0 B(\mathcal{D}, \mathcal{N}')$.*

Proof. This follows form combinatorial arguments with the differential formula for $\xi_\gamma(\mathbf{a})$ shown in [FJ]

$$d\xi_\gamma(\mathbf{a}) = \rho(\mathbf{a}) - \sum_{0 \leqslant i < j \leqslant n} \xi_\gamma(a_0; \dots, a_i, a_{j+1}, \dots; a_{n+1}) \rho(a_i; \dots; a_{j+1}). \quad \square$$

Lemma 6.2. *Assume $a_0 \neq a_1$ and $a_n \neq a_{n+1}$. The image of $\xi_\gamma(\mathbf{a}) \in \mathcal{D}^0(n)$ under the integration map λ_0 ([BK]) is given by the iterated integral:*

$$(-x)^n \int_\gamma \frac{dt}{t - a_1} \circ \cdots \circ \frac{dt}{t - a_n}.$$

Furthermore $\lambda Z_\gamma(\mathbf{a}) \in H^0 B(\mathcal{N}'_{\mathbb{C}})$ (λ was defined in theorem 5.3) is:

$$\sum \int_\gamma \frac{dt}{t - a_{i_1}} \circ \dots \frac{dt}{t - a_{i_k}} \prod_{j=0}^{k} \mathbb{I}(a_{i_j}; \dots; a_{i_{j+1}}),$$

where the sum is taken over all indices $0 = i_0 < i_1 < \cdots < i_k < i_{k+1} = n + 1$ with $k = 0, 1, 2, \dots$.

Proof. For reasons of type, the only term in $\sum_{k=1}^{n}\tau_k(\mathbf{a})$ which contributes in $\lambda_0(\xi_\gamma(\mathbf{a}))$ is $\tau_n(\mathbf{a})$ which is

$$\tau_n(\mathbf{a}) = (\delta\eta_n(\mathbf{a}))\cdot\Gamma\cdot(\delta\Gamma)^{n-1} + (-1)^n\eta_n(\mathbf{a})\cdot(\delta\Gamma)^n.$$

The first part of this sum gives zero integral for reasons of type. Note that $\eta_n(\mathbf{a})$ is $\mathrm{Alt}(f(s_1,a_1),\ldots,f(s_n,a_n))$, where

$$f(s_i,a_i) = \begin{cases} 1-\frac{\gamma(s_i)}{a_i} & \text{if } a_i\neq 0,\\ \gamma(s_i) & \text{if } a_i = 0\end{cases}.$$

Therefore:

$$\int_{\eta_n(\mathbf{a})}\frac{dz_1}{z_1}\wedge\cdots\wedge\frac{dz_n}{z_n} = \int_\gamma\frac{dt}{t-a_1}\circ\cdots\circ\frac{dt}{t-a_n}.$$

Note that this integral is convergent due to the assumption of the lemma. This shows that

$$x^n(2\pi i)^{-n}\int_{\tau_n(\mathbf{a})}\frac{dz_1}{z_1}\wedge\cdots\wedge\frac{dz_{2n}}{z_{2n}} = (-x)^n\int_\gamma\frac{dt}{t-a_1}\circ\cdots\circ\frac{dt}{t-a_n}.$$

This proves the first part of the lemma. To prove the second part, note that if in the set of all admissible decompositions one fixes $P_1 = (a_0,a_{i_1},\ldots,a_{i_k},a_{n+1})$ then

$$\sum_D[\rho(P_2)|\ldots|\rho(P_k)] = \prod_{j=0}^{k}\mathbb{I}(a_{i_j};\ldots;a_{i_{j+1}}).$$

This together with the first part of the lemma, letting $x=1$ finishes the proof. $\square$

The MHTS on the pro-unipotent fundamental torsor $\Pi^{uni}(\mathbb{A}^1-S;a_0,a_{n+1})$ is given by:

$$\Pi_{dR}(\mathbb{A}^1-S;a_0,a_{n+1}) := \mathbb{Q}\langle\langle X_s\rangle\rangle_{s\in S}\quad \mathrm{weight}(X_s):=-2,$$
$$\Pi_B(\mathbb{A}-S;a_0;a_{n+1}) := \mathrm{Im}\left(\Pi^{uni}(\mathbb{A}^1-S;a_0,a_{n+1})\xrightarrow{\Phi}\mathbb{C}\langle\langle X_s\rangle\rangle_{s\in S}\right).$$

Here the map Φ is defined by

$$\gamma\mapsto\sum\left(\int_\gamma\frac{dt}{t-s_1}\circ\cdots\circ\frac{dt}{t-s_n}\right)X_{s_1}\cdots X_{s_n}$$

and the sum is taken over $n=0,1,\ldots$ and $s_i\in S$. The Hodge analogue of the iterated integral $\int_\gamma\frac{dt}{t-a_1}\circ\cdots\circ\frac{dt}{t-a_n}$ is the framed MHTS given above for $\Pi := \Pi^{uni}(\mathbb{A}^1-S;a_0,a_{n+1})$, together with frames $1\in\Pi_0$, $(X_{a_1}\cdots X_{a_n})'\in\mathrm{Hom}(\Pi_{-2n},\mathbb{Q})$. Here $\{(X_{s_1}\cdots X_{s_m})'\}_{s_i\in S}$ is the dual basis. We denote this by $\mathbb{I}^{\mathcal{H}}(a_0;a_1,\ldots,a_n;a_{n+1})$.

Let $\mathbf{a} = (a_0; a_1, \ldots; a_{n+1})$ be a generic sequence in S such that $a_0 \neq a_1$ and $a_n \neq a_{n+1}$. Then it is shown in [FJ] that the pair $(\mathcal{N}_{\mathbf{a}}, \mathcal{D}_{\mathbf{a}})$ is admissible in the sense of [BK].

Theorem 6.3. *The Hodge realization of* $\mathbb{I}(\mathbf{a})$ *is equal to* $(-1)^n \mathbb{I}^{\mathcal{H}}(\mathbf{a})$.

Proof. We may assume that $a_0 \neq a_1$ and $a_n \neq a_{n+1}$. We need a special care for a treatment of divergent case (cf.[FJ]). Let $(J_{\mathbf{a}}, \mathbb{I}(\mathbf{a}), \epsilon) := \mathrm{Real}'_{MHTS}(\mathbb{I}(\mathbf{a}))$. We define a map $\Pi(n) \longrightarrow J_{\mathbf{a}}$ by:

$$X_{b_1} \cdots X_{b_k} \mapsto (-1)^k \sum \prod_{j=0}^{k} \mathbb{I}(a_{i_j}; a_{i_j+1}, \ldots; a_{i_{j+1}})$$

where the sum is taken over all indices $0 = i_0 < i_1 < \cdots < i_k < i_{k+1} = n+1$ such that $a_{i_j} = b_j$. This obviously extends to a graded linear map from $\Pi(n) \longrightarrow J_{\mathbf{a}}$ which respects the corresponding frames. We need to show that for a path γ from a_0 to a_{n+1}:

$$\sum \int_\gamma \frac{dt}{t-a_{i_1}} \circ \ldots \frac{dt}{t-a_{i_k}} \prod_{j=0}^{k} \mathbb{I}(a_{i_j}; \ldots; a_{i_{j+1}}),$$

where the sum is taken over all indices $0 = i_0 < i_1 < \cdots < i_k < i_{k+1} = n+1$ with $k = 0, 1, 2, \ldots,$ belongs to the Betti space of J_a. But this element is the image of $Z_\gamma(\mathbf{a})$ under the map λ according to the lemma 6.2. This completes the proof of the equivalence of the two framed MHTS's. □

Our results gives the following examples.

Example 6.4. (1) For a,b,$c \in \mathbb{C}$ in generic position, the tree in the figure 1 gives a framed mixed Tate motive of the integral $\int_a^b \frac{dt}{t-c}$

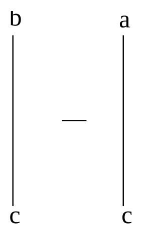

FIGURE 1. The element of $H^0(B(\mathcal{T}_S))$ corresponding to $\int_a^b \frac{dt}{t-c}$.

(2) For a,b,c,$d \in \mathbb{C}$ in generic position, the tree in the figure 2 gives a framed mixed Tate motive of the integral $It \int_a^b \frac{dt}{t-c} \circ \frac{dt}{t-d}$

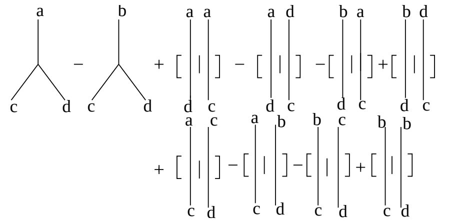

FIGURE 2. The element of $H^0(B(\mathcal{T}_S))$ corresponding to $\int_a^b \frac{dt}{t-c} \circ \frac{dt}{t-d}$.

Example 6.5. (1) For $a \in \mathbb{C}$ and $k \in \mathbb{N}$ the framed mixed Tate motive associated to the polylogarithm $Li_k(a) = \sum_n \frac{a^n}{n^k}$ is given by the following cycles [BK]:

$$\rho_k(a) + [\rho_{k-1}(a)|\rho_1(1-a)] + [\rho_{k-2}(a)|\rho_1(1-a)|\rho_1(1-a)] + \cdots \\ + [\rho_1(a)|\rho_1(1-a)|\cdots|\rho_1(1-a)]$$

where

$$\rho_k(a) = (-1)^{\frac{k(k-1)}{2}} \mathrm{Alt}\left(x_1, \cdots, x_{k-1}, 1-x_1, 1-\frac{x_2}{x_1}, \cdots, 1-\frac{x_{k-1}}{x_{k-2}}, 1-\frac{a}{x_{k-1}}\right).$$

(2) For $n,\ m \in \mathbb{N}$ and $a,\ b \in \mathbb{C}$ with $b \neq 1$ and $ab \neq 1$ the framed mixed Tate motive associated to the double polylogarithm $Li_{n,m}(a,b) = \sum_{0<k<l} \frac{a^k b^l}{k^n l^m}$ is given by the following cycles:

$$\sum_{k\geqslant 0, l\geqslant 0} (-1)^{n+k} \binom{n+m-k-l-2}{n-l-1} \rho_{a,b}(l, n+m-l-k-2, k) + [\rho_b(n)|\rho_a(m)]$$
$$+ (-1)^n \binom{n+m-2}{n-1} \left[\mathrm{Alt}\{1-ab\} \;\middle|\; \rho_{\frac{1}{a}}(n+m-1) - \rho_b(n+m-1)\right]$$
$$+ \sum_{i=1}^{m-1} (-1)^n \binom{m+n-i-2}{m-i-1} [\rho_{ab}(i+1)|\rho_{\frac{1}{a}}(m+n-i-1)]$$
$$+ \sum_{j=1}^{n-1} (-1)^{n+j+1} \binom{m+n-j-2}{n-j-1} [\rho_{ab}(j+1)|\rho_b(m+n-j-1)].$$

where

$$\begin{aligned}\rho_{a,b}(n,m,k) =& (-1)^{m+n} \mathrm{Alt}\Big(1-\frac{1}{x_1}, 1-\frac{x_1}{x_2}, \ldots, 1-\frac{x_{k+n}}{x_{k+n+1}}, 1-abx_{k+n+1}, \\ & 1-\frac{x_{k+1}}{x_{k+n+2}}, 1-\frac{x_{k+n+2}}{x_{k+n+3}}, \ldots, 1-\frac{x_{k+n+m}}{x_{k+n+m+1}}, 1-bx_{k+n+m+1}, \\ & x_{k+n+m+1}, x_{k+n+m}, \ldots, \widehat{x_{k+1}}, \ldots, x_1\Big),\end{aligned}$$

$$\rho_a(n) = (-1)^{n-1}\mathrm{Alt}\Big(1-\frac{1}{x_1}, 1-\frac{x_1}{x_2}, \ldots, 1-\frac{x_{n-2}}{x_{n-1}}, 1-ax_{n-1}, x_{n-1}, \ldots, x_1\Big).$$

Because of the condition $b \neq 1$ and $ab \neq 1$, this cycle does not provide a cycle representing the double zeta value $\zeta(n,m) = \sum_{0<k<l} \frac{1}{k^n l^m}$.

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